

MATCHABLE NUMBERS

NATHAN MCNEW AND CARL POMERANCE

ABSTRACT. We say a natural number n is matchable if there is a bijection from the set of $\tau(n)$ divisors of n to the set $\{1, 2, \dots, \tau(n)\}$, where corresponding numbers are relatively prime. We show that the set of matchable numbers has an asymptotic density, which we compute, and we show that every squarefree number is matchable. We also present some related unsolved problems.

1. INTRODUCTION

Given two finite sets of integers of the same cardinality, a bijection ψ between them is said to be a coprime matching if for each x in the domain of ψ , x and $\psi(x)$ are coprime. Alternatively, one can consider the bipartite graph from one set to the other where there is an edge whenever the numbers on the edge's vertices are coprime: a coprime matching is a perfect matching in this graph.

There have been several papers on coprime matchings over the years, mostly where the two sets are intervals of consecutive integers. For example, in [5], Pomerance and Selfridge proved a conjecture of D. J. Newman that there is always a coprime matching from $\{1, 2, \dots, n\}$ to any other interval of n consecutive integers. This was generalized by Bohman and Peng [1] to some cases where the intervals are arbitrarily placed on the number line, and they showed a connection to the notorious lonely runner conjecture. Their paper was subsequently improved in [3], generalized to a counting problem in [4], with further progress by McNew [2] and Sah and Sawhney [9].

In this paper we consider a problem of Récaman [6], where one of the sets continues to be an initial interval of consecutive integers, but the other set is generally not an interval, rather it is the set of divisors of a number n . More precisely, for a positive integer n let $D(n)$ denote the set of divisors of n , and $\tau(n) = |D(n)|$. We say an integer n is *matchable* if there is a coprime matching between $\{1, 2, \dots, \tau(n)\}$ and $D(n)$.

One might think at first that every number is matchable, and this holds for $n = 1, 2, \dots, 7$. However, 8 is not matchable, nor is any subsequent multiple of 4. The proof is easy: If $4 \mid n$, then at least $2/3$ of the members of $D(n)$ are even, but fewer than $2/3$ of the members of $\{1, 2, \dots, k\}$ are odd when $k > 3$. Since even divisors must be mapped to odd numbers in a coprime matching, proper multiples of 4 are seen to be not matchable. This can be generalized to other primes as well, see below.

Say a number n is an M-number if it is not divisible by any p^p with p prime. For example, every squarefree number is an M-number. It is easy to see that the set of M-numbers

possesses an asymptotic density, which density is

$$\alpha = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = 0.72199023441955 \dots$$

Among the comments in [6] we find the conjecture of König and Alekseyev that every M-number is matchable, and few non-M-numbers are matchable, and in particular, the asymptotic density of the set of matchable numbers is α . In this paper we prove that the asymptotic density of the symmetric difference of the set of matchable numbers and the set of M-numbers is 0, so that the density of the set of matchable numbers is indeed α .

We agree with the conjecture of König and Alekseyev that every M-number is matchable. Towards a proof we show at least that every squarefree number is matchable. Probably our techniques can be extended to the remaining M-numbers.

For each prime p let

$$M_p = \prod_{q \leq p} q^{q-1},$$

where q runs over primes. It is easy to see that each M_p is matchable. Indeed, for $1 \leq j \leq \tau(M_p)$, we map j to

$$\psi(j) := \prod_{q \leq p} q^{(j \bmod q)}.$$

To see this note that since $\tau(M_p) = \prod_{q \leq p} q$, the Chinese remainder theorem shows that each integer $j \in [1, \tau(M_p)]$ corresponds to a unique vector $(j \bmod 2, j \bmod 3, \dots, j \bmod p)$. Further, for $q \leq p$, $q \mid j$ if and only if $q \nmid \psi(j)$.

The set of M-numbers is precisely the set of all divisors of the numbers M_p as p varies. As noted in [6], if one can prove that all of the divisors of a matchable number are themselves matchable, we would immediately have the corollary that every M-number is matchable. Unfortunately, we did not find a way to make this elegant plan work.

2. PRELIMINARY RESULTS

We generalize the result that if $4 \mid n$ and $n > 4$, then n is not matchable.

Proposition 1. *Suppose $p^2 \mid n$ for some prime p , and let $0 \leq r < p$ be the remainder when $\tau(n)$ is divided by p . If $\tau(n) > r(p+1)$, then n is not matchable.*

Proof. Suppose p^k is the largest power of p dividing n , with $k \geq 2$. We can partition the $\tau(n)$ divisors of n into $k+1$ sets each having size $\tau(n/p^k)$ according to the power to which p appears as a factor. Only one of those sets will contain integers coprime to p , and so the total number of divisors of n coprime to p is $\frac{\tau(n)}{k+1} \leq \frac{\tau(n)}{p+1}$. On the other hand, the number of integers in $[1, \tau(n)]$ divisible by p is

$$\left\lfloor \frac{\tau(n)}{p} \right\rfloor = \frac{\tau(n) - r}{p}. \tag{1}$$

Since $\tau(n) > r(p+1)$ we find that $(\tau(n) - r)(p+1) > \tau(n)p$ and thus the quantity in (1) is strictly greater than $\frac{\tau(n)}{p+1}$, the upper bound we just found for the number of divisors of n coprime to p . Thus, there are too few divisors of n coprime to p to match with these integers up to $\tau(n)$ divisible by p . \square

Corollary 1. *If $p^p | n$ for some prime p , and $\tau(n) \geq p^2$, then n is not matchable.*

Let $\omega(n)$ denote the number of different primes that divide n . Note that $\tau(n) \geq 2^{\omega(n)}$, with equality if and only if n is squarefree.

Corollary 2. *The upper density of the set of matchable integers is at most α .*

Proof. It suffices to show that the set of matchable numbers that are not M-numbers has asymptotic density 0. The set of integers divisible by some p^p for $p \geq N$ has density $\ll N^{-N}$. If $p^p | n$ and n is matchable, then Corollary 1 implies that $\omega(n) \leq 2 \log p / \log 2$. So, if n is matchable and not an M-number, it is either divisible by some p^p for $p \geq N$ or $\omega(n) \leq 2 \log N / \log 2$.

The counting function to x of the latter numbers n is $\ll x(\log \log x)^{2 \log N / \log 2} / \log x$, so for N fixed, the set has density 0. Putting the two together, the upper density is $\leq N^{-N}$, and since N is arbitrary, the corollary is proved. \square

We remark that if we let $N = \log \log x$ in the proof, we have the counting function to x of the set of matchable numbers that are not M-numbers is $\leq x / (\log x)^{1+o(1)}$ as $x \rightarrow \infty$. This result is best possible since every number $27p$ with p prime is matchable (as is easily checked), yet not an M-number.

Our plan is to first prove that squarefree numbers with at least 45 prime factors are matchable, and then by a somewhat different method, we prove it for squarefree numbers with fewer than 45 prime factors. Finally, we extend the argument to M-numbers with sufficiently many prime-power divisors not of the form p^{p-1} and not divisible by the square of any large prime, and use a density argument to finish.

We conclude with some open problems and a discussion of strongly matchable numbers. These are numbers n such that there is a coprime matching between $D(n)$ and every coprime arithmetic progression of $\tau(n)$ integers.

3. SQUAREFREE NUMBERS WITH MANY PRIME FACTORS

Lemma 1. *If $2n$ is matchable, then so is n .*

Proof. We may assume n is odd. A coprime matching for $2n$ pairs $D(2n) = D(n) \cup 2D(n)$ with $[1, \tau(2n)] = [1, 2\tau(n)]$. Since the even divisors $2D(n)$ must be paired with odd integers in $[1, 2\tau(n)]$, the odd divisors $D(n)$ are paired with the even integers $\{2, 4, \dots, 2\tau(n)\}$. Dividing by 2, this gives a coprime matching of $D(n)$ with $[1, \tau(n)]$, so n is matchable. \square

Theorem 1. *Every squarefree number having at least 45 prime factors is matchable.*

We now introduce a notion that will be used to track error bounds in counting.

Definition 1. For an integer $k \geq 1$, we say that a set S of integers is a k -AP combination if it can be constructed as follows:

- (1) A single arithmetic progression is a 1-AP combination.
- (2) If S_1 is a k_1 -AP combination, S_2 is a k_2 -AP combination, and $S_1 \cap S_2 = \emptyset$, then $S_1 \cup S_2$ is a $(k_1 + k_2)$ -AP combination.
- (3) If S_1 is a k_1 -AP combination, $S_2 \subseteq S_1$ is a k_2 -AP combination, then $S_1 \setminus S_2$ is a $(k_1 + k_2)$ -AP combination.

Note that since we don't assume the constituent arithmetic progressions are nonempty, any set S that is a k -AP combination is also a k' -AP combination for any $k' > k$.

Lemma 2. *If S is a k -AP combination whose constituent arithmetic progressions all have common differences coprime to d , then the number of elements of S divisible by d is $|S|/d + \theta$ where $|\theta| \leq k$.*

Proof. We proceed by induction on the construction of S . For the base case, let S be an arithmetic progression of length m with common difference q where $\gcd(d, q) = 1$. If $m \geq d$, the elements of S form a complete residue system modulo d , and among any d consecutive terms, exactly one is divisible by d . So, the count of d -multiples is $m/d + \theta$ with $|\theta| \leq 1$.

For the inductive step, suppose $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$. Then the count of d -multiples in S equals

$$(|S_1|/d + \theta_1) + (|S_2|/d + \theta_2) = |S|/d + (\theta_1 + \theta_2),$$

where $|\theta_1| \leq k_1$, $|\theta_2| \leq k_2$, so $|\theta_1 + \theta_2| \leq k_1 + k_2$.

If $S_2 \subseteq S_1$ and $S = S_1 \setminus S_2$, then the count of d -multiples in S equals

$$(|S_1|/d + \theta_1) - (|S_2|/d + \theta_2) = |S|/d + (\theta_1 - \theta_2),$$

where $|\theta_1 - \theta_2| \leq k_1 + k_2$. □

Lemma 3. *Suppose $p_1 < p_2 < \dots < p_j$ are primes with $p_1 = 2$ and I is an interval of L consecutive integers where $2^j \mid L$ and $L \geq 4^j$. Then I can be partitioned into 2^j sets $A_{v,j}$ of size $L/2^j$, parametrized by divisors v of $m_j := p_1 p_2 \dots p_j$, such that every member of $A_{v,j}$ is coprime to v . Moreover, each $A_{v,j}$ is a 2^{j-1} -AP combination whose constituent AP's have common differences dividing m_j , and hence for any d coprime to m_j , the number of elements of $A_{v,j}$ divisible by d is within 2^{j-1} of $|A_{v,j}|/d$.*

Proof. The error bound of 2^{j-1} follows from Lemma 2. For notational simplicity we assume that $I = \{1, 2, \dots, L\}$. We prove the existence of the stated partition by induction on j .

For $j = 1$, since $p_1 = 2$ by assumption, we partition I into $A_{1,1}$, the even integers in I , and $A_{2,1}$, the odd integers. Each is a single arithmetic progression, hence a 1-AP combination.

For $j = 2$, let $p = p_2$ be the second prime. We construct the four sets as follows. Let x be chosen so that the number of odd integers in $I \cap [1, x]$ that are not divisible by p equals $L/4$. To see that such an x exists, note that the majority of odd integers up to L are not divisible by p . Indeed, the number of odd elements of I divisible by p is $\leq L/2p + 1$, and this is $< L/4$ by the assumption $L \geq 4^j$. Set

$$\begin{aligned} A_{2p,2} &= \{\text{odd } i \in I : i \leq x, p \nmid i\}, \\ A_{2,2} &= \{\text{odd } i \in I : p \mid i, i \leq x\} \cup \{\text{odd } i \in I : i > x\}. \end{aligned}$$

Similarly, let y be chosen so that the number of even integers in $I \cap [1, y]$ not divisible by p equals $L/4$, and set

$$\begin{aligned} A_{p,2} &= \{\text{even } i \in I : i \leq y, p \nmid i\}, \\ A_{1,2} &= \{\text{even } i \in I : p \mid i, i \leq y\} \cup \{\text{even } i \in I : i > y\}. \end{aligned}$$

Each of $A_{2p,2}$ and $A_{p,2}$ is the difference of two arithmetic progressions, hence a 2-AP combination by rule (3). Each of $A_{2,2}$ and $A_{1,2}$ is the disjoint union of two arithmetic progressions, hence a 2-AP combination by rule (2). Thus each set is a 2-AP combination.

For $j \geq 3$, assume the result holds for $j - 1$: each $A_{v,j-1}$ is a 2^{j-2} -AP combination. For each $v \mid m_{j-1}$, we partition $A_{v,j-1}$ into $A_{v,j}$ and $A_{p_j v,j}$ using a cutoff as follows. Let z_v be chosen so that the number of elements of $A_{v,j-1}$ that are $\leq z_v$ and not divisible by p_j equals $L/2^j$. Set

$$\begin{aligned} A_{p_j v,j} &= \{i \in A_{v,j-1} : i \leq z_v, p_j \nmid i\}, \\ A_{v,j} &= \{i \in A_{v,j-1} : p_j \mid i, i \leq z_v\} \cup \{i \in A_{v,j-1} : i > z_v\}. \end{aligned}$$

Every element of $A_{p_j v,j}$ is coprime to p_j (and was already coprime to v), so is coprime to $p_j v$. Every element of $A_{v,j}$ was already coprime to v .

We now verify the sizes. By the induction hypothesis, the number of p_j -multiples in $A_{v,j-1}$ is within 2^{j-2} of $|A_{v,j-1}|/p_j = L/(p_j 2^{j-1})$, hence at most

$$\frac{L}{p_j 2^{j-1}} + 2^{j-2} \leq \frac{L}{3 \cdot 2^{j-1}} + \frac{L}{2^{j+2}} < \frac{L}{2^j},$$

using $p_j \geq 3$ and $L \geq 4^j$. Thus there are enough non- p_j -multiples in $A_{v,j-1}$ to fill $A_{p_j v,j}$ to size $L/2^j$, and $A_{v,j}$ receives the remaining $L/2^j$ elements.

It remains to show each new set is a 2^{j-1} -AP combination. By induction, we know that each set $A_{v,j-1}$ is a 2^{j-2} -AP combination.

A key observation is that intersecting with $\{i : i \leq z_v\}$ or $\{i : i > z_v\}$ preserves the AP-combination structure of a set. If T is a k -AP combination, then $\{i \in T : i \leq z_v\}$ and $\{i \in T : i > z_v\}$ are each k -AP combinations. (We simply truncate each constituent arithmetic progression. This could result in some of them being empty.) Similarly, restricting to p_j -multiples preserves the AP-combination structure, since if T is a k -AP combination, then $\{i \in T : p_j \mid i\}$ is a k -AP combination (just replace each arithmetic progression by the sub-arithmetic progression of its p_j -multiples).

Now consider each of the sets used to construct $A_{v,j}$ and $A_{p_j v,j}$. First, $\{i \in A_{v,j-1} : p_j \mid i, i \leq z_v\}$ is just $A_{v,j-1}$ restricted to p_j -multiples and then to $\{i \leq z_v\}$. So, it has the same combination structure as $A_{v,j-1}$, hence is a 2^{j-2} -AP combination. Similarly $\{i \in A_{v,j-1} : i > z_v\}$ is $A_{v,j-1}$ restricted to $\{i > z_v\}$, hence also a 2^{j-2} -AP combination. Since $A_{v,j}$ is the disjoint union of these two it is a 2^{j-1} -AP combination.

For $A_{p_j v,j}$, we note that

$$\begin{aligned} A_{p_j v,j} &= \{i \in A_{v,j-1} : p_j \nmid i, i \leq z_v\} \\ &= \{i \in A_{v,j-1} : i \leq z_v\} \setminus \{i \in A_{v,j-1} : p_j \mid i, i \leq z_v\}. \end{aligned}$$

As argued above, each of these sets is a 2^{j-2} -AP combination, and the latter is a subset of the former, so their difference is a 2^{j-1} -AP combination. This completes the induction. \square

Proof of Theorem 1. Let u be a squarefree number satisfying the hypotheses with $\ell = \omega(u)$. If u is odd, then $2u$ is even with $\omega(2u) = \ell + 1 \geq 46$ prime factors. If we can show that $2u$ is matchable, then u is matchable by Lemma 1. Thus we may assume u is even, so that $2 = p_1 < p_2 < \dots < p_\ell$ where $u = p_1 p_2 \dots p_\ell$.

For an integer $j \leq \omega(u)/2$ to be chosen later, we let $m_j = p_1 p_2 \dots p_j$ and $n = u/m_j$, so that m_j is the product of the j smallest primes dividing u while n contains the rest. We then apply Lemma 3 using these j smallest prime factors and $I = [1, \tau(u)] = [1, 2^{j+\omega(n)}]$. Since $4^j \leq \tau(u)$, the lemma allows us to produce a partition of I into sets $A_{v,j}$ as described in the lemma, each having size $\tau(n)$.

Thus it now suffices to show that there are one-to-one correspondences between $D(n)$ and each of the sets $A_{v,j}$ with corresponding numbers relatively prime. Indeed, for $v \mid m_j$, the correspondence can instead be between $A_{v,j}$ and $vD(n)$, keeping the coprime property. Then, as $D(u)$ is the disjoint union of the sets $vD(n)$ as v runs over all of the divisors of m_j , we can piece together these matchings and so have a coprime matching of all divisors of $u = m_j n$ to $[1, \tau(u)]$.

To show the existence of these one-to-one correspondences we note that any divisor $d \mid n$ is coprime to m_j , so by Lemma 3, the number of integers in $A_{v,j}$ divisible by d is within $\kappa := 2^{j-1}$ of $\tau(n)/d$.

We choose j as follows. For $\ell \geq 68$ we take $j = \lfloor \sqrt{\ell} \rfloor$ and for $45 \leq \ell \leq 67$ we take $j = 4$ except $j = 3$ when $\ell \in \{46, 47, 48\}$ and $j = 5$ when $\ell = 52$. Note that in every case we have $j \leq \sqrt{\ell}$. With these choices, $\omega(n) = \ell - j \geq 41$ and one can verify that

$$f(n) := \sum_{p \mid n} \frac{1}{p} = \sum_{i=j+1}^{\ell} \frac{1}{p_i} \leq \sum_{i=j+1}^{\ell} \frac{1}{P_i} < \frac{93}{100}, \quad (2)$$

where P_i is the i -th prime. To see this in the case that $j = \lfloor \sqrt{\ell} \rfloor$, we use that $P_i > i \log i$ (see [7]), so

$$f(n) < \frac{1}{\sqrt{68} \log \sqrt{68}} + \int_{\sqrt{\ell}}^{\ell} \frac{dt}{t \log t} < 0.06 + \log 2 < 0.76.$$

We use Hall's theorem for the bipartite graph on $A = A_{v,j}$ to $D(n)$ where there is an edge precisely when $a \in A$ and $d \mid n$ are coprime. Suppose that $S \subset A$ and that $s \in S$ minimizes $k = \omega((s, n))$. We wish to show that $|S|$ is bounded above by the size of the neighborhood $N(S)$ of S . If $k = 0$ then the neighborhood of S is all of $D(n)$, so the condition holds.

Assume that $k \geq 1$. We have

$$\begin{aligned} |S| &\leq \sum_{\substack{d \mid n \\ \omega(d)=k \\ d \leq \tau(u)}} \sum_{\substack{a \in A \\ d \mid a}} 1 \leq \sum_{\substack{d \mid n \\ \omega(d)=k}} \left(\frac{\tau(n)}{d} + \kappa \right) \\ &\leq \tau(n) \frac{f(n)^k}{k!} + \kappa \binom{\omega(n)}{k} \\ &< \frac{\tau(n)}{2^k} \left(\frac{(93/50)^k}{k!} + 2^{2j-1+k-\ell+\log_2 \binom{\omega(n)}{k}} \right). \end{aligned} \quad (3)$$

Since the primes dividing n are all at least P_{j+1} , a divisor d of n with $\omega(d) = k$ satisfies $d \geq P_{j+1} P_{j+2} \cdots P_{j+k}$. For such a d to divide any element of $A \subset [1, \tau(u)]$, we need $d \leq \tau(u) = 2^\ell$. Thus we only need consider $k \leq \bar{k}$, where \bar{k} is the largest integer with $P_{j+1} P_{j+2} \cdots P_{j+\bar{k}} \leq 2^\ell$.

3.1. Case $k \geq 4$. For Hall's condition to hold, since the neighborhood of S has size at least $\tau(n)/2^k$, it suffices if

$$\frac{(93/50)^k}{k!} + 2^{2j-1+k-\ell+\log_2 \binom{\ell-j}{k}} < 1 \quad (4)$$

for each k with $4 \leq k \leq \bar{k}$. Note that $(93/50)^4/24 < 0.50$, so we need only show that the second term is at most $\frac{1}{2}$.

Let $E_k := 2j - 1 + k - \ell + \log_2 \binom{\omega(n)}{k}$ denote the exponent, so the second term equals 2^{E_k} .

Large ℓ ($\ell \geq 192$). Set $j = \lfloor \sqrt{\ell} \rfloor$. We first bound \bar{k} . Since $j = \lfloor \sqrt{\ell} \rfloor \geq 13$ for $\ell \geq 192$, we have $P_{j+1} \geq P_{14} = 43$. The constraint $\prod_{i=1}^k P_{j+i} \leq 2^\ell$ implies $43^k < 2^\ell$, giving $\bar{k} < \ell / \log_2 43 < 0.185\ell$. With $\omega(n) = \ell - j > \ell - \sqrt{\ell}$, we have $\bar{k}/\omega(n) < 0.20$ for $\ell \geq 192$.

With the entropy function $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$, writing $c = k/\omega(n) < 0.20$, we have $H(c) < 0.722$, so $c + H(c) < 0.922$. Thus

$$\begin{aligned} E_k &\leq 2j - 1 + (c + H(c))\omega(n) - \ell \\ &< 2\sqrt{\ell} - 1 + 0.922(\ell - \sqrt{\ell}) - \ell \\ &= -0.078\ell + 1.078\sqrt{\ell} - 1 < -1.03 \end{aligned}$$

for $\ell \geq 192$. Since $(93/50)^4/4! < 0.50$ and $2^{E_k} < 0.50$, we have (4) for $4 \leq k \leq \bar{k}$.

Moderate ℓ ($68 \leq \ell < 192$). We use $j = \lfloor \sqrt{\ell} \rfloor$ and verify (4) by direct computation. For each ℓ in this range, we find

$$\bar{k} = \max\{k : \prod_{i=j+1}^{j+k} P_i < 2^\ell\}.$$

Then, for each k with $4 \leq k \leq \bar{k}$, we compute $E_k = k + 2j - 1 - \ell + \log_2 \binom{\ell-j}{k}$ and verify that $(93/50)^k/k! + 2^{E_k} < 1$.

Small ℓ ($45 \leq \ell \leq 67$). Here we take (as above) $j = 4$ when $\ell = 45$, $j = 3$ for $46 \leq \ell \leq 48$, $j = 4$ for $49 \leq \ell \leq 67$ (except $j = 5$ when $\ell = 52$). As above, we verify (4) directly for all $4 \leq k \leq \bar{k}$. This concludes the case $k \geq 4$ for all $\ell \geq 45$.

3.2. Case $k = 3$. Here the neighborhood of S has at least $\tau(n)/8$ elements. Let $a_1 \in S$ with $(a_1, n) = q_1 q_2 q_3$ for distinct primes q_1, q_2, q_3 dividing n .

First, suppose that the triple $\{q_1, q_2, q_3\}$ is unique (every $a \in S$ has $(a, n) = q_1 q_2 q_3$ or $\omega((a, n)) > 3$). Then

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ q_1 q_2 q_3 | a}} 1 + \sum_{\substack{d|n \\ \omega(d)=4}} \left(\frac{\tau(n)}{d} + \kappa \right) \\ &\leq \frac{\tau(n)}{q_1 q_2 q_3} + \kappa + \tau(n) \frac{f(n)^4}{24} + \kappa \binom{\omega(n)}{4} \\ &< \frac{\tau(n)}{8} \left(\frac{8}{q_1 q_2 q_3} + \frac{8f(n)^4}{24} + 2^{2\sqrt{\ell}+2-\ell} \left(1 + \binom{\ell}{4} \right) \right), \end{aligned}$$

using that $\tau(n) = 2^{\ell-j}$. Since $q_1 q_2 q_3 \geq 7 \cdot 11 \cdot 13 = 1001$ for $j = 3$ and $f(n) < 0.93$, the expression in parentheses is at most $8/1001 + 8(0.93)^4/24 + 2^{2\sqrt{\ell}+2-\ell} \binom{\ell}{4} < 0.26 + 2^{2\sqrt{\ell}+2-\ell} \binom{\ell}{4}$, which is < 1 when $\ell \geq 27$.

Now suppose S contains at least 2 elements a_1, a_2 with $\omega((a_i, n)) = 3$, say $(a_1, n) = q_1 q_2 q_3$ and $(a_2, n) = q_4 q_5 q_6$, with $\{q_1, q_2, q_3\} \neq \{q_4, q_5, q_6\}$. The two triples share at most 2 primes,

so by inclusion-exclusion $|N(S)| \geq 2 \cdot \tau(n)/8 - \tau(n)/16 = 3\tau(n)/16$. Also,

$$\begin{aligned} |S| &\leq \tau(n) \frac{f(n)^3}{6} + \kappa \binom{\omega(n)}{3} \\ &< \frac{3\tau(n)}{16} \left(\frac{16f(n)^3}{18} + \frac{16}{3} \cdot 2^{2\sqrt{\ell}-1-\ell} \binom{\ell}{3} \right). \end{aligned}$$

Since $f(n) < 0.93$, we have $16(0.93)^3/18 < 0.72$, and $\frac{16}{3} \cdot 2^{2\sqrt{\ell}-1-\ell} \binom{\ell}{3} < 0.28$ for $\ell \geq 25$. This completes the case $k = 3$.

3.3. Case $k = 2$. Here the neighborhood of S has at least $\tau(n)/4$ elements. Let $a_1 \in S$ with $(a_1, n) = q_1q_2$ for distinct primes $q_1, q_2 \mid n$.

First, suppose that the pair $\{q_1, q_2\}$ is unique (every $a \in S$ has $(a, n) = q_1q_2$ or $\omega((a, n)) > 2$). Then

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ q_1q_2 \mid a}} 1 + \sum_{\substack{d \mid n \\ \omega(d)=3}} \left(\frac{\tau(n)}{d} + \kappa \right) \leq \frac{\tau(n)}{q_1q_2} + \kappa + \tau(n) \frac{f(n)^3}{6} + \kappa \binom{\omega(n)}{3} \\ &< \frac{\tau(n)}{4} \left(\frac{4}{q_1q_2} + \frac{4f(n)^3}{6} + 2^{2\sqrt{\ell}+1-\ell} \left(1 + \binom{\ell}{3} \right) \right). \end{aligned}$$

Since $q_1q_2 \geq 77$ for $j \geq 3$ and $f(n) < 0.93$, we have $4/77 + 4(0.93)^3/6 < 0.59$, and $2^{2\sqrt{\ell}+1-\ell} (1 + \binom{\ell}{3}) < 0.33$ for $\ell \geq 23$.

Now suppose S contains 2 elements a_1, a_2 with $\omega((a_i, n)) = 2$, say $(a_1, n) = q_1q_2$ and $(a_2, n) = q_3q_4$, with $\{q_1, q_2\} \neq \{q_3, q_4\}$. Assume also that every other $a \in S$ has either $(a, n) = q_1q_2$, $(a, n) = q_3q_4$ or $\omega((a, n)) \geq 3$. The pairs share at most one prime, so by inclusion-exclusion $|N(S)| \geq 2 \cdot \tau(n)/4 - \tau(n)/8 = 3\tau(n)/8$.

Also,

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ q_1q_2 \mid a}} 1 + \sum_{\substack{a \in A \\ q_3q_4 \mid a}} 1 + \sum_{\substack{d \mid n \\ \omega(d) \geq 3}} \left(\frac{\tau(n)}{d} + \kappa \right) \\ &< \frac{3\tau(n)}{8} \left(\frac{8}{3} \cdot \frac{2}{77} + \frac{8f(n)^3}{18} + \frac{8}{3} \cdot 2^{2\sqrt{\ell}-1-\ell} \left(2 + \binom{\ell}{3} \right) \right). \end{aligned}$$

Since $f(n) < 0.93$, we have $16/(3 \cdot 77) + 8(0.93)^3/18 < 0.43$, and $\frac{8}{3} \cdot 2^{2\sqrt{\ell}-1-\ell} (2 + \binom{\ell}{3}) < 0.5$ for $\ell \geq 21$.

So now assume that there are at least 3 different values of (a, n) for $a \in S$ with exactly 2 prime factors. If the 3 values are q_1q_2, q_3q_4, q_5q_6 , then the case when one prime is shared among all 3 numbers gives the smallest size for $N(S)$ and that size is $(7/16)\tau(n)$. Then

$$|S| < \frac{7\tau(n)}{16} \left(\frac{16f(n)^2}{14} + \frac{16}{7} \cdot 2^{2\sqrt{\ell}-1-\ell} \binom{\omega(n)}{2} \right) < \frac{7\tau(n)}{16} (0.99 + 0.01),$$

for $\ell \geq 26$, completing the case $k = 2$.

3.4. **Case $k = 1$.** Here the neighborhood of S has at least $\tau(n)/2$ elements, so we may assume that $|S| \geq \tau(n)/2$. Let $a_1 \in S$ with $(a_1, n) = q_1$ for some prime $q_1|n$.

First, suppose that q_1 is unique (every $a \in S$ has $(a, n) = q_1$ or $\omega((a, n)) \geq 2$). Then

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ q_1|a}} 1 + \sum_{\substack{a \in A, q_1 \nmid a \\ \omega((a, n)) \geq 2}} 1 \leq \frac{\tau(n)}{q_1} + \kappa + \sum_{\substack{d|n, q_1 \nmid d \\ \omega(d)=2}} \left(\frac{\tau(n)}{d} + \kappa \right) \\ &\leq \frac{\tau(n)}{q_1} + \kappa + \tau(n) \frac{(f(n) - 1/q_1)^2}{2} + \kappa \binom{\omega(n) - 1}{2} \\ &< \tau(n) \left(\frac{f(n)^2}{2} + \frac{1 - f(n)}{q_1} + \frac{1}{2q_1^2} \right) + 2^{j-1} \left(1 + \binom{\omega(n) - 1}{2} \right). \end{aligned}$$

Using $f(n) < 93/100$ and $q_1 \geq P_{j+1} \geq 7$ for $j \geq 3$, the coefficient of $\tau(n)$ is at most $0.432 + 0.07/7 + 0.5/49 < 0.453$. Since $\tau(n) = 2^{\ell-j}$,

$$\begin{aligned} |S| &< 0.453\tau(n) + 2^{j-1} \left(1 + \binom{\omega(n) - 1}{2} \right) \\ &\leq \frac{\tau(n)}{2} \left(0.906 + 2^{2\sqrt{\ell}-\ell} \left(1 + \binom{\ell - 4}{2} \right) \right). \end{aligned}$$

This is $< \tau(n)/2$ when $\ell \geq 30$.

Now suppose S contains elements a_i with $(a_i, n) = q_i$ for $r \geq 2$ distinct primes q_1, \dots, q_r dividing n . The neighborhood of S contains all divisors of n coprime to at least one of q_1, \dots, q_r .

If $r = 2$, this neighborhood has cardinality $\frac{3}{4}\tau(n)$. We have

$$\begin{aligned} |S| &\leq \sum_{i=1}^2 \left(\frac{\tau(n)}{q_i} + \kappa \right) + \sum_{\substack{d|n \\ \omega(d)=2}} \left(\frac{\tau(n)}{d} + \kappa \right) \\ &< \tau(n) \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{f(n)^2}{2} \right) + 2^{j-1} \left(2 + \binom{\omega(n)}{2} \right). \end{aligned}$$

Since $q_1, q_2 \geq P_{j+1}$ and $f(n) < 93/100$, the first term is at most $2/7 + 0.433 < 0.72\tau(n)$ for $j \geq 3$. Thus,

$$|S| < \frac{3\tau(n)}{4} \left(0.96 + \frac{4}{3} \cdot 2^{2\sqrt{\ell}-1-\ell} \left(2 + \binom{\ell - 3}{2} \right) \right).$$

This is $< \frac{3}{4}\tau(n)$ when $\ell \geq 22$.

If $r = 3$, the neighborhood has cardinality at least $\frac{7}{8}\tau(n)$, and

$$\begin{aligned} |S| &< \tau(n) \left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{f(n)^2}{2} \right) + 2^{j-1} \left(3 + \binom{\omega(n)}{2} \right) \\ &< \frac{7\tau(n)}{8} \left(\frac{0.862}{7/8} + \frac{8}{7} \cdot 2^{2\sqrt{\ell}-1-\ell} \left(3 + \binom{\ell - 3}{2} \right) \right), \end{aligned}$$

using that q_1, q_2, q_3 are distinct primes ≥ 7 for $j \geq 3$ and $f(n) < 93/100$. Our estimate for $|S|$ is $< \frac{7}{8}\tau(n)$ for $\ell \geq 35$.

If $r \geq 4$, the neighborhood has cardinality at least $\frac{15}{16}\tau(n)$, and

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ \omega((a,n)) \geq 1}} 1 \leq \sum_{q|n} \sum_{\substack{a \in A \\ q|a}} 1 \leq \tau(n)f(n) + \kappa\omega(n) \\ &< \frac{93}{100}\tau(n) + 2^{j-1}\omega(n) < \frac{15\tau(n)}{16} \left(\frac{93/100}{15/16} + \frac{16}{15} \cdot 2^{2\sqrt{\ell}-1-\ell} \right). \end{aligned}$$

This is $< \frac{15}{16}\tau(n)$ for $\ell \geq 13$. Hall's condition holds, completing the proof. \square

4. SQUAREFREE NUMBERS WITH FEW PRIME FACTORS

In this section we prove the following theorem.

Theorem 2. *Every squarefree number with at most 44 prime factors is matchable.*

We illustrate the argument for $\ell = 24$, the smallest case that exhibits the full complexity, and then discuss the modifications for other $\ell \leq 44$. We first establish Proposition 2 at the end of this section for $\ell = 24$: we show there is a coprime matching from $D(u)$ to the odd integers in $[1, 2^{25}]$, where u is odd, squarefree, and with 24 prime factors.

Let $u = q_1 q_2 \cdots q_{24}$ be the product of any 24 distinct odd primes $3 \leq q_1 < q_2 < \cdots < q_{24}$, so $\tau(u) = 2^{24} = 16,777,216$. We wish to show there is a coprime matching between the 2^{24} odd integers in $[1, 2^{25}]$ and $D(u)$. By Hall's theorem it suffices to show that for every set S of odd integers in $[1, 2^{25}]$, the neighborhood

$$N(S) = \{d \mid u : \gcd(d, a) = 1 \text{ for some } a \in S\}$$

satisfies $|N(S)| \geq |S|$.

Let $c_i(u)$ denote the number of odd integers in $[1, 2^{25}]$ sharing exactly i prime factors with u ,

$$c_i(u) := \#\{\text{odd } a \in [1, 2^{25}] : \omega((a, u)) = i\}$$

and $c_{\geq i}(u) = \sum_{i' \geq i} c_{i'}(u)$ the count of those odd integers sharing at least i prime factors with u . A key observation is that the counts c_i are only made larger if the odd primes comprising u are replaced with smaller odd primes.

Monotonicity. Let $n = 3 \cdot 5 \cdot 7 \cdots 97$ denote the product of the *smallest* 24 odd primes, so $q_i \geq p_i$ where p_i is the i -th odd prime. If $a = q_{j_1} q_{j_2} \cdots q_{j_i} b$ is any odd integer in $[1, 2^{25}]$ counted by $c_i(u)$, with $(b, u) = 1$, then it is simple to check that the mapping

$$q_{j_1} q_{j_2} \cdots q_{j_i} b \rightarrow p_{j_1} p_{j_2} \cdots p_{j_i} b$$

is an injection from integers counted by $c_i(u)$ to those counted by $c_i(n)$. Thus, $c_i(u) \leq c_i(n)$.

The rest of the proof will rely heavily on the counts $c_i(n)$ and $c_{\geq i}(n)$, which we will write as c_i and $c_{\geq i}$, respectively. We will also occasionally need counts for the number of integers having a fixed greatest common divisor with u (respectively n).

As with the counts c_i , the number of odd integers in the interval $[1, 2^{25}]$ whose gcd with u is $q_{j_1} q_{j_2} \cdots q_{j_i}$ is majorized by the number of such integers whose gcd with n is $p_{j_1} p_{j_2} \cdots p_{j_i}$. We denote $\gcd_d := \#\{\text{odd } a \in [1, 2^{25}] : (a, n) = d\}$.

We record in Table 1 the computed values of c_i for $3 \leq \ell \leq 44$, and in Table 2 the computed values of \gcd_d for $d = 105, 15, 21, 3, 5$ (all computations were performed using exact values; large entries in the tables are rounded up, as noted in the captions). The former table also

contains a column, ω_{\max} containing the largest value of i such that c_i is nonzero, and the latter table includes a column x_3 , containing the count of odd integers in $[1, 2^{25}]$ which are either divisible by 3 or counted by $c_{\geq 3}$, whose purpose will be explained shortly. In the latter table, values are only included when needed in the analogue of the argument described below (blank entries are not necessarily 0).

We will use frequently the observation that if $k := \min_{a \in S} \omega((a, u))$ then

$$|N(S)| \geq 2^{24-k} = \frac{\tau(u)}{2^k}.$$

Let S be a nonempty set of odd integers in $[1, 2^{25}]$; we verify Hall's condition by considering the possible values of $k = \min_{s \in S} \omega((s, u))$.

By summing the values of c_i in Table 1, in the row for $\ell = 24$ we find that

$$c_{\geq 5} = 88,525, \quad c_{\geq 4} = 485,129, \quad c_{\geq 3} = 2,151,882, \quad c_{\geq 2} = 6,377,708, \quad c_{\geq 1} = 12,741,251.$$

4.1. Step 1: Cases $k \geq 4$. First, suppose $k \geq 5$. Then by the monotonicity described above we have $|S| \leq c_{\geq 5} = 88,525$. On the other hand, since $\omega_{\max} = 7$ in the row $\ell = 24$ of Table 1, we have $\omega((s, u)) \leq 7$ for all s , and thus $|N(S)| \geq |N(s)| \geq 2^{24-7} = 131,072$. So $|S| < |N(S)|$ and Hall's condition holds.

If $k = 4$ then every element $s \in S$ has $\omega((s, u)) \geq 4$ so $|S| \leq c_{\geq 4} = 485,129$. Since at least one element has $\omega((s, u)) = 4$ we find that $|N(S)| \geq 2^{24-4} = 1,048,576 > 485,129 \geq |S|$, so again, Hall's condition holds.

4.2. Step 2: Case $k = 3$. Every element of S has $\omega \geq 3$, so $|S| \leq c_{\geq 3} = 2,151,882$, while $|N(S)| \geq 2^{21} = 2,097,152$. Since $2,151,882 > 2,097,152$ we cannot immediately conclude that Hall's condition holds.

Let $s \in S$ be such that $\omega((s, u)) = 3$ and let $d = (s, u)$. Then $d \geq 105 = 3 \times 5 \times 7$ and, by the monotonicity mentioned above we have $\gcd_d \leq \gcd_{105} = 83,729$.

If d were unique, then every element of S sharing exactly 3 prime factors with u would need to have greatest common divisor d with u . Then we would have

$$|S| \leq \gcd_d + c_{\geq 4}(u) \leq \gcd_{105} + c_{\geq 4} = 83,729 + 485,129 < 2^{21} \leq |N(S)|,$$

and so Hall's condition would be satisfied. So we suppose d is not unique, namely there is a second element, s' having $\omega((s', u)) = 3$ but $(s, u) \neq (s', u)$. In this case, since (s, u) and (s', u) can share at most two primes, considering the neighborhood of just s and s' we find, by inclusion-exclusion that

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{21} + 2^{21} - 2^{20} = \frac{3}{16} \cdot 2^{24} = 3,145,728.$$

But $|S| \leq c_{\geq 3} = 2,151,882 < 3,145,728 \leq |N(S)|$, hence Hall's condition holds when $k = 3$.

4.3. Step 3: Case $k = 2$. Every element of S has $\omega \geq 2$, so $|S| \leq c_{\geq 2} = 6,377,708$, while $|N(S)| \geq 2^{22} = 4,194,304$. Again, we cannot immediately conclude using Hall's theorem, but we can assume $|S| > |N(S)| \geq 2^{22}$.

Let $s \in S$ such that $\omega((s, u)) = 2$ and let $e = (s, u)$. Then $e \geq 15 = 3 \times 5$ and, by monotonicity, $\gcd_e \leq \gcd_{15} = 504,881$. If e were unique, we would find that

$$|S| \leq \gcd_{15} + c_{\geq 3} = 504,881 + 2,151,882 = 2,656,763 < 2^{22} \leq |N(S)|,$$

satisfying Hall's condition, and so we assume that e is not unique, namely there exists a second $s' \in S$ with $\omega((s', u)) = 2$ but $(s', u) = e' \neq e$. Since e and e' share at most one prime factor, we can update our lower bound for the neighborhood $N(S)$ to

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{22} + 2^{22} - 2^{21} = \frac{3}{8} \times 2^{24} = 6,291,456. \quad (5)$$

Note, that this is still less than $c_{\geq 2}$. Now, if e and e' were the only two such divisors, then

$$\begin{aligned} |S| &\leq \gcd_e + \gcd_{e'} + c_{\geq 3} \leq \gcd_{15} + \gcd_{21} + c_{\geq 3} \\ &= 504,881 + 336,514 + 2,151,882 = 2,993,277 < 6,291,456 \leq |N(S)|. \end{aligned}$$

Again, Hall's condition is satisfied in this case, leaving us with the possibility that there is a third $s'' \in S$, with $\omega((s'', u)) = 2$ and where $(s'', u) = e'' \notin \{e, e'\}$. In this case the neighborhood of S is minimized in the situation when e, e', e'' all share a single prime factor, in which case it is bounded below by

$$|N(S)| \geq |N(s) \cup N(s') \cup N(s'')| \geq 3 \cdot 2^{22} - 3 \cdot 2^{21} + 2^{20} = 3 \cdot 2^{21} + 2^{20} = \frac{7}{16} \cdot 2^{24} = 7,340,032.$$

But $|S| \leq c_{\geq 2} = 6,377,708 < 7,340,032 \leq |N(S)|$. Hence, in every case Hall's condition holds when $k = 2$.

4.4. Step 4: Case $k = 1$. Every element of S has $\omega \geq 1$, so $|S| \leq c_{\geq 1} = 12,741,251$, while $|N(S)| \geq 2^{23} = 8,388,608$, so we will again need to work with the specific gcds.

Proceeding as above, we suppose $s \in S$ has $\omega((s, u)) = 1$, with $(s, u) = q$ and suppose that q is unique. Then

$$|S| \leq \gcd_3 + c_{\geq 2} = 2,019,785 + 6,377,708 = 8,397,493 > 2^{23}.$$

Note that unlike in previous steps, this bound does not allow us to conclude (yet) that there is another s' and another prime q' distinct from q with $(s', u) = q'$. So we consider again those elements of S sharing two prime factors with u , one of which is q .

Let x_q denote the total number of odd numbers in $[1, 2^{25}]$ which are either divisible by q or counted by $c_{\geq 3}$. As with other statistics, this count is majorized by the count $x_3 = 6,334,949$ which is included in Table 2.

Since this count is smaller than our bound $|N(S)| \geq 2^{23}$, we would be done if S consisted only of elements counted by x_3 . So we suppose that S contains s' , not counted by x_q . Then $\omega((s', u)) \leq 2$ and $q \nmid s'$. In this case, we can update our lower bound on $|N(S)|$. This quantity is smallest when s' shares precisely two prime factors with u , neither of which is q , in which case we find that

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{23} + 2^{22} - 2^{21} = 5 \cdot 2^{21} = \frac{5}{8} \cdot 2^{24} = 10,485,760.$$

Since this quantity now exceeds $\gcd_3 + c_{\geq 2}$, we find that Hall's criterion is necessarily satisfied unless $(s', u) = q'$ where $q' \neq q$ is a different prime factor. Now our lower bound for $|N(S)|$ improves to

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{23} + 2^{23} - 2^{22} = 3 \cdot 2^{22} = \frac{3}{4} \cdot 2^{24} = 12,582,912.$$

Using this bound, we now return to our original line of argumentation. This bound exceeds

$$\begin{aligned} c_{\geq 2} + \gcd_q + \gcd_{q'} &\leq c_{\geq 2} + \gcd_3 + \gcd_5 \\ &= 6,377,708 + 2,019,785 + 1,010,179 = 9,407,672 < 12,582,912. \end{aligned}$$

From this, we see that Hall's condition will be satisfied unless S contains a third element s'' with $\omega((s'', u)) = 1$ and $(s'', u) = q''$ for some prime divisor $q'' \neq q, q'$ of u .

With three elements $s, s',$ and s'' all having mutually distinct prime gcds with u , inclusion-exclusion gives

$$|N(S)| \geq |N(s) \cup N(s') \cup N(s'')| \geq 3 \cdot 2^{23} - 3 \cdot 2^{22} + 2^{21} = \frac{7}{8} \cdot 2^{24} = 14,680,064.$$

But $|S| \leq c_{\geq 1} = 12,741,251 < 14,680,064 \leq |N(S)|$, so Hall's condition holds when $k = 1$.

4.5. Step 5: Case $k = 0$. If the minimum value of $\omega((s, u))$ over $s \in S$ is 0, then some $s \in S$ is coprime to u , so $N(S) = D(u)$ and $|N(S)| = 2^{24} \geq |S|$ and Hall's condition holds immediately.

In all cases we have $|N(S)| \geq |S|$, so Hall's condition holds and a perfect matching exists. Since the argument used only upper bounds on census counts, and these upper bounds hold for any $u = q_1 \cdots q_{24}$ with $q_i \geq p_i$, the result applies to every squarefree product of 24 distinct odd primes.

4.6. Adjustments for other $\ell \leq 44$. Essentially the same argument can be carried through using any ℓ in place of 24, for $3 \leq \ell \leq 44$. The only important adjustment is to change the computed $c_{\geq j}$ values and \gcd_d values according to the entries in Table 1 and Table 2.

For values of $\ell < 24$ some of the steps above can be omitted, for example when $\ell < 23$, the argument involving \gcd_{21} is unnecessary. After considering $c_{\geq 3} + \gcd_{15}$ and using it to conclude that there exists $s' \in S$ with $\omega((s', u)) \leq 2$, $(s', u) \neq (s, u)$ it is already possible to conclude directly that $|N(S)| > |S|$, since in this case we find, as in (5) that $|N(S)| \geq \frac{3}{8} \cdot 2^\ell$, which is already greater than $c_{\geq 2}$. When a quantity is unneeded for the argument, it is omitted from Table 2.

For values of $\ell > 24$, sometimes additional steps are necessary in the initial Step 1. For example when $\ell = 40$, the maximum value of $\omega((s, u))$ over $s \in S$ is 10 (as noted in Table 1 in the ω_{\max} column). Since $2^{40-10} = 1,073,741,824 \geq c_{\geq 7}$, Hall's condition is satisfied for any $k \geq 7$. One can then check for each $4 \leq k < 7$ that $2^{40-k} \geq c_{\geq k}$ so the condition is met for each of these k as well. Putting this all together, we have shown the following.

Proposition 2. *For any $\ell \leq 44$ and u the product of ℓ distinct odd primes, there exists a coprime matching between $D(u)$ and the odd integers in $[1, 2^{\ell+1}]$.*

An identical Hall's theorem argument establishes Proposition 3, with the bipartite graph now having $D(u)$ matched to all integers in $[1, 2^\ell]$ (rather than just odd integers in $[1, 2^{\ell+1}]$), and using the values in Table 3 and Table 4 in place of Tables 1 and 2.

Proposition 3. *For any $\ell \leq 44$ and u the product of ℓ distinct odd primes, there exists a coprime matching between $D(u)$ and the integers in $[1, 2^\ell]$.*

We can now combine these propositions to give a proof of Theorem 2.

Proof of Theorem 2. If u is squarefree, odd, and has at most 44 prime factors, then u is matchable by Proposition 3. If u is even and has $\ell \leq 44$ prime factors, write $u = 2u'$; we create a matching as follows. By Proposition 2 applied to u' (which has $\ell - 1$ odd prime factors), there exists a coprime matching of the divisors of u' to the odd integers in $[1, 2^\ell]$. We associate to each even divisor $2d$ of u the odd integer matched to d by this proposition. Then, by Proposition 3 applied to u' , there exists a coprime matching of $D(u')$ to the integers in $[1, 2^{\ell-1}]$; for every odd divisor d of u (which is also a divisor of u'), we match d to $2a$, where $a \in [1, 2^{\ell-1}]$ is the integer matched to d .

The first construction matches the $2^{\ell-1}$ even divisors of u bijectively to the $2^{\ell-1}$ odd integers in $[1, 2^\ell]$, and the second matches the $2^{\ell-1}$ odd divisors bijectively to the $2^{\ell-1}$ even integers in $[2, 2^\ell]$; together they give a perfect matching from $D(u)$ to $[1, 2^\ell] = [1, \tau(u)]$. Coprimality is preserved: $(2d, a) = (d, a) = 1$ since a is odd, and $(d, 2a) = (d, a) = 1$ since d is odd. \square

5. M-NUMBERS

Say a positive integer is an M -number if it is not divisible by any p^p for p prime, i.e., $v_p(n) \leq p - 1$ for all primes p . Call a prime $p \mid n$ *tight* if $v_p(n) = p - 1$, and write $n = n_T n_R$ where $n_T = \prod_{p \mid n, p \text{ tight}} p^{p-1}$ and $n_R = n/n_T$. Set $r = \text{rad}(n_T) = \tau(n_T)$.

We conjecture that every M -number is matchable. In this section we explain how to generalize the proofs of Lemma 3 and Theorem 1 to show that every M -number with sufficiently many non-tight prime factors and not divisible by the square of any large non-tight prime is matchable (Theorem 3), which implies in particular that the set of non-matchable M -numbers has asymptotic density 0 (Corollary 3).

The key tool is a partition lemma analogous to Lemma 3, but adapted to handle the tight and non-tight primes separately. Let $p_1 < p_2 < \dots$ denote the odd primes in order, and write $a_i = v_{p_i}(n_R)$ for each i . For a parameter $j \geq 0$, let m_j be the p_j -smooth part of n_R (i.e., $m_j = \prod_{i \leq j, a_i \geq 1} p_i^{a_i}$), and set $n' = n_R/m_j$ and $K = \prod_{i \leq j, a_i \geq 1} 2a_i$.

Lemma 4. *Let n be an M -number with n_T, n_R, r, m_j, n' as above. Assume $\tau(n') \geq 4^j \cdot r^j$. Then $[1, \tau(n)]$ can be partitioned into $\tau(n_T) \cdot \tau(m_j)$ sets A_d (one for each $d \mid n_T m_j$), each of size $\tau(n')$, with every element of A_d coprime to d . Moreover each A_d is a K -AP combination with common differences dividing rm_j , so for any $e \mid n'$ the count of elements of A_d divisible by e is within K of $\tau(n')/e$.*

The proof follows the same plan as Lemma 3, with two modifications to handle tight primes and primes of m_j with exponent greater than 1.

The tight primes are handled first and all at once using an argument akin to the one in the Introduction with M_p . Since $v_p(n) = p - 1$ for each prime $p \mid r$, the Chinese remainder theorem gives a bijection $\rho: D(n_T) \rightarrow \{0, 1, \dots, r - 1\}$ defined by $\rho(d_T) \equiv v_p(d_T) \pmod{p}$ for each $p \mid r$. We partition $[1, \tau(n)] = [1, r\tau(n_R)]$ into $\tau(n_T) = r$ residue classes modulo r , pairing d_T with the class $\rho(d_T) \pmod{r}$. Each class is an arithmetic progression of length $\tau(n_R)$ and common difference r , and for any $a \equiv \rho(d_T) \pmod{r}$, the coprimality $(a, d_T) = 1$ holds because $p \mid a$ if and only if $p \nmid d_T$ for each $p \mid r$. This plays exactly the role of the even/odd split on the prime 2 in Lemma 3, producing $\tau(n_T)$ equal-length arithmetic progressions, one per divisor of n_T , with no contribution to the size of K .

The primes of m_j are then handled by the same inductive construction as in Lemma 3, applied to each of these arithmetic progressions. The one difference is that a prime $p_i \mid m_j$ with exponent $a_i > 1$ requires an $(a_i + 1)$ -way split rather than a binary one: we choose a_i cutpoints to make a_i sets of non-multiples of p_i of size $\tau(n')$. The key point is that the M -number condition gives $a_i < p_i - 1$ (the tight case $a_i = p_i - 1$ has already been handled), and this is precisely what guarantees enough non- p_i -multiples to fill each set, the same role played by the bound $p_j \geq 3$ in Lemma 3. Unfolding the recursion gives

$$K = \prod_{i \leq j, a_i \geq 1} 2a_i \leq \prod_{i \leq j} 2(p_i - 2), \quad (6)$$

and this bound grows roughly as $(2j)^j$ rather than the squarefree growth of 2^{j-1} , which means that Theorem 3 will require more prime factors than Theorem 1 (though we don't show a numerically explicit bound here).

Theorem 3. *Every M -number n with sufficiently many non-tight prime factors that is not divisible by the square of any non-tight prime $q > p_j$, where $j = \lfloor \sqrt{\omega(n_R)} \rfloor$, is matchable.*

The proof follows closely that of Theorem 1, using Lemma 4 in place of Lemma 3. Let $\ell = \omega(n_R)$ be the number of non-tight prime factors of n and $j = \lfloor \sqrt{\ell} \rfloor$. Lemma 4 partitions $[1, \tau(n)]$ into sets A_d , one for each $d \mid n_T m_j$, each of size $\tau(n')$ and forming a K -AP combination with common differences dividing rm_j . For each d we apply Hall's theorem to match $D(n')$ to A_d , exactly as in the proof of Theorem 1, with K playing the role of $\kappa = 2^{j-1}$ and n' playing the role of n .

There are two important differences from the squarefree case. The first is in the construction of m_j . In Theorem 1, m_j is always the product of the first j primes of u . Here, we take m_j to be only the p_j -smooth part of n_R . This is necessary in order to ensure that the bound (6) holds. If we instead took the first j prime factors of n_R regardless of size, those primes could be large, and the M -number condition would then permit large exponents, inflating K . With this change we are still able to ensure that $f(n') < 0.93$.

The second difference is in the size of the error term K in the AP-combination divisibility counts: the bound $K \leq \prod_{i \leq j} 2(p_i - 2)$ grows roughly as $(2j)^j$ rather than 2^{j-1} , so more prime factors are needed for $K/\tau(n')$ to become negligible and all of the Hall conditions to hold. The hypothesis that n is not divisible by the square of any non-tight prime $q > p_j$ ensures that $n' = n_R/m_j$ is squarefree, so the arguments required to ensure that Hall's theorem applies in the proof of Theorem 1 for various values of k can be used nearly verbatim. In particular, $\tau(n') = 2^{\omega(n')}$ and each of the neighborhood bounds $|N(S)| \geq \tau(n')/2^k$ hold as in the squarefree case. We just need larger values of ℓ to overcome the larger value of K .

Once we have obtained the matchings $\phi_d: D(n') \rightarrow A_d$, we can construct the coprime matching of $D(n)$ to $[1, \tau(n)]$ via $\psi(d \cdot e) = \phi_d(e)$, as in Theorem 1.

Since the set of M -numbers having fewer than any fixed number of prime factors has asymptotic density zero, as do those with a repeated large non-tight prime factor, we obtain the following corollary.

Corollary 3. *Every M -number is matchable except possibly for a set of asymptotic density zero.*

With Corollary 2 we have the following.

Corollary 4. *The set of matchable numbers has asymptotic density α .*

6. STRONGLY MATCHABLE NUMBERS

Recall that we say n is strongly matchable if for each coprime arithmetic progression of $\tau(n)$ integers there is a coprime matching to $D(n)$.

Conjecture 1. *A number n is strongly matchable if and only if it is an M-number.*

One would think that our techniques for matchable numbers could be applied here but there is a difficulty. In the proof of Theorem 1 we strongly used that we are mapping $D(n)$ to an interval of small numbers, namely $[1, \tau(n)]$, but with strongly matchable, the interval is not only generalized to a coprime arithmetic progression, it can be anywhere on the number line. The latter condition is the difficulty. We at least have a few results in the direction of the conjecture.

Proposition 4. *Every strongly matchable number is an M-number.*

Proof. We prove the contrapositive. Suppose n is not an M-number, and so $p^p \mid n$ for some prime p . Then at least $p/(p+1)$ of the members of $D(n)$ are divisible by p . On average as a coprime arithmetic progression of $\tau(n)$ integers varies, exactly $(p-1)/p$ of the integers in the set are coprime to p , so there is at least one such coprime arithmetic progression of length $\tau(n)$ with $< p/(p+1)$ of its members coprime to p . We cannot coprimely match $D(n)$ with this interval, so n is not strongly matchable. \square

Proposition 5. *If n is strongly matchable and not divisible by the prime p , then $p^{p-1}n$ is strongly matchable.*

Proof. We have $\tau(p^{p-1}n) = p\tau(n)$. Let I be a coprime arithmetic progression of length $p\tau(n)$, say $I = \{i_1, i_2, \dots, i_{p\tau(n)}\}$. For $j = 1, 2, \dots, p$, let I_j be the subsequence $(i_{j+kp})_k$ of length $\tau(n)$. At most one of these subsequences has its terms all divisible by p , and the other subsequences are all coprime to p . If there is some j where all the terms of I_j are divisible by p , denote this j by j_0 , otherwise let $j_0 = 1$. There are coprime matchings ψ_j from $D(n)$ to I_j for each j . We construct a coprime matching from $D(p^{p-1}n)$ to I as follows. For $D(n)$, we already have ψ_{j_0} . For $p^k D(n)$, $1 \leq k \leq p-1$, $k \neq j_0$, we map it to one of the subsequences I_j not used via ψ_j ; that is, we map $p^k d \in p^k D(n)$ to $\psi_j(d)$. The union of these maps gives a coprime matching from $D(p^{p-1}n)$ to I , completing the proof. \square

Proposition 6. *The set of strongly matchable numbers has lower asymptotic density greater than $4/11$.*

Proof. We first note that if n is strongly matchable and p is a prime with $p \nmid n$ and $p > 2\tau(n)$, then pn is also strongly matchable. Indeed, take an arbitrary coprime arithmetic progression I of $\tau(pn) = 2\tau(n)$ integers. There are coprime matchings of $D(n)$ to both the first half of I and the second half of I , say ψ_1, ψ_2 . Not both of these halves contain a multiple of p , so map $pD(n)$ to a half not containing a multiple of p via $p\psi_i$, and use the unadorned injection for the other half.

Let S_j be the set of odd squarefree numbers s with $\omega(s) = j$ and each prime factor of s is $< 2^j$, and let S be the union of the sets S_j . Suppose that $n > 1$ is an odd squarefree number not divisible by any member of S . Then

$$n = q_1 q_2 \dots q_k, \quad 3 \leq q_1 < q_2 < \dots < q_k, \quad \text{each } q_j \text{ prime,} \quad \text{each } q_j \geq 2^j.$$

Indeed, if not and $q_j < 2^j$ for some j , then $q_1 q_2 \dots q_j \in S_j$, contradicting our assumption that n is not divisible by any member of S .

We claim that any odd squarefree n not divisible by any element of S is strongly matchable. First, $n = 1$ is strongly matchable, and any odd prime q is strongly matchable by the argument above, since $q > 2\tau(1)$. Now suppose that $n_j := q_1 q_2 \dots q_j$ is strongly matchable. Since $\tau(n_j) = 2^j$ and $q_{j+1} > 2^{j+1}$, it follows by the argument above that $n_j q_{j+1}$ is strongly matchable. Induction completes the argument that n is strongly matchable.

It remains to show that such numbers n comprise a set of positive lower density. The reciprocal sum of the primes $q \leq 2^j$ is $\log j + O(1)$, and in fact from [8, (3.18)] and a calculation, this sum is $< \log j$ when $j \geq 4$. Since $\sum_{p \leq 7} 1/p > 1.17619$, it follows that

$$\sum_{s \in S'_j} \frac{1}{s} \leq \frac{(\log j - 1.17619)^j}{j!}, \quad j \geq 4,$$

where S'_j is the set of $s \in S_j$ with all prime factors > 8 . Let T denote the set of squarefree numbers n with all prime factors > 8 . Then T has an asymptotic density equal to

$$\frac{6}{\pi^2} \prod_{p \leq 7} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^2}\right)^{-1} = 0.221640\dots$$

If $s \in S$ divides some $n \in T$, then each prime factor of s is > 8 , so for some $j \geq 4$, we have $s \in S_j$, and so $s \in S'_j$. The upper asymptotic density of the set of multiples of the elements in $\cup_{j \geq 4} S'_j$ is at most

$$\sum_{j \geq 4} \frac{(\log j - 1.17619)^j}{j!} < 0.000331239.$$

Let T' denote the subset of T consisting of numbers not divisible by any member of S , so that every member of T' is strongly matchable. The lower asymptotic density of T' is greater than 0.2213. We can boost this using Proposition 5, getting the lower density of the set of strongly matchable numbers at least 0.3694. \square

REFERENCES

- [1] T. Bohman and F. Peng, Coprime mappings and lonely runners, *Mathematika* **62** (2022), 784–804.
- [2] N. McNew, Permutations and the divisor graph of $[1, n]$, *Mathematika* **63** (2023), 51–67.
- [3] C. Pomerance, Coprime matchings, *Integers* **22** (2022), #A2, 9 pp.
- [4] C. Pomerance, Coprime permutations, *Integers* **22** (2022), #83, 20 pp.
- [5] C. Pomerance and J. L. Selfridge, Proof of D. J. Newman’s coprime mapping conjecture, *Mathematika* **27** (1980), 69–83.
- [6] B. Récaman, Coprime matching of an integer’s $d(n)$ divisors with the set of the first $d(n)$ integers, *mathoverflow*, 2022.
- [7] B. Rosser, The n th prime is greater than $n \log n$, *Proc Lond. Math. Soc. (2)*, **45** (1939), 21–44.
- [8] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.
- [9] A. Sah and M. Sawhney, Enumerating coprime permutations, *Mathematika* **68** (2022), 1120–1134.

Table 1: Census of values of $\omega((s, n))$ for odd $s \in [1, 2^{\ell+1}]$ where n is the product of the first ℓ odd primes; large values are rounded up.

ℓ	ω_{\max}	c_0	c_1	c_2	c_3	c_4	c_5	c_6	$c_{\geq 7}$
3	2	3	4	1					
4	2	7	7	2					
5	2	13	11	8					
6	3	25	21	17	1				
7	3	47	43	33	5				
8	3	89	95	56	16				
9	3	164	210	95	43				
10	4	309	441	176	94	4			
11	4	597	878	376	179	18			
12	4	1166	1736	798	341	55			
13	5	2293	3376	1758	612	152	1		
14	5	4505	6612	3758	1138	364	7		
15	5	8897	12940	7892	2233	768	38		
16	5	17558	25510	16243	4553	1540	132		
17	6	34585	50650	32767	9755	2921	393	1	
18	6	68151	100919	65561	21015	5468	1021	9	
19	6	134282	201536	130617	45041	10387	2370	55	
20	6	264692	402354	260661	95132	20367	5157	213	
21	6	522290	803185	521116	197833	41521	10487	720	
22	7	1031482	1601975	1045031	405967	87226	20541	2076	6
23	7	2039192	3193416	2100106	825167	185875	39443	5363	46
24	7	4035965	6363543	4225826	1666753	396604	75606	12689	230
25	7	7992094	12678222	8506329	3361015	840077	147784	28017	894
26	8	15830224	25256133	17121427	6780185	1762060	297171	58718	2946
27	8	31367217	50318652	34440697	13693052	3657862	613215	118465	8568
28	8	62163303	1.00×10^8	69234321	27697616	7530374	1290469	233192	22820
29	8	1.23×10^8	2.00×10^8	1.39×10^8	56080479	15404730	2739542	454761	56236
30	8	2.45×10^8	3.98×10^8	2.79×10^8	1.13×10^8	31329808	5803412	887715	129521
31	9	4.85×10^8	7.94×10^8	5.60×10^8	2.30×10^8	63616039	12232414	1760793	283999
32	9	9.64×10^8	1.58×10^9	1.12×10^9	4.65×10^8	1.29×10^8	25579607	3565554	598677
33	9	1.91×10^9	3.16×10^9	2.25×10^9	9.40×10^8	2.62×10^8	53108556	7367039	1228836
34	9	3.80×10^9	6.30×10^9	4.52×10^9	1.90×10^9	5.32×10^8	1.09×10^8	15416636	2473356
35	9	7.55×10^9	1.26×10^{10}	9.06×10^9	3.83×10^9	1.08×10^9	2.24×10^8	32498453	4948093
36	10	1.50×10^{10}	2.51×10^{10}	1.82×10^{10}	7.73×10^9	2.20×10^9	4.59×10^8	68545789	9907571
37	10	2.98×10^{10}	5.00×10^{10}	3.64×10^{10}	1.56×10^{10}	4.47×10^9	9.36×10^8	1.44×10^8	19982844
38	10	5.93×10^{10}	9.99×10^{10}	7.30×10^{10}	3.14×10^{10}	9.07×10^9	1.91×10^9	3.01×10^8	40742931
39	10	1.18×10^{11}	1.99×10^{11}	1.46×10^{11}	6.33×10^{10}	1.84×10^{10}	3.89×10^9	6.26×10^8	83930709
40	10	2.35×10^{11}	3.98×10^{11}	2.93×10^{11}	1.28×10^{11}	3.74×10^{10}	7.93×10^9	1.29×10^9	1.74×10^8
41	11	4.66×10^{11}	7.93×10^{11}	5.87×10^{11}	2.57×10^{11}	7.57×10^{10}	1.62×10^{10}	2.66×10^9	3.64×10^8
42	11	9.28×10^{11}	1.58×10^{12}	1.18×10^{12}	5.17×10^{11}	1.53×10^{11}	3.29×10^{10}	5.46×10^9	7.61×10^8
43	11	1.85×10^{12}	3.16×10^{12}	2.36×10^{12}	1.04×10^{12}	3.11×10^{11}	6.71×10^{10}	1.12×10^{10}	1.59×10^9
44	11	3.67×10^{12}	6.31×10^{12}	4.72×10^{12}	2.10×10^{12}	6.29×10^{11}	1.37×10^{11}	2.29×10^{10}	3.32×10^9
45	11	7.31×10^{12}	1.26×10^{13}	9.46×10^{12}	4.22×10^{12}	1.27×10^{12}	2.79×10^{11}	4.68×10^{10}	6.91×10^9

Table 2: Selected gcd counts $\text{gcd}_d = \#\{\text{odd } s \in [1, 2^{\ell+1}] : (s, n) = d\}$ and auxiliary count $x_3 = \#\{\text{odd } s \in [1, 2^{\ell+1}] : 3 \mid s \text{ or } \omega((s, n)) \geq 3\}$, where n is the product of the first ℓ odd primes; large values are rounded up. Only values needed in the proof are shown; blank entries are not necessarily zero.

ℓ	gcd_{105}	gcd_{15}	gcd_{21}	gcd_3	x_3	gcd_5
3				2		
4				3		
5		2		5		
6		3		10		
7		5		21		
8		9		42		
9		17		86		
10		36		166		
11		76		315		
12		152		604		
13		300		1164		
14		590		2256		
15		1139		4416		
16		2218		8682		
17		4314		17 139		
18		8453		33 877		
19		16 639		66 979		
20		32 846		132 281		
21		64 979		261 372		130 677
22		128 676		516 379		258 258
23	42 293	254 834	169 795	1 020 848		510 604
24	83 729	504 881	336 514	2 019 785	6 334 949	1 010 179
25	166 004	1 000 144	666 745	3 998 146	12 706 706	1 999 526
26	329 275	1 980 869	1 320 714	7 916 785	25 481 743	3 958 891
27	653 169	3 923 935	2 616 309	15 683 688	51 096 769	7 842 216
28	1 295 341	7 773 941	5 183 268	31 078 505	1.02×10^8	15 539 148
29	2 568 728	15 406 877	10 272 217	61 607 914	2.06×10^8	30 803 093
30	5 097 322	30 566 423	20 378 662	1.22×10^8	4.12×10^8	61 123 344
31	10 115 856	60 661 064	40 441 466	2.43×10^8	8.26×10^8	1.21×10^8
32	20 080 727	1.20×10^8	80 288 746	4.82×10^8	1.66×10^9	2.41×10^8
33	39 866 096	2.39×10^8	1.59×10^8	9.57×10^8	3.32×10^9	4.78×10^8
34	79 187 622	4.75×10^8	3.17×10^8	1.90×10^9	6.66×10^9	9.50×10^8
35	1.57×10^8	9.44×10^8	6.29×10^8	3.78×10^9	1.34×10^{10}	1.89×10^9
36	3.13×10^8	1.88×10^9	1.25×10^9	7.50×10^9	2.68×10^{10}	3.75×10^9
37	6.21×10^8	3.73×10^9	2.49×10^9	1.49×10^{10}	5.36×10^{10}	7.46×10^9
38	1.24×10^9	7.41×10^9	4.94×10^9	2.97×10^{10}	1.08×10^{11}	1.48×10^{10}
39	2.46×10^9	1.47×10^{10}	9.83×10^9	5.90×10^{10}	2.15×10^{11}	2.95×10^{10}
40	4.89×10^9	2.93×10^{10}	1.95×10^{10}	1.17×10^{11}	4.32×10^{11}	5.86×10^{10}
41	9.72×10^9	5.83×10^{10}	3.89×10^{10}	2.33×10^{11}	8.65×10^{11}	1.17×10^{11}
42	1.93×10^{10}	1.16×10^{11}	7.73×10^{10}	4.64×10^{11}	1.73×10^{12}	2.32×10^{11}
43	3.85×10^{10}	2.31×10^{11}	1.54×10^{11}	9.23×10^{11}	3.47×10^{12}	4.62×10^{11}
44	7.65×10^{10}	4.59×10^{11}	3.06×10^{11}	1.84×10^{12}	6.96×10^{12}	9.18×10^{11}
45	1.52×10^{11}	9.14×10^{11}	6.09×10^{11}	3.66×10^{12}	1.39×10^{13}	1.83×10^{12}

Table 3: Census of values of $\omega((s, n))$ for $s \in [1, 2^\ell]$ where n is the product of the first ℓ odd primes; large values are rounded up.

ℓ	ω_{\max}	c_0	c_1	c_2	c_3	c_4	c_5	c_6	$c_{\geq 7}$
3	1	4	4						
4	2	6	9	1					
5	2	11	18	3					
6	2	22	30	12					
7	3	44	51	32	1				
8	3	87	91	72	6				
9	3	171	180	138	23				
10	3	328	375	251	70				
11	4	626	793	451	174	4			
12	4	1200	1646	847	381	22			
13	4	2316	3359	1653	785	79			
14	5	4510	6717	3407	1507	242	1		
15	5	8832	13321	7145	2823	639	8		
16	5	17400	26245	15033	5318	1494	46		
17	5	34338	51657	31407	10240	3247	183		
18	6	67840	101977	64647	20478	6606	595	1	
19	6	134032	202022	131428	42198	12911	1687	10	
20	6	264639	401506	264780	88485	24820	4281	65	
21	6	522702	799799	530538	186141	47696	9993	283	
22	6	1032593	1595114	1060794	389735	93243	21796	1029	
23	7	2041220	3182621	2121272	808354	186821	45105	3209	6
24	7	4038813	6350266	4246629	1659760	382912	89884	8900	52
25	7	7995366	12665960	8515547	3381389	797705	175613	22568	284
26	7	15832644	25252891	17098634	6854406	1673928	341895	53269	1197
27	8	31366915	50335662	34359252	13851260	3510598	671330	118470	4241
28	8	62157666	1.00×10^8	69065837	27961797	7329469	1341027	251722	13225
29	8	1.23×10^8	2.00×10^8	1.39×10^8	56440743	15199047	2730221	515906	37449
30	8	2.45×10^8	3.99×10^8	2.79×10^8	1.14×10^8	31245966	5636269	1027549	97383
31	8	4.85×10^8	7.95×10^8	5.60×10^8	2.30×10^8	63883643	11755588	2023659	237081
32	9	9.64×10^8	1.58×10^9	1.12×10^9	4.64×10^8	1.30×10^8	24606119	3981833	546323
33	9	1.91×10^9	3.16×10^9	2.25×10^9	9.38×10^8	2.64×10^8	51472118	7908459	1205696
34	9	3.80×10^9	6.30×10^9	4.52×10^9	1.90×10^9	5.36×10^8	1.07×10^8	15904855	2563508
35	9	7.55×10^9	1.26×10^{10}	9.06×10^9	3.83×10^9	1.09×10^9	2.22×10^8	32484581	5314501
36	9	1.50×10^{10}	2.51×10^{10}	1.82×10^{10}	7.73×10^9	2.21×10^9	4.57×10^8	67141601	10821108
37	10	2.98×10^{10}	5.00×10^{10}	3.64×10^{10}	1.56×10^{10}	4.47×10^9	9.38×10^8	1.40×10^8	21818440
38	10	5.93×10^{10}	9.98×10^{10}	7.30×10^{10}	3.14×10^{10}	9.07×10^9	1.92×10^9	2.92×10^8	43904888
39	10	1.18×10^{11}	1.99×10^{11}	1.46×10^{11}	6.33×10^{10}	1.84×10^{10}	3.91×10^9	6.10×10^8	88614835
40	10	2.35×10^{11}	3.98×10^{11}	2.93×10^{11}	1.28×10^{11}	3.73×10^{10}	7.97×10^9	1.27×10^9	1.80×10^8
41	10	4.66×10^{11}	7.93×10^{11}	5.87×10^{11}	2.57×10^{11}	7.57×10^{10}	1.62×10^{10}	2.63×10^9	3.69×10^8
42	11	9.28×10^{11}	1.58×10^{12}	1.18×10^{12}	5.18×10^{11}	1.53×10^{11}	3.30×10^{10}	5.44×10^9	7.60×10^8
43	11	1.85×10^{12}	3.16×10^{12}	2.36×10^{12}	1.04×10^{12}	3.10×10^{11}	6.72×10^{10}	1.12×10^{10}	1.57×10^9
44	11	3.67×10^{12}	6.31×10^{12}	4.72×10^{12}	2.10×10^{12}	6.28×10^{11}	1.37×10^{11}	2.30×10^{10}	3.27×10^9
45	11	7.31×10^{12}	1.26×10^{13}	9.46×10^{12}	4.22×10^{12}	1.27×10^{12}	2.79×10^{11}	4.70×10^{10}	6.81×10^9

Table 4: Selected gcd counts $\text{gcd}_d = \#\{s \in [1, 2^\ell] : (s, n) = d\}$ and auxiliary count $x_3 = \#\{s \in [1, 2^\ell] : 3 \mid s \text{ or } \omega((s, n)) \geq 3\}$, where n is the product of the first ℓ odd primes; large values are rounded up. Only values needed in the proof are shown; blank entries are not necessarily zero.

ℓ	gcd_{105}	gcd_{15}	gcd_{21}	gcd_3	x_3	gcd_5
3				2		
4				4		
5				7		
6				11		
7		7		19		
8		12		37		
9		20		75		
10		35		154		
11		68		310		
12		138		612		
13		280		1195		
14		566		2320		
15		1135		4504		
16		2241		8783		
17		4400		17 182		
18		8607		33 788		
19		16 846		66 658		
20		33 048		131 710		
21		65 061		260 517		130 105
22		128 425		515 466		257 532
23		254 093	169 296	1 020 164		509 903
24	84 153	503 484	335 445	2 020 025	6 320 878	1 009 955
25	166 227	998 109	665 082	4 000 127	12 695 486	2 000 257
26	328 933	1 978 571	1 318 640	7 921 325	25 484 436	3 961 321
27	651 867	3 922 276	2 614 483	15 691 232	51 130 089	7 846 881
28	1 292 724	7 774 396	5 182 819	31 088 550	1.03×10^8	15 546 187
29	2 564 674	15 411 506	10 274 822	61 618 338	2.06×10^8	30 811 418
30	5 092 434	30 577 232	20 386 319	1.22×10^8	4.12×10^8	61 130 339
31	10 111 570	60 679 387	40 455 703	2.43×10^8	8.27×10^8	1.21×10^8
32	20 079 847	1.20×10^8	80 309 360	4.82×10^8	1.66×10^9	2.41×10^8
33	39 872 679	2.39×10^8	1.59×10^8	9.57×10^8	3.32×10^9	4.78×10^8
34	79 206 028	4.75×10^8	3.17×10^8	1.90×10^9	6.66×10^9	9.50×10^8
35	1.57×10^8	9.44×10^8	6.29×10^8	3.78×10^9	1.33×10^{10}	1.89×10^9
36	3.13×10^8	1.88×10^9	1.25×10^9	7.50×10^9	2.68×10^{10}	3.75×10^9
37	6.21×10^8	3.73×10^9	2.49×10^9	1.49×10^{10}	5.36×10^{10}	7.46×10^9
38	1.24×10^9	7.41×10^9	4.94×10^9	2.97×10^{10}	1.08×10^{11}	1.48×10^{10}
39	2.46×10^9	1.47×10^{10}	9.83×10^9	5.90×10^{10}	2.15×10^{11}	2.95×10^{10}
40	4.89×10^9	2.93×10^{10}	1.95×10^{10}	1.17×10^{11}	4.32×10^{11}	5.86×10^{10}
41	9.72×10^9	5.83×10^{10}	3.89×10^{10}	2.33×10^{11}	8.65×10^{11}	1.17×10^{11}
42	1.93×10^{10}	1.16×10^{11}	7.73×10^{10}	4.64×10^{11}	1.73×10^{12}	2.32×10^{11}
43	3.85×10^{10}	2.31×10^{11}	1.54×10^{11}	9.23×10^{11}	3.47×10^{12}	4.62×10^{11}
44	7.65×10^{10}	4.59×10^{11}	3.06×10^{11}	1.84×10^{12}	6.96×10^{12}	9.18×10^{11}
45	1.52×10^{11}	9.14×10^{11}	6.09×10^{11}	3.66×10^{12}	1.39×10^{13}	1.83×10^{12}
46	3.03×10^{11}	1.82×10^{12}	1.21×10^{12}	7.28×10^{12}	2.79×10^{13}	3.64×10^{12}

ACKNOWLEDGMENTS

We thank Gerry Myerson for telling us about matchable numbers, and Bernardo Récamán for his encouragement.

DEPARTMENT OF MATHEMATICS, TOWSON UNIVERSITY, TOWSON, MD 21252, USA
E-mail address: `nmcnew@towson.edu`

MATHEMATICS DEPARTMENT, DARTMOUTH COLLEGE, HANOVER, NH 03755, USA
E-mail address: `carlp@math.dartmouth.edu`