MATCHABLE NUMBERS

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Abstract. We say a natural number n is matchable if there is a bijection from the set of $\tau(n)$ divisors of n to the set $\{1, 2, \ldots, \tau(n)\}$, where corresponding numbers are relatively prime. We show that the set of matchable numbers has an asymptotic density, which we compute, and we show that every squarefree number is matchable. We also present some related unsolved problems.

1. Introduction

Given two finite sets of integers of the same cardinality, a bijection ψ between them is said to be a coprime matching if for each x in the domain of ψ, x and $\psi(x)$ are coprime. Alternatively, one can consider the bipartite graph from one set to the other where there is an edge whenever the numbers on the edge's vertices are coprime: a coprime matching is a perfect matching in this graph.

There have been several papers on coprime matchings over the years, mostly where the two sets are intervals of consecutive integers. For example, in [5], Pomerance and Selfridge proved a conjecture of D. J. Newman that there is always a coprime matching from $\{1, 2, \ldots, n\}$ to any other interval of n consecutive integers. This was generalized by Bohman and Peng [1] to some cases where the intervals are arbitrarily placed on the number line, and they showed a connection to the notorious lonely runner conjecture. Their paper was subsequently improved in [3] and generalized to a counting problem in [4], with further progress by McNew [2] and Sah and Sawhney [9].

In this paper we consider a problem of Recamán [6], where one of the sets continues to be an initial interval of consecutive integers, but the other set is generally not an interval, rather it is the set of divisors of a number n. More precisely, for a positive integer n let $D(n)$ denote the set of divisors of n, and $\tau(n) = |D(n)|$. We say an integer n is *matchable* if there is a coprime matching between $\{1, 2, \ldots, \tau(n)\}$ and $D(n)$.

One might think at first that every number is matchable, and this holds for $n = 1, 2, \ldots, 7$. However, 8 is not matchable, nor is any subsequent multiple of 4. The proof is easy: If $4 \mid n$, then at least $2/3$ of the members of $D(n)$ are even, but fewer than $2/3$ of the members of $\{1, 2, \ldots, k\}$ are odd when $k > 3$. Since even divisors must be mapped to odd numbers in a coprime matching, proper multiples of 4 are seen to be not matchable. This can be generalized to other primes as well, see below.

Say a number n is an M-number if it is not divisible by any p^p with p prime. For example, every squarefree number is an M-number. It is easy to see that the set of M-numbers

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possesses an asymptotic density, which is

$$\alpha = \prod_{p\,\mathrm{prime}} \left(1 - \frac{1}{p^p}\right) = 0.72199023441955\ldots.$$

Among the comments in [6] we find the conjecture of König and Alekseyev that every M-number is matchable, and few non-M-numbers are matchable, and in particular, the asymptotic density of the set of matchable numbers is α. In this paper we prove that the asymptotic density of the symmetric difference of the set of matchable numbers and the set of M-numbers is 0, so that the density of the set of matchable numbers is indeed α.

We agree with the conjecture of König and Alekseyev that every M-number is matchable. Towards a proof we show at least that every squarefree number is matchable. Probably our techniques can be extended to the remaining M-numbers.

For each prime p let

$$M_p = \prod_{q \le p} q^{q-1},$$

where q runs over primes. It is easy to see that each M_p is matchable. Indeed, for $1 \le j \le \tau(M_p)$, we map j to

$$\psi(j) := \prod_{q \le p} q^{(j \bmod q)}.$$

To see this note that since $\tau(M_p) = \prod_{q \le p} q$, the Chinese remainder theorem shows that each integer $j \in [1, \tau(M_p)]$ corresponds to a unique vector $(j \bmod 2, j \bmod 3, \ldots, j \bmod p)$. Further, for $q \le p$, $q \mid j$ if and only if $q \nmid \psi(j)$.

The set of M-numbers is precisely the set of all divisors of the numbers M_p as p varies. As noted in [6], if one can prove that all of the divisors of a matchable number are themselves matchable, we would immediately have the corollary that every M-number is matchable. Unfortunately, we did not find a way to make this elegant plan work.

2. Preliminary results

We generalize the result that if $4 \mid n$ and $n > 4$, then n is not matchable.

Proposition 1. *Suppose $p^p | n$ for some prime p, and let $0 \le r < p$ be the remainder when $\tau(n)$ is divided by p. If $\tau(n) > r(p+1)$, then n is not matchable.*

Proof. Suppose p^k is the largest power of p dividing n, with $k \ge p$. We can partition the $\tau(n)$ divisors of n into $k+1$ sets each having size $\tau(n/p^k)$ according to the power to which p appears as a factor. Only one of those sets will contain integers coprime to p, and so the total number of divisors of n coprime to p is $\frac{\tau(n)}{k+1} \le \frac{\tau(n)}{p+1}$. On the other hand, the number of integers in $[1, \tau(n)]$ divisible by p is

$$\left\lfloor \frac{\tau(n)}{p} \right\rfloor = \frac{\tau(n) - r}{p}. \tag{1}$$

Since $\tau(n) > r(p+1)$ we find that $(\tau(n) - r)(p+1) > \tau(n)p$ and thus the quantity in (1) is strictly greater than $\frac{\tau(n)}{p+1}$, the upper bound we just found for the number of divisors of n coprime to p. Thus, there are too few divisors of n coprime to p to match with these integers up to $\tau(n)$ divisible by p. □

Corollary 1. *If $p^p|n$ for some prime p, and $\tau(n) \geq p^2$, then n is not matchable.*

Let $\omega(n)$ denote the number of different primes that divide n. Note that $\tau(n) \geq 2^{\omega(n)}$, with equality if and only if n is squarefree.

Corollary 2. *The upper density of the set of matchable integers is at most α.*

Proof. It suffices to show that the set of matchable numbers that are not M-numbers has asymptotic density 0. The set of integers divisible by some p^p for $p \geq N$ has density $\ll N^{-N}$. If $p^p \,|\, n$ and n is matchable, then Corollary 1 implies that $\omega(n) \leq 2\log p/\log 2$. So, if n is matchable and not an M-number, it is either divisible by some p^p for $p \geq N$ or $\omega(n) \leq 2\log N/\log 2$.

The counting function to x of the latter numbers n is $\ll x(\log\log x)^{2\log N/\log 2}/\log x$, so for N fixed, the set has density 0. Putting the two together, the upper density is $\leq N^{-N}$, and since N is arbitrary, the corollary is proved. □

We remark that if we let $N = \log\log x$ in the proof, we have the counting function to x of the set of matchable numbers that are not M-numbers is $\leq x/(\log x)^{1+o(1)}$ as $x \to \infty$. This result is best possible since every number $27p$ with p prime is matchable (as is easily checked), yet not an M-number.

Our plan is to first prove that squarefree numbers with at least 45 prime factors are matchable, and then by a somewhat different method, we prove it for squarefree numbers with fewer than 45 prime factors. Finally, we extend the argument to M-numbers with sufficiently many prime-power divisors not of the form p^{p-1} and not divisible by the square of any large prime, and use a density argument to finish.

We conclude with some open problems and a discussion of strongly matchable numbers. These are numbers n such that there is a coprime matching between $D(n)$ and every coprime arithmetic progression of $\tau(n)$ integers.

3. Squarefree numbers with many prime factors

Lemma 1. *If $2n$ is matchable, then so is n.*

Proof. We may assume n is odd. A coprime matching for $2n$ pairs $D(2n) = D(n) \cup 2D(n)$ with $[1, \tau(2n)] = [1, 2\tau(n)]$. Since the even divisors $2D(n)$ must be paired with odd integers in $[1, 2\tau(n)]$, the odd divisors $D(n)$ are paired with the even integers $\{2, 4, \ldots, 2\tau(n)\}$. Dividing by 2, this gives a coprime matching of $D(n)$ with $[1, \tau(n)]$, so n is matchable. □

Theorem 1. *Every squarefree number having at least 45 prime factors is matchable.*

We now introduce a notion that will be used to track error bounds in counting.

Definition 1. For an integer $k \geq 1$, we say that a set S of integers is a *k-AP combination* if it can be constructed as follows:

(1) A single arithmetic progression is a 1-AP combination.
(2) If S_1 is a k_1-AP combination, S_2 is a k_2-AP combination, and $S_1 \cap S_2 = \emptyset$, then $S_1 \cup S_2$ is a $(k_1 + k_2)$-AP combination.
(3) If S_1 is a k_1-AP combination, $S_2 \subseteq S_1$ is a k_2-AP combination, then $S_1 \setminus S_2$ is a $(k_1 + k_2)$-AP combination.

Note that since we don't assume the constituent arithmetic progressions are nonempty, any set S that is a k-AP combination is also a k'-AP combination for any $k' > k$.

Lemma 2. *If S is a k-AP combination whose constituent arithmetic progressions all have common differences coprime to d, then the number of elements of S divisible by d is $|S|/d+\theta$ where $|\theta| \le k$.*

Proof. We proceed by induction on the construction of S. For the base case, let S be an arithmetic progression of length m with common difference q where $\gcd(d, q) = 1$. If $m \ge d$, the elements of S form a complete residue system modulo d, and among any d consecutive terms, exactly one is divisible by d. So, the count of d-multiples is $m/d + \theta$ with $|\theta| \le 1$.

For the inductive step, suppose $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$. Then the count of d-multiples in S equals

$$(|S_1|/d + \theta_1) + (|S_2|/d + \theta_2) = |S|/d + (\theta_1 + \theta_2),$$

where $|\theta_1| \le k_1$, $|\theta_2| \le k_2$, so $|\theta_1 + \theta_2| \le k_1 + k_2$.

If $S_2 \subseteq S_1$ and $S = S_1 \setminus S_2$, then the count of d-multiples in S equals

$$(|S_1|/d + \theta_1) - (|S_2|/d + \theta_2) = |S|/d + (\theta_1 - \theta_2),$$

where $|\theta_1 - \theta_2| \le k_1 + k_2$. □

Lemma 3. *Suppose $p_1 < p_2 < \cdots < p_j$ are primes with $p_1 = 2$ and I is an interval of L consecutive integers where $2^j \mid L$ and $L \ge 4^j$. Then I can be partitioned into 2^j sets $A_{v,j}$ of size $L/2^j$, parametrized by divisors v of $m_j := p_1p_2 \dots p_j$, such that every member of $A_{v,j}$ is coprime to v. Moreover, each $A_{v,j}$ is a 2^{j-1}-AP combination whose constituent AP's have common differences dividing m_j, and hence for any d coprime to m_j, the number of elements of $A_{v,j}$ divisible by d is within 2^{j-1} of $|A_{v,j}|/d$.*

Proof. The error bound of 2^{j-1} follows from Lemma 2. We prove the existence of the stated partition by induction on j.

For $j = 1$, since $p_1 = 2$ by assumption, we partition I into $A_{1,1}$, the even integers in I, and $A_{2,1}$, the odd integers. Each is a single arithmetic progression, hence a 1-AP combination.

For $j = 2$, let $p = p_2$ be the second prime. We construct the four sets as follows. Let x be chosen so that the number of odd integers in $I \cap [1, x]$ that are not divisible by p equals $L/4$. To see that such an x exists, note that the majority of odd integers in I are not divisible by p. Indeed, the number of odd elements of I divisible by p is $\le L/(2p) + 1$, and this is $< L/4$ by the assumption $L \ge 4^j$. Set

$$A_{2p,2} = \{\text{odd } i \in I : i \le x,\, p \nmid i\},$$
$$A_{2,2} = \{\text{odd } i \in I : p \mid i,\, i \le x\} \cup \{\text{odd } i \in I : i > x\}.$$

Similarly, let y be chosen so that the number of even integers in $I \cap [1, y]$ not divisible by p equals $L/4$, and set

$$A_{p,2} = \{\text{even } i \in I : i \le y,\, p \nmid i\},$$
$$A_{1,2} = \{\text{even } i \in I : p \mid i,\, i \le y\} \cup \{\text{even } i \in I : i > y\}.$$

Each of $A_{2p,2}$ and $A_{p,2}$ is the difference of two arithmetic progressions, hence a 2-AP combination by rule (3). Each of $A_{2,2}$ and $A_{1,2}$ is the disjoint union of two arithmetic progressions, hence a 2-AP combination by rule (2). Thus each set is a 2-AP combination.

For $j \geq 3$, assume the result holds for $j-1$: each $A_{v,j-1}$ is a 2^{j-2}-AP combination. For each $v \mid m_{j-1}$, we partition $A_{v,j-1}$ into $A_{v,j}$ and $A_{p_jv,j}$ using a cutoff as follows. Let z_v be chosen so that the number of elements of $A_{v,j-1}$ that are $\leq z_v$ and not divisible by p_j equals $L/2^j$. Set

$$A_{p_jv,j} = \{i \in A_{v,j-1} : i \leq z_v,\, p_j \nmid i\},$$
$$A_{v,j} = \{i \in A_{v,j-1} : p_j \mid i,\, i \leq z_v\} \cup \{i \in A_{v,j-1} : i > z_v\}.$$

Every element of $A_{p_jv,j}$ is coprime to p_j (and was already coprime to v), so is coprime to p_jv. Every element of $A_{v,j}$ was already coprime to v.

We now verify the sizes. By the induction hypothesis, the number of p_j-multiples in $A_{v,j-1}$ is within 2^{j-2} of $|A_{v,j-1}|/p_j = L/(p_j 2^{j-1})$, hence at most

$$\frac{L}{p_j 2^{j-1}} + 2^{j-2} \leq \frac{L}{3 \cdot 2^{j-1}} + \frac{L}{2^{j+2}} < \frac{L}{2^j},$$

using $p_j \geq 3$ and $L \geq 4^j$. Thus there are enough non-p_j-multiples in $A_{v,j-1}$ to fill $A_{p_jv,j}$ to size $L/2^j$, and $A_{v,j}$ receives the remaining $L/2^j$ elements.

It remains to show each new set is a 2^{j-1}-AP combination. By induction, we know that each set $A_{v,j-1}$ is a 2^{j-2}-AP combination.

A key observation is that intersecting with $\{i : i \leq z_v\}$ or $\{i : i > z_v\}$ preserves the AP-combination structure of a set. If T is a k-AP combination, then $\{i \in T : i \leq z_v\}$ and $\{i \in T : i > z_v\}$ are each k-AP combinations. (We simply truncate each constituent arithmetic progression. This could result in some of them being empty.) Similarly, restricting to p_j-multiples preserves the AP-combination structure, since if T is a k-AP combination, then $\{i \in T : p_j \mid i\}$ is a k-AP combination (just replace each arithmetic progression by the sub-arithmetic progression of its p_j-multiples).

Now consider each of the sets used to construct $A_{v,j}$ and $A_{p_jv,j}$. First, $\{i \in A_{v,j-1} : p_j \mid i,\, i \leq z_v\}$ is just $A_{v,j-1}$ restricted to p_j-multiples and then to $\{i \leq z_v\}$. So, it has the same combination structure as $A_{v,j-1}$, hence is a 2^{j-2}-AP combination. Similarly $\{i \in A_{v,j-1} : i > z_v\}$ is $A_{v,j-1}$ restricted to $\{i > z_v\}$, hence also a 2^{j-2}-AP combination. Since $A_{v,j}$ is the disjoint union of these two it is a 2^{j-1}-AP combination.

For $A_{p_jv,j}$, we note that

$$\begin{aligned} A_{p_jv,j} &= \{i \in A_{v,j-1} : p_j \nmid i,\, i \leq z_v\} \\ &= \{i \in A_{v,j-1} : i \leq z_v\} \setminus \{i \in A_{v,j-1} : p_j \mid i,\, i \leq z_v\}. \end{aligned}$$

As argued above, each of these sets is a 2^{j-2}-AP combination, and the latter is a subset of the former, so their difference is a 2^{j-1}-AP combination. This completes the induction. □

Proof of Theorem 1. Let u be a squarefree number satisfying the hypotheses with $\ell = \omega(u)$. If u is odd, then $2u$ is even with $\omega(2u) = \ell + 1 \geq 46$ prime factors. If we can show that $2u$ is matchable, then u is matchable by Lemma 1. Thus we may assume u is even, so that $2 = p_1 < p_2 < \cdots < p_\ell$ where $u = p_1 p_2 \cdots p_\ell$.

For an integer $j \leq \omega(u)/2$ to be chosen later, we let $m_j = p_1 p_2 \cdots p_j$ and $n = u/m_j$, so that m_j is the product of the j smallest primes dividing u while n contains the rest. We then apply Lemma 3 using these j smallest prime factors and $I = [1, \tau(u)] = [1, 2^{j+\omega(n)}]$. Since $4^j \leq \tau(u)$, the lemma allows us to produce a partition of I into sets $A_{v,j}$ as described in the lemma, each having size $\tau(n)$.

Thus it now suffices to show that there are one-to-one correspondences between $D(n)$ and each of the sets $A_{v,j}$ with corresponding numbers relatively prime. Indeed, for $v \,|\, m_j$, the correspondence can instead be between $A_{v,j}$ and $vD(n)$, keeping the coprime property. Then, as $D(u)$ is the disjoint union of the sets $vD(n)$ as v runs over all of the divisors of m_j, we can piece together these matchings and so have a coprime matching of all divisors of $u = m_j n$ to $[1, \tau(u)]$.

To show the existence of these one-to-one correspondences we note that any divisor $d \mid n$ is coprime to m_j, so by Lemma 3, the number of integers in $A_{v,j}$ divisible by d is within 2^{j-1} of $\tau(n)/d$.

We choose j as follows. For $\ell \geq 68$ we take $j = \lfloor \sqrt{\ell} \rfloor$ and for $45 \leq \ell \leq 67$ we take $j = 4$ except $j = 3$ when $\ell \in \{46, 47, 48\}$ and $j = 5$ when $\ell = 52$. Note that in every case we have $j \leq \sqrt{\ell}$. With these choices, $\omega(n) = \ell - j \geq 41$ and one can verify that

$$f(n) := \sum_{p \,|\, n} \frac{1}{p} = \sum_{i=j+1}^{\ell} \frac{1}{p_i} \leq \sum_{i=j+1}^{\ell} \frac{1}{P_i} < \frac{93}{100}, \tag{2}$$

where P_i is the i-th prime. To see this in the case that $j = \lfloor \sqrt{\ell} \rfloor$, we use that $P_i > i \log i$ (see [7]), so

$$f(n) < \frac{1}{\sqrt{68} \log \sqrt{68}} + \int_{\sqrt{\ell}}^{\ell} \frac{dt}{t \log t} < 0.06 + \log 2 < 0.76.$$

We use Hall's theorem for the bipartite graph on $A = A_{v,j}$ to $D(n)$ where there is an edge precisely when $a \in A$ and $d \,|\, n$ are coprime. Suppose that $S \subset A$ and that $s \in S$ minimizes $k = \omega((s,n))$. We wish to show that $|S|$ is bounded above by the size of the neighborhood $N(S)$ of S. If $k = 0$ then the neighborhood of S is all of $D(n)$, so the condition holds.

Assume that $k \geq 1$. Since each element s of S is in $A \subset [1, \tau(u)]$ and is divisible by some $d \mid n$ with $\omega(d) = k$, we have

$$|S| \leq \sum_{\substack{d \,|\, n \\ \omega(d)=k \\ d \leq \tau(u)}} \sum_{\substack{a \in A \\ d \,|\, a}} 1 \leq \sum_{\substack{d \,|\, n \\ \omega(d)=k}} \left(\frac{\tau(n)}{d} + 2^{j-1} \right).$$

By the multinomial theorem, we have

$$\sum_{\substack{d \,|\, n \\ \omega(d)=k}} \frac{1}{d} \leq \frac{f(n)^k}{k!},$$

so that

$$\begin{aligned} |S| &\leq \tau(n) \frac{f(n)^k}{k!} + 2^{j-1} \binom{\omega(n)}{k} \\ &< \frac{\tau(n)}{2^k} \left(\frac{(93/50)^k}{k!} + 2^{2j-1+k-\ell+\log_2 \binom{\omega(n)}{k}} \right). \end{aligned} \tag{3}$$

Since the primes dividing n are all at least P_{j+1}, a divisor d of n with $\omega(d) = k$ satisfies $d \geq P_{j+1} P_{j+2} \cdots P_{j+k}$. For such a d to divide any element of $A \subset [1, \tau(u)]$, we need $d \leq \tau(u) = 2^\ell$, as we have seen, so we only need consider $k \leq \overline{k}$, where $\overline{k}$ is the largest integer with $P_{j+1} P_{j+2} \cdots P_{j+\overline{k}} \leq 2^\ell$.

3.1. **Case $k \geq 4$.** For Hall's condition to hold, since the neighborhood of S has size at least $\tau(n)/2^k$, it suffices if

$$\frac{(93/50)^k}{k!} + 2^{2j-1+k-\ell+\log_2\binom{\omega(n)}{k}} < 1 \tag{4}$$

for each k with $4 \leq k \leq \overline{k}$. Note that $(93/50)^4/24 < 0.50$, so we need only show that the second term is at most $\frac{1}{2}$.

Let $E_k := 2j - 1 + k - \ell + \log_2\binom{\omega(n)}{k}$ denote the exponent, so the second term equals 2^{E_k}.

Large ℓ ($\ell \geq 192$). Set $j = \lfloor\sqrt{\ell}\rfloor$. We first bound $\overline{k}$. Since $j = \lfloor\sqrt{\ell}\rfloor \geq 13$ for $\ell \geq 192$, we have $P_{j+1} \geq P_{14} = 43$. The constraint $\prod_{i=1}^{k} P_{j+i} \leq 2^\ell$ implies $43^k < 2^\ell$, giving $\overline{k} < \ell/\log_2 43 < 0.185\ell$. With $\omega(n) = \ell - j > \ell - \sqrt{\ell}$, we have $\overline{k}/\omega(n) < 0.20$ for $\ell \geq 192$.

Using the entropy function $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ and the standard bound $\log_2\binom{a}{b} < a \cdot H(b/a)$, writing $c = k/\omega(n) < 0.20$ and using that H is increasing on $(0, \frac{1}{2})$, we have $H(c) < H(0.20) < 0.722$, so $c + H(c) < 0.922$. Thus

$$\begin{aligned} E_k &\leq 2j - 1 + (c + H(c))\omega(n) - \ell \\ &< 2\sqrt{\ell} - 1 + 0.922(\ell - \sqrt{\ell}) - \ell \\ &= -0.078\ell + 1.078\sqrt{\ell} - 1 < -1.03 \end{aligned}$$

for $\ell \geq 192$. Since $(93/50)^4/4! < 0.50$ and $2^{E_k} < 0.50$, we have (4) for $4 \leq k \leq \overline{k}$.

Moderate ℓ ($68 \leq \ell < 192$). We use $j = \lfloor\sqrt{\ell}\rfloor$ and verify (4) by direct computation. For each ℓ in this range, we find

$$\overline{k} = \max\{k : \prod_{i=j+1}^{j+k} P_i < 2^\ell\}.$$

Then, for each k with $4 \leq k \leq \overline{k}$, we compute $E_k = k + 2j - 1 - \ell + \log_2\binom{\ell-j}{k}$ and verify that $(93/50)^k/k! + 2^{E_k} < 1$.

Small ℓ ($45 \leq \ell \leq 67$). Here we take (as above) $j = 4$ when $\ell = 45$, $j = 3$ for $46 \leq \ell \leq 48$, $j = 4$ for $49 \leq \ell \leq 67$ (except $j = 5$ when $\ell = 52$). As above, we verify (4) directly for all $4 \leq k \leq \overline{k}$. This concludes the case $k \geq 4$ for all $\ell \geq 45$.

3.2. **Case $k = 3$.** Here the neighborhood of S has at least $\tau(n)/8$ elements. Let $a_1 \in S$ with $(a_1, n) = q_1q_2q_3$ for distinct primes q_1, q_2, q_3 dividing n.

First, suppose that the triple $\{q_1, q_2, q_3\}$ is unique (every $a \in S$ has $(a, n) = q_1q_2q_3$ or $\omega((a, n)) > 3$). Then

$$\begin{aligned} |S| &\leq \sum_{\substack{a \in A \\ q_1q_2q_3 \mid a}} 1 + \sum_{\substack{d \mid n \\ \omega(d)=4}} \left(\frac{\tau(n)}{d} + 2^{j-1}\right) \\ &\leq \frac{\tau(n)}{q_1q_2q_3} + 2^{j-1} + \tau(n)\frac{f(n)^4}{24} + 2^{j-1}\binom{\omega(n)}{4} \\ &< \frac{\tau(n)}{8}\left(\frac{8}{q_1q_2q_3} + \frac{8f(n)^4}{24} + 2^{2\sqrt{\ell}+2-\ell}\left(1 + \binom{\ell}{4}\right)\right), \end{aligned}$$

using that $\tau(n) = 2^{\ell-j}$. Since $q_1q_2q_3 \geq 7 \cdot 11 \cdot 13 = 1001$ for $j = 3$ and $f(n) < 0.93$, the expression in parentheses is at most $8/1001 + 8(0.93)^4/24 + 2^{2\sqrt{\ell}+2-\ell}\binom{\ell}{4} < 0.26 + 2^{2\sqrt{\ell}+2-\ell}\binom{\ell}{4}$, which is < 1 when $\ell \geq 27$.

Now suppose S contains at least 2 elements a_1, a_2 with $\omega((a_i, n)) = 3$, say $(a_1, n) = q_1q_2q_3$ and $(a_2, n) = q_4q_5q_6$, with $\{q_1, q_2, q_3\} \neq \{q_4, q_5, q_6\}$. The two triples share at most 2 primes, so by inclusion-exclusion $|N(S)| \geq 2 \cdot \tau(n)/8 - \tau(n)/16 = 3\tau(n)/16$. Also,

$$\begin{aligned}|S| &\leq \tau(n)\frac{f(n)^3}{6} + 2^{j-1}\binom{\omega(n)}{3} \\ &< \frac{3\tau(n)}{16}\left(\frac{16f(n)^3}{18} + \frac{16}{3} \cdot 2^{2\sqrt{\ell}-1-\ell}\binom{\ell}{3}\right).\end{aligned}$$

Since $f(n) < 0.93$, we have $16(0.93)^3/18 < 0.72$, and $\frac{16}{3} \cdot 2^{2\sqrt{\ell}-1-\ell}\binom{\ell}{3} < 0.28$ for $\ell \geq 25$. This completes the case $k = 3$.

3.3. **Case $k = 2$.** Here the neighborhood of S has at least $\tau(n)/4$ elements. Let $a_1 \in S$ with $(a_1, n) = q_1q_2$ for distinct primes $q_1, q_2 \mid n$.

First, suppose that the pair $\{q_1, q_2\}$ is unique (every $a \in S$ has $(a, n) = q_1q_2$ or $\omega((a, n)) > 2$). Then

$$\begin{aligned}|S| &\leq \sum_{\substack{a\in A\\ q_1q_2|a}} 1 + \sum_{\substack{d|n\\ \omega(d)=3}} \left(\frac{\tau(n)}{d} + 2^{j-1}\right) \leq \frac{\tau(n)}{q_1q_2} + 2^{j-1} + \tau(n)\frac{f(n)^3}{6} + 2^{j-1}\binom{\omega(n)}{3} \\ &< \frac{\tau(n)}{4}\left(\frac{4}{q_1q_2} + \frac{4f(n)^3}{6} + 2^{2\sqrt{\ell}+1-\ell}\left(1 + \binom{\ell}{3}\right)\right).\end{aligned}$$

Since $q_1q_2 \geq 77$ for $j \geq 3$ and $f(n) < 0.93$, we have $4/77 + 4(0.93)^3/6 < 0.59$, and $2^{2\sqrt{\ell}+1-\ell}(1 + \binom{\ell}{3}) < 0.33$ for $\ell \geq 23$.

Now suppose S contains 2 elements a_1, a_2 with $\omega((a_i, n)) = 2$, say $(a_1, n) = q_1q_2$ and $(a_2, n) = q_3q_4$, with $\{q_1, q_2\} \neq \{q_3, q_4\}$. Assume also that every other $a \in S$ has either $(a, n) = q_1q_2$, $(a, n) = q_3q_4$ or $\omega((a, n)) \geq 3$. The pairs share at most one prime, so by inclusion-exclusion $|N(S)| \geq 2 \cdot \tau(n)/4 - \tau(n)/8 = 3\tau(n)/8$.

Also,

$$\begin{aligned}|S| &\leq \sum_{\substack{a\in A\\ q_1q_2|a}} 1 + \sum_{\substack{a\in A\\ q_3q_4|a}} 1 + \sum_{\substack{d|n\\ \omega(d)\geq 3}} \left(\frac{\tau(n)}{d} + 2^{j-1}\right) \\ &< \frac{3\tau(n)}{8}\left(\frac{8}{3} \cdot \frac{2}{77} + \frac{8f(n)^3}{18} + \frac{8}{3} \cdot 2^{2\sqrt{\ell}-1-\ell}\left(2 + \binom{\ell}{3}\right)\right).\end{aligned}$$

Since $f(n) < 0.93$, we have $16/(3 \cdot 77) + 8(0.93)^3/18 < 0.43$, and $\frac{8}{3} \cdot 2^{2\sqrt{\ell}-1-\ell}(2 + \binom{\ell}{3}) < 0.5$ for $\ell \geq 21$.

So now assume that there are at least 3 different values of (a, n) for $a \in S$ with exactly 2 prime factors. If the 3 values are q_1q_2, q_3q_4, q_5q_6, then the case when one prime is shared among all 3 numbers gives the smallest size for $N(S)$ and that size is $(7/16)\tau(n)$. Then

$$|S| < \frac{7\tau(n)}{16}\left(\frac{16f(n)^2}{14} + \frac{16}{7} \cdot 2^{2\sqrt{\ell}-1-\ell}\binom{\omega(n)}{2}\right) < \frac{7\tau(n)}{16}(0.99 + 0.01),$$

for $\ell \geq 26$, completing the case $k = 2$.

3.4. **Case $k = 1$.** Here the neighborhood of S has at least $\tau(n)/2$ elements, so we may assume that $|S| \geq \tau(n)/2$. Let $a_1 \in S$ with $(a_1, n) = q_1$ for some prime $q_1 | n$.

First, suppose that q_1 is unique (every $a \in S$ has $(a, n) = q_1$ or $\omega((a, n)) \geq 2$). Then

$$|S| \leq \sum_{\substack{a\in A\\ q_1|a}} 1 + \sum_{\substack{a\in A,\, q_1\nmid a\\ \omega((a,n))\geq 2}} 1 \leq \frac{\tau(n)}{q_1} + 2^{j-1} + \sum_{\substack{d|n,\, q_1\nmid d\\ \omega(d)=2}} \left(\frac{\tau(n)}{d} + 2^{j-1}\right)$$
$$\leq \frac{\tau(n)}{q_1} + 2^{j-1} + \tau(n)\frac{(f(n) - 1/q_1)^2}{2} + 2^{j-1}\binom{\omega(n)-1}{2}$$
$$< \tau(n)\left(\frac{f(n)^2}{2} + \frac{1 - f(n)}{q_1} + \frac{1}{2q_1^2}\right) + 2^{j-1}\left(1 + \binom{\omega(n)-1}{2}\right).$$

Using $f(n) < 93/100$ and $q_1 \geq P_{j+1} \geq 7$ for $j \geq 3$, the coefficient of $\tau(n)$ is at most $0.432 + 0.07/7 + 0.5/49 < 0.453$. Since $\tau(n) = 2^{\ell-j}$,

$$|S| < 0.453\tau(n) + 2^{j-1}\left(1 + \binom{\omega(n)-1}{2}\right)$$
$$\leq \frac{\tau(n)}{2}\left(0.906 + 2^{2\sqrt{\ell}-\ell}\left(1 + \binom{\ell-4}{2}\right)\right).$$

This is $< \tau(n)/2$ when $\ell \geq 30$.

Now suppose S contains elements a_i with $(a_i, n) = q_i$ for $r \geq 2$ distinct primes $q_1, \ldots, q_r$ dividing n. The neighborhood of S contains all divisors of n coprime to at least one of $q_1, \ldots, q_r$.

If $r = 2$, this neighborhood has cardinality $\frac{3}{4}\tau(n)$. We have

$$|S| \leq \sum_{i=1}^{2}\left(\frac{\tau(n)}{q_i} + 2^{j-1}\right) + \sum_{\substack{d\,|\,n\\ \omega(d)=2}}\left(\frac{\tau(n)}{d} + 2^{j-1}\right)$$
$$< \tau(n)\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{f(n)^2}{2}\right) + 2^{j-1}\left(2 + \binom{\omega(n)}{2}\right).$$

Since $q_1, q_2 \geq P_{j+1}$ and $f(n) < 93/100$, the first term is at most $2/7 + 0.433 < 0.72\tau(n)$ for $j \geq 3$. Thus,

$$|S| < \frac{3\tau(n)}{4}\left(0.96 + \frac{4}{3}\cdot 2^{2\sqrt{\ell}-1-\ell}\left(2 + \binom{\ell-3}{2}\right)\right).$$

This is $< \frac{3}{4}\tau(n)$ when $\ell \geq 22$.

If $r = 3$, the neighborhood has cardinality at least $\frac{7}{8}\tau(n)$, and

$$|S| < \tau(n)\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{f(n)^2}{2}\right) + 2^{j-1}\left(3 + \binom{\omega(n)}{2}\right)$$
$$< \frac{7\tau(n)}{8}\left(\frac{0.862}{7/8} + \frac{8}{7}\cdot 2^{2\sqrt{\ell}-1-\ell}\left(3 + \binom{\ell-3}{2}\right)\right),$$

using that q_1, q_2, q_3 are distinct primes ≥ 7 for $j \geq 3$ and $f(n) < 93/100$. Our estimate for $|S|$ is $< \frac{7}{8}\tau(n)$ for $\ell \geq 35$.

If $r \geq 4$, the neighborhood has cardinality at least $\frac{15}{16}\tau(n)$, and

$$|S| \leq \sum_{\substack{a \in A \\ \omega((a,n)) \geq 1}} 1 \leq \sum_{q|n} \sum_{\substack{a \in A \\ q \,|\, a}} 1 \leq \tau(n) f(n) + 2^{j-1}\omega(n)$$
$$< \frac{93}{100}\tau(n) + 2^{j-1}\omega(n) < \frac{15\tau(n)}{16}\left(\frac{93/100}{15/16} + \frac{16}{15} \cdot 2^{2\sqrt{\ell}-1-\ell}\ell\right).$$

This is $< \frac{15}{16}\tau(n)$ for $\ell \geq 13$. Hall's condition holds, completing the proof. □

4. Squarefree numbers with few prime factors

In this section we prove the following theorem.

Theorem 2. *Every squarefree number with at most* 44 *prime factors is matchable.*

We illustrate the argument for $\ell = 24$, the smallest case that exhibits the full complexity, and then discuss the modifications for other $\ell \leq 44$. We first establish Proposition 2 at the end of this section for $\ell = 24$: we show there is a coprime matching from $D(u)$ to the odd integers in $[1, 2^{25}]$, where u is odd, squarefree, and with 24 prime factors.

Let $u = q_1 q_2 \cdots q_{24}$ be the product of any 24 distinct odd primes $3 \leq q_1 < q_2 < \cdots < q_{24}$, so $\tau(u) = 2^{24} = 16{,}777{,}216$. We wish to show there is a coprime matching between the 2^{24} odd integers in $[1, 2^{25}]$ and $D(u)$. By Hall's theorem it suffices to show that for every set S of odd integers in $[1, 2^{25}]$, the neighborhood

$$N(S) \ = \ \{\, d \mid u : \gcd(d, a) = 1 \text{ for some } a \in S \,\}$$

satisfies $|N(S)| \geq |S|$.

Let $c_i(u)$ denote the number of odd integers in $[1, 2^{25}]$ sharing exactly i prime factors with u,

$$c_i(u) \coloneqq \#\{\text{odd } a \in [1, 2^{25}] : \omega((a, u)) = i\}$$

and $c_{\geq i}(u) = \sum_{i' \geq i} c_{i'}(u)$ the count of those odd integers sharing at least i prime factors with u. A key observation is that the counts $c_{\geq i}(u)$ are only made larger if the odd primes comprising u are replaced with smaller odd primes.

Lemma 4. *Let u be a squarefree odd number as above and let p, q be odd primes with $q \mid u$, $p \nmid u$, and $p < q$. Then $c_{\geq i}(u) \leq c_{\geq i}(u \cdot p/q)$ for all $i \geq 1$.*

Proof. Set $v = u/q$, so $u = vq$ and $u \cdot p/q = vp$. We construct an injection f from integers counted by $c_{\geq i}(vq)$ to those counted by $c_{\geq i}(vp)$. For each integer a counted by $c_{\geq i}(vq)$, set

$$f(a) = \begin{cases} a & \text{if } \omega((v,a)) \geq i, \text{ or } pq \mid a \text{ and } \omega((v,a)) = i-1, \\ a \cdot p^j/q^j & \text{otherwise (where } q^j \parallel a). \end{cases}$$

Note that if a falls into the second case above, we must have $\omega((v,a)) = i - 1$ and $p \nmid a$. In each of the conditions for the first case above we find that $a = f(a)$ is counted by both $c_{\geq i}(vq)$ and $c_{\geq i}(vp)$. Note that in the second case above, since $p < q$ we have $f(a) < a \leq 2^{25}$ and $f(a)$ odd, and since $p, q \nmid v$ the primes of v dividing $f(a)$ are the same as those dividing a, so $\omega((vp, f(a))) = (i-1) + 1 = i$. Thus f maps into the target set.

For injectivity note that in the second case, since $q \nmid f(a)$, $\omega((vq, f(a))) = i - 1 < i$, so the images $f(a)$ lie outside the domain and cannot collide with the images from the first-case. □

Let $n = 3 \cdot 5 \cdot 7 \cdots 97$ denote the product of the *smallest* 24 odd primes, so $q_i \geq p_i$ where p_i is the i-th odd prime. Applying Lemma 4 repeatedly, each time replacing a prime of u with a smaller prime of n, gives $c_{\geq i}(u) \leq c_{\geq i}(n)$ for each $i \geq 1$.

The rest of the proof will rely heavily on the counts $c_i(n)$ and $c_{\geq i}(n)$, which we will write as c_i and $c_{\geq i}$, respectively. We will also occasionally need counts for the number of integers having a fixed greatest common divisor with u (respectively n).

As with the counts $c_{\geq i}$, the number of odd integers in the interval $[1, 2^{25}]$ whose gcd with u is $q_{j_1} q_{j_2} \cdots q_{j_i}$ is majorized by the number of such integers whose gcd with n is $p_{j_1} p_{j_2} \cdots p_{j_i}$. We denote $\gcd_d := \#\{\text{odd } a \in [1, 2^{25}] : (a, n) = d\}$.

We record in Table 1 the computed values of c_i for $3 \leq \ell \leq 44$, and in Table 2 the computed values of $\gcd_d$ for $d = 105, 15, 21, 3, 5$ (all computations[1] were performed using exact values; large entries in the tables are rounded up, as noted in the captions). The former table also contains a column, $\omega_{\max}$ containing the largest value of i such that c_i is nonzero, and the latter table includes a column x_3, containing the count of odd integers in $[1, 2^{25}]$ which are either divisible by 3 or counted by $c_{\geq 3}$, whose purpose will be explained shortly. In the latter table, values are only included when needed in the analogue of the argument described below (blank entries are not necessarily 0).

We will use frequently the observation that if $k := \min_{a \in S} \omega((a, u))$ then

$$|N(S)| \;\geq\; 2^{24-k} \;=\; \frac{\tau(u)}{2^k}.$$

Let S be a nonempty set of odd integers in $[1, 2^{25}]$; we verify Hall's condition by considering the possible values of $k = \min_{s \in S} \omega((s, u))$.

By summing the values of c_i in Table 1, in the row for $\ell = 24$ we find that

$$c_{\geq 5} = 88{,}525, \quad c_{\geq 4} = 485{,}129, \quad c_{\geq 3} = 2{,}151{,}882, \quad c_{\geq 2} = 6{,}377{,}708, \quad c_{\geq 1} = 12{,}741{,}251.$$

4.1. **Step 1: Cases $k \geq 4$.** First, suppose $k \geq 5$. Then by the monotonicity described above we have $|S| \leq c_{\geq 5} = 88{,}525$. On the other hand, since $\omega_{\max} = 7$ in the row $\ell = 24$ of Table 1, we have $\omega((s, u)) \leq 7$ for all s, and thus $|N(S)| \geq |N(s)| \geq 2^{24-7} = 131{,}072$. So $|S| < |N(S)|$ and Hall's condition holds.

If $k = 4$ then every element $s \in S$ has $\omega((s, u)) \geq 4$ so $|S| \leq c_{\geq 4} = 485{,}129$. Since at least one element has $\omega((s, u)) = 4$ we find that $|N(S)| \geq 2^{24-4} = 1{,}048{,}576 > 485{,}129 \geq |S|$, so again, Hall's condition holds.

4.2. **Step 2: Case $k = 3$.** Every element of S has $\omega \geq 3$, so $|S| \leq c_{\geq 3} = 2{,}151{,}882$, while $|N(S)| \geq 2^{21} = 2{,}097{,}152$. Since $2{,}151{,}882 > 2{,}097{,}152$ we cannot immediately conclude that Hall's condition holds.

Let $s \in S$ be such that $\omega((s, u)) = 3$ and let $d = (s, u)$. Then $d \geq 105 = 3 \times 5 \times 7$ and, by the monotonicity mentioned above we have $\gcd_d \leq \gcd_{105} = 83{,}729$.

If d were unique, then every element of S sharing exactly 3 prime factors with u would need to have greatest common divisor d with u. Then we would have

$$|S| \leq \gcd_d + c_{\geq 4}(u) \leq \gcd_{105} + c_{\geq 4} = 83{,}729 + 485{,}129 < 2^{21} \leq |N(S)|,$$

[1]Python code used to generate the numbers in these tables is included with the arXiv version of this paper.

and so Hall's condition would be satisfied. So we suppose d is not unique, namely there is a second element, s' having $\omega((s',u)) = 3$ but $(s,u) \neq (s',u)$. In this case, since (s,u) and (s',u) can share at most two primes, considering the neighborhood of just s and s' we find, by inclusion-exclusion that

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{21} + 2^{21} - 2^{20} = \frac{3}{16} \cdot 2^{24} = 3{,}145{,}728.$$

But $|S| \leq c_{\geq 3} = 2{,}151{,}882 < 3{,}145{,}728 \leq |N(S)|$, hence Hall's condition holds when $k = 3$.

4.3. **Step 3: Case $k = 2$.** Every element of S has $\omega \geq 2$, so $|S| \leq c_{\geq 2} = 6{,}377{,}708$, while $|N(S)| \geq 2^{22} = 4{,}194{,}304$. Again, we cannot immediately conclude using Hall's theorem, but we can assume $|S| > |N(S)| \geq 2^{22}$.

Let $s \in S$ such that $\omega((s,u)) = 2$ and let $e = (s,u)$. Then $e \geq 15 = 3 \times 5$ and, by monotonicity, $\gcd_e \leq \gcd_{15} = 504{,}881$. If e were unique, we would find that

$$|S| \leq \gcd_{15} + c_{\geq 3} = 504{,}881 + 2{,}151{,}882 = 2{,}656{,}763 < 2^{22} \leq |N(S)|,$$

satisfying Hall's condition, and so we assume that e is not unique, namely there exists a second $s' \in S$ with $\omega((s',u)) = 2$ but $(s',u) = e' \neq e$. Since e and e' share at most one prime factor, we can update our lower bound for the neighborhood $N(S)$ to

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{22} + 2^{22} - 2^{21} = \frac{3}{8} \times 2^{24} = 6{,}291{,}456. \quad (5)$$

Note that this is still less than $c_{\geq 2}$. Now, if e and e' were the only two such divisors, then

$$\begin{aligned} |S| &\leq \gcd_e + \gcd_{e'} + c_{\geq 3} \leq \gcd_{15} + \gcd_{21} + c_{\geq 3} \\ &= 504{,}881 + 336{,}514 + 2{,}151{,}882 = 2{,}993{,}277 < 6{,}291{,}456 \leq |N(S)|. \end{aligned}$$

Again, Hall's condition is satisfied in this case, leaving us with the possibility that there is a third $s'' \in S$, with $\omega((s'',u)) = 2$ and where $(s'',u) = e'' \notin \{e, e'\}$. In this case the neighborhood of S is minimized in the situation when e, e', e'' all share a single prime factor, in which case it is bounded below by

$$|N(S)| \geq |N(s) \cup N(s') \cup N(s'')| \geq 3 \cdot 2^{22} - 3 \cdot 2^{21} + 2^{20} = 3 \cdot 2^{21} + 2^{20} = \frac{7}{16} \cdot 2^{24} = 7{,}340{,}032.$$

But $|S| \leq c_{\geq 2} = 6{,}377{,}708 < 7{,}340{,}032 \leq |N(S)|$. Hence, in every case Hall's condition holds when $k = 2$.

4.4. **Step 4: Case $k = 1$.** Every element of S has $\omega \geq 1$, so $|S| \leq c_{\geq 1} = 12{,}741{,}251$, while $|N(S)| \geq 2^{23} = 8{,}388{,}608$, so we will again need to work with the specific gcds.

Proceeding as above, we suppose $s \in S$ has $\omega((s,u)) = 1$, with $(s,u) = q$ and suppose that q is unique. Then

$$|S| \leq \gcd_3 + c_{\geq 2} = 2{,}019{,}785 + 6{,}377{,}708 = 8{,}397{,}493 > 2^{23}.$$

Note that unlike in previous steps, this bound does not allow us to conclude (yet) that there is another s' and another prime q' distinct from q with $(s',u) = q'$. So we consider again those elements of S sharing two prime factors with u, one of which is q.

Let x_q denote the total number of odd numbers in $[1, 2^{25}]$ which are either divisible by q or counted by $c_{\geq 3}$. As with other statistics, this count is majorized by the count $x_3 = 6{,}334{,}949$ which is included in Table 2.

Since this count is smaller than our bound $|N(S)| \geq 2^{23}$, we would be done if S consisted only of elements counted by x_3. So we suppose that S contains s', not counted by x_q. Then $\omega((s', u)) \leq 2$ and $q \nmid s'$. In this case, we can update our lower bound on $|N(S)|$. This quantity is smallest when s' shares precisely two prime factors with u, neither of which is q, in which case we find that

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{23} + 2^{22} - 2^{21} = 5 \cdot 2^{21} = \frac{5}{8} \cdot 2^{24} = 10{,}485{,}760.$$

Since this quantity now exceeds $\gcd_3 + c_{\geq 2}$, we find that Hall's criterion is necessarily satisfied unless $(s', u) = q'$ where $q' \neq q$ is a different prime factor. Now our lower bound for $|N(S)|$ improves to

$$|N(S)| \geq |N(s) \cup N(s')| = 2^{23} + 2^{23} - 2^{22} = 3 \cdot 2^{22} = \frac{3}{4} \cdot 2^{24} = 12{,}582{,}912.$$

Using this bound, we now return to our original line of argumentation. This bound exceeds

$$\begin{aligned} c_{\geq 2} + \gcd_q + \gcd_{q'} &\leq c_{\geq 2} + \gcd_3 + \gcd_5 \\ &= 6{,}377{,}708 + 2{,}019{,}785 + 1{,}010{,}179 = 9{,}407{,}672 < 12{,}582{,}912. \end{aligned}$$

From this, we see that Hall's condition will be satisfied unless S contains a third element s'' with $\omega((s'', u)) = 1$ and $(s'', u) = q''$ for some prime divisor $q'' \neq q, q'$ of u.

With three elements s, s', and s'' all having mutually distinct prime gcds with u, inclusion-exclusion gives

$$|N(S)| \geq |N(s) \cup N(s') \cup N(s'')| \geq 3 \cdot 2^{23} - 3 \cdot 2^{22} + 2^{21} = \frac{7}{8} \cdot 2^{24} = 14{,}680{,}064.$$

But $|S| \leq c_{\geq 1} = 12{,}741{,}251 < 14{,}680{,}064 \leq |N(S)|$, so Hall's condition holds when $k = 1$.

4.5. **Step 5: Case $k = 0$.** If the minimum value of $\omega((s, u))$ over $s \in S$ is 0, then some $s \in S$ is coprime to u, so $N(S) = D(u)$ and $|N(S)| = 2^{24} \geq |S|$ and Hall's condition holds immediately.

In all cases we have $|N(S)| \geq |S|$, so Hall's condition holds and a perfect matching exists. Since the argument used only upper bounds on census counts, and these upper bounds hold for any $u = q_1 \cdots q_{24}$ with $q_i \geq p_i$, the result applies to every squarefree product of 24 distinct odd primes.

4.6. **Adjustments for other $\ell \leq 44$.** Essentially the same argument can be carried through using any ℓ in place of 24, for $3 \leq \ell \leq 44$. The only important adjustment is to change the computed $c_{\geq j}$ values and $\gcd_d$ values according to the entries in Table 1 and Table 2.

For values of $\ell < 24$ some of the steps above can be omitted, for example when $\ell < 23$, the argument involving $\gcd_{21}$ is unnecessary. After considering $c_{\geq 3} + \gcd_{15}$ and using it to conclude that there exists $s' \in S$ with $\omega((s', u)) \leq 2$, $(s', u) \neq (s, u)$ it is already possible to conclude directly that $|N(S)| > |S|$, since in this case we find, as in (5) that $|N(S)| \geq \frac{3}{8} \cdot 2^{\ell}$, which is already greater than $c_{\geq 2}$. When a quantity is unneeded for the argument, it is omitted from Table 2.

For values of $\ell > 24$, sometimes additional steps are necessary in the initial Step 1. For example when $\ell = 40$, the maximum value of $\omega((s, u))$ over $s \in S$ is 10 (as noted in Table 1 in the $\omega_{\max}$ column). Since $2^{40-10} = 1{,}073{,}741{,}824 \geq c_{\geq 7}$, Hall's condition is satisfied for any $k \geq 7$. One can then check for each $4 \leq k < 7$ that $2^{40-k} \geq c_{\geq k}$ so the condition is met for each of these k as well. Putting this all together, we have shown the following.

Proposition 2. *For any $\ell \leq 44$ and u the product of ℓ distinct odd primes, there exists a coprime matching between $D(u)$ and the odd integers in $[1, 2^{\ell+1}]$.*

An identical Hall's theorem argument establishes Proposition 3, with the bipartite graph now having $D(u)$ matched to all integers in $[1, 2^{\ell}]$ (rather than just odd integers in $[1, 2^{\ell+1}]$), and using the values in Table 3 and Table 4 in place of Tables 1 and 2.

Proposition 3. *For any $\ell \leq 44$ and u the product of ℓ distinct odd primes, there exists a coprime matching between $D(u)$ and the integers in $[1, 2^{\ell}]$.*

We can now combine these propositions to give a proof of Theorem 2.

Proof of Theorem 2. If u is squarefree, odd, and has at most 44 prime factors, then u is matchable by Proposition 3. If u is even and has $\ell \leq 44$ prime factors, write $u = 2u'$; we create a matching as follows. By Proposition 2 applied to u' (which has $\ell - 1$ odd prime factors), there exists a coprime matching of the divisors of u' to the odd integers in $[1, 2^{\ell}]$. We associate to each even divisor $2d$ of u the odd integer matched to d by this proposition. Then, by Proposition 3 applied to u', there exists a coprime matching of $D(u')$ to the integers in $[1, 2^{\ell-1}]$; for every odd divisor d of u (which is also a divisor of u'), we match d to $2a$, where $a \in [1, 2^{\ell-1}]$ is the integer matched to d.

The first construction matches the $2^{\ell-1}$ even divisors of u bijectively to the $2^{\ell-1}$ odd integers in $[1, 2^{\ell}]$, and the second matches the $2^{\ell-1}$ odd divisors bijectively to the $2^{\ell-1}$ even integers in $[2, 2^{\ell}]$; together they give a perfect matching from $D(u)$ to $[1, 2^{\ell}] = [1, \tau(u)]$. Coprimality is preserved: $(2d, a) = (d, a) = 1$ since a is odd, and $(d, 2a) = (d, a) = 1$ since d is odd. □

5. M-numbers

Say a positive integer is an M-number if it is not divisible by any p^p for p prime, i.e., $v_p(n) \leq p - 1$ for all primes p. Call a prime $p \mid n$ *tight* if $v_p(n) = p - 1$, and write $n = n_T n_R$ where $n_T = \prod_{p \mid n, p \text{ tight}} p^{p-1}$ and $n_R = n/n_T$. Set $r = \mathrm{rad}(n_T) = \tau(n_T)$. Note that $(n_T, n_R) = 1$: any prime p dividing both would satisfy $v_p(n) \geq p$, contradicting the M-number condition.

We conjecture that every M-number is matchable. In this section we explain how to generalize the proofs of Lemma 3 and Theorem 1 to show that every M-number with sufficiently many non-tight prime factors and not divisible by the square of any large non-tight prime is matchable (Theorem 3), which implies in particular that the set of non-matchable M-numbers has asymptotic density 0 (Corollary 3).

The key tool is a partition lemma analogous to Lemma 3, but adapted to handle the tight and non-tight primes separately. (Note that in Lemma 3 the only tight prime is 2, which is why 2 is handled separately.) Let $p_1 < p_2 < \cdots < p_\ell$ denote the non-tight prime factors of n_R in increasing order, where $\ell = \omega(n_R)$, and set $a_i = v_{p_i}(n)$ for each i. For a parameter j with $0 \leq j \leq \ell$, let $m_j = \prod_{i \leq j} p_i^{a_i}$, set $n' = n_R/m_j$, and $K = 2^j$.

Lemma 5. *Let n be an* M*-number with n_T, n_R, r, m_j, n' as above. Assume $\tau(n') \geq 4^j$. Then $[1, \tau(n)]$ can be partitioned into $\tau(n_T) \cdot \tau(m_j)$ sets A_d (one for each $d \mid n_T m_j$), each of* *size $\tau(n')$, with every element of A_d coprime to d. Moreover each A_d is a K-AP combination with common differences dividing rm_j, so for any $e \mid n'$ the count of elements of A_d divisible by e is within K of $\tau(n')/e$.*

Proof. We follow the same plan as Lemma 3, with two modifications to handle tight primes and primes of m_j with exponent greater than 1.

The tight primes are handled first and all at once using an argument akin to the one in the introduction with M_p. Since $v_p(n) = p - 1$ for each prime $p \mid r$, the Chinese remainder theorem gives a bijection $\rho\colon D(n_T) \to \{0, 1, \ldots, r-1\}$ defined by $\rho(d) \equiv v_p(d) \pmod p$ for each $p \mid r$. We partition $[1, \tau(n)] = [1, r\tau(n_R)]$ into $\tau(n_T) = r$ residue classes modulo r, pairing d with the class $\rho(d) \pmod r$. Each class is an arithmetic progression of length $\tau(n_R)$ and common difference r, and for any $a \equiv \rho(d) \pmod r$ for $d \mid n_T$, the coprimality $(a, d) = 1$ holds because $p \mid a$ if and only if $p \nmid d$ for each $p \mid r$. This plays exactly the role of the even/odd split on the prime 2 in Lemma 3, producing $\tau(n_T)$ equal-length arithmetic progressions, one per divisor of n_T, with no contribution to the size of K.

The primes of m_j are then handled inductively. After processing $p_1, \ldots, p_{i-1}$, we have $r \cdot \tau(m_{i-1})$ sets (one per divisor of $n_T m_{i-1}$), each a 2^{i-1}-AP combination of size $L' = \tau(m_j)\tau(n')/\tau(m_{i-1})$ with constituent APs having common differences coprime to p_i. We then split each current set S into $a_i + 1$ equal subsets of size $L'/(a_i+1)$, one for each $v_{p_i}(d) \in \{0, \ldots, a_i\}$, with those for $v_{p_i}(d) \geq 1$ consisting of non-multiples of p_i. Choose a cutpoint z so that the non-multiples of p_i in S before z number exactly $a_i \cdot L'/(a_i+1)$. By Lemma 2, the count of p_i-multiples in S is within 2^{i-1} of L'/p_i, hence at most $L'/p_i + 2^{i-1}$. The M-number condition $a_i < p_i - 1$ and the hypothesis $\tau(n') \geq 4^j$ give

$$\frac{L'}{a_i+1} - \frac{L'}{p_i} \geq \frac{L'}{p_i(a_i+1)} \geq \frac{\tau(n')}{p_i} \geq \frac{4^j}{p_j} > 2^{j-1} \geq 2^{i-1},$$

using that $p_i - a_i - 1 \geq 1$, $L'/(a_i+1) = \tau(m_j)\tau(n')/\tau(m_i) \geq \tau(n')$, and $p_i \leq p_j$. Thus $L'/p_i + 2^{i-1} < L'/(a_i+1)$, so there are more than $a_i L'/(a_i+1)$ non-multiples of p_i in S and the cutpoint z is well-defined. The non-multiples before z and the complementary set (multiples before z together with everything after z) are each 2^i-AP combinations by the same set-difference and union arguments as in Lemma 3. The non-multiples before z are then divided into a_i equal contiguous parts by $a_i - 1$ further interval cuts; restricting a 2^i-AP combination to $\{a \leq z'\}$ or $\{a > z'\}$ preserves the AP-combination number (as in Lemma 3), so each part remains a 2^i-AP combination. After all j steps the sets are 2^j-AP combinations of size $\tau(n')$, giving $K = 2^j$. ☐

Theorem 3. *Let n be an* M*-number with non-tight primes $p_1 < p_2 < \cdots < p_\ell$ in increasing order. If $\ell \geq 44$ and $v_{p_i}(n) = 1$ for $\ell \geq i > j := \lfloor \sqrt{\omega(n_R)} \rfloor$, then n is matchable.*

Proof. The proof follows closely that of Theorem 1, using Lemma 5 in place of Lemma 3. Lemma 5 partitions $[1, \tau(n)]$ into sets A_d, one for each $d \mid n_T m_j$, each of size $\tau(n')$ and forming a K-AP combination with common differences dividing rm_j. (The hypothesis $\tau(n') \geq 4^j$ holds because $\tau(n') = 2^{\ell-j} \geq 2^{2j} = 4^j$, since $\ell - j \geq 2j$ once $\ell \geq 9$.)

The hypothesis that n is not divisible by the square of any non-tight prime $q > p_j$ ensures that n' is squarefree. For each $d \mid n_T m_j$ we apply Hall's theorem to match $D(n')$ to A_d. Setting $\tilde{\ell} = \ell + 1$ and $\tilde{\jmath} = j + 1$ gives $K = 2^{\tilde{\jmath}-1}$ and $\omega(n') = \tilde{\ell} - \tilde{\jmath}$, so the Hall's theorem arguments of Theorem 1 apply verbatim with $(\tilde{\ell}, \tilde{\jmath})$ in place of (ℓ, j), covering all $\ell \geq 44$.

(The shift $\tilde{\ell} = \ell + 1$ reflects that the prime 2, here a tight prime contributing to n_T, plays the same structural role as $p_1 = 2$ in Theorem 1. In both cases it contributes to neither K nor to f, and the bound $f(n') \le \sum_{i=\tilde{j}+1}^{\tilde{\ell}} 1/P_i < 0.93$ matches that theorem's bound precisely.)

Once we have obtained the matchings $\phi_d \colon D(n') \to A_d$, we can construct the coprime matching of $D(n)$ to $[1, \tau(n)]$ via $\psi(d \cdot e) = \phi_d(e)$, as in Theorem 1. □

Since the set of M-numbers having fewer than any fixed number of non-tight prime factors has asymptotic density zero, as do those with a repeated large non-tight prime factor, we obtain the following corollary.

Corollary 3. *Every* M*-number is matchable except possibly for a set of asymptotic density zero.*

With Corollary 2 we have the following.

Corollary 4. *The set of matchable numbers has asymptotic density α.*

6. Strongly matchable numbers

Recall that we say n is strongly matchable if for each coprime arithmetic progression of $\tau(n)$ integers there is a coprime matching to $D(n)$.

Conjecture 1. *A number n is strongly matchable if and only if it is an* M*-number.*

One would think that our techniques for matchable numbers could be applied here but there is a difficulty. In the proof of Theorem 1 we strongly used that we are mapping $D(n)$ to an interval of small numbers, namely $[1, \tau(n)]$, but with strongly matchable, the interval is not only generalized to a coprime arithmetic progression, it can be anywhere on the number line. The latter condition is the difficulty. We at least have a few results in the direction of the conjecture.

Proposition 4. *Every strongly matchable number is an* M*-number.*

Proof. We prove the contrapositive. Suppose n is not an M-number, and so $p^p \mid n$ for some prime p. Then at least $p/(p+1)$ of the members of $D(n)$ are divisible by p. Let I be a coprime arithmetic progression of length $\tau(n)$. Each shift $I+m$ is again a coprime arithmetic progression of length $\tau(n)$. On average as m varies, exactly $(p-1)/p$ of the integers in the set are coprime to p, so there is at least one such m where $I+m$ has less than $p/(p+1)$ of its members coprime to p. We cannot coprimely match $D(n)$ with $I+m$, so n is not strongly matchable. □

Proposition 5. *If n is strongly matchable and not divisible by the prime p, then $p^{p-1}n$ is strongly matchable.*

Proof. We have $\tau(p^{p-1}n) = p\tau(n)$. Let I be a coprime arithmetic progression of length $p\tau(n)$, say $I = \{i_1, i_2, \ldots, i_{p\tau(n)}\}$. For $j = 1, 2, \ldots, p$, let I_j be the subsequence $(i_{j+kp})_k$ of length $\tau(n)$. At most one of these subsequences has its terms all divisible by p, and the other subsequences are all coprime to p. If there is some j where all the terms of I_j are divisible by p, denote this j by j_0, otherwise let $j_0 = 1$. There are coprime matchings ψ_j from $D(n)$ to I_j for each j. We construct a coprime matching from $D(p^{p-1}n)$ to I as follows. For $D(n)$, we already have ψ_{j_0}. For $p^k D(n)$, $1 \le k \le p-1$, $k \ne j_0$, we map it to one of the subsequences

I_j not used via ψ_j; that is, we map $p^k d \in p^k D(n)$ to $\psi_j(d)$. The union of these maps gives a coprime matching from $D(p^{p-1}n)$ to I, completing the proof. $\square$

Proposition 6. *The set of strongly matchable numbers has lower asymptotic density greater than* $4/11$.

Proof. We first note that if n is strongly matchable and p is a prime with $p \nmid n$ and $p > 2\tau(n)$, then pn is also strongly matchable. Indeed, take an arbitrary coprime arithmetic progression I of $\tau(pn) = 2\tau(n)$ integers. There are coprime matchings of $D(n)$ to both the first half of I and the second half of I, say ψ_1, ψ_2. Not both of these halves contain a multiple of p, so map $pD(n)$ to a half not containing a multiple of p via $p\psi_i$, and use the unadorned injection for the other half.

Let S_j be the set of odd squarefree numbers s with $\omega(s) = j$ and each prime factor of s is $< 2^j$, and let S be the union of the sets S_j. Suppose that $n > 1$ is an odd squarefree number not divisible by any member of S. Then

$$n = q_1 q_2 \dots q_k, \quad 3 \le q_1 < q_2 < \cdots < q_k, \quad \text{each } q_j \text{ prime}, \quad \text{each } q_j \ge 2^j.$$

Indeed, if not and $q_j < 2^j$ for some j, then $q_1 q_2 \dots q_j \in S_j$, contradicting our assumption that n is not divisible by any member of S.

We claim that any odd squarefree n not divisible by any element of S is strongly matchable. First, $n = 1$ is strongly matchable, and any odd prime q is strongly matchable by the argument above, since $q > 2\tau(1)$. Now suppose that $n_j := q_1 q_2 \cdots q_j$ is strongly matchable. Since $\tau(n_j) = 2^j$ and $q_{j+1} > 2^{j+1}$, it follows by the argument above that $n_j q_{j+1}$ is strongly matchable. Induction completes the argument that n is strongly matchable.

It remains to show that such numbers n comprise a set of positive lower density. The reciprocal sum of the primes $q \le 2^j$ is $\log j + O(1)$, and in fact from [8, (3.18)] and a calculation, this sum is $< \log j$ when $j \ge 4$. Since $\sum_{p \le 7} 1/p > 1.17619$, it follows that

$$\sum_{s \in S'_j} \frac{1}{s} \le \frac{(\log j - 1.17619)^j}{j!}, \quad j \ge 4,$$

where S'_j is the set of $s \in S_j$ with all prime factors > 8. Let T denote the set of squarefree numbers n with all prime factors > 8. Then T has an asymptotic density equal to

$$\frac{6}{\pi^2} \prod_{p \le 7} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^2}\right)^{-1} = 0.221640\dots.$$

If $s \in S$ divides some $n \in T$, then each prime factor of s is > 8, so for some $j \ge 4$, we have $s \in S_j$, and so $s \in S'_j$. The upper asymptotic density of the set of multiples of the elements in $\cup_{j \ge 4} S'_j$ is at most

$$\sum_{j \ge 4} \frac{(\log j - 1.17619)^j}{j!} < 0.000331239.$$

Let T' denote the subset of T consisting of numbers not divisible by any member of S, so that every member of T' is strongly matchable. The lower asymptotic density of T' is greater than 0.2213. We can boost this using Proposition 5. For each $n \in T'$ and each $p \in \{2, 3, 5, 7\}$, we have $p \nmid n$ (since all prime factors of n exceed 8), so Proposition 5 shows that $p^{p-1}n$ is also strongly matchable. Taking potential prime factors of this form into account improves the lower bound for the lower density of the set of strongly matchable numbers to at least

$$0.2213 \cdot \left(1+\frac{1}{2}\right)\left(1+\frac{1}{9}\right)\left(1+\frac{1}{5^4}\right)\left(1+\frac{1}{7^6}\right) > 0.3694 > \frac{4}{11}.$$

□

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Table 1: Census of values of $\omega((s,n))$ for odd $s \in [1, 2^{\ell+1}]$ where n is the product of the first ℓ odd primes; large values are rounded up.

ℓ	$\omega_{\max}$	c_0	c_1	c_2	c_3	c_4	c_5	c_6	$c_{\geq 7}$
3	2	3	4	1					
4	2	7	7	2					
5	2	13	11	8					
6	3	25	21	17	1				
7	3	47	43	33	5				
8	3	89	95	56	16				
9	3	164	210	95	43				
10	4	309	441	176	94	4			
11	4	597	878	376	179	18			
12	4	1166	1736	798	341	55			
13	5	2293	3376	1758	612	152	1		
14	5	4505	6612	3758	1138	364	7		
15	5	8897	12 940	7892	2233	768	38		
16	5	17 558	25 510	16 243	4553	1540	132		
17	6	34 585	50 650	32 767	9755	2921	393	1	
18	6	68 151	100 919	65 561	21 015	5468	1021	9	
19	6	134 282	201 536	130 617	45 041	10 387	2370	55	
20	6	264 692	402 354	260 661	95 132	20 367	5157	213	
21	6	522 290	803 185	521 116	197 833	41 521	10 487	720	
22	7	1 031 482	1 601 975	1 045 031	405 967	87 226	20 541	2076	6
23	7	2 039 192	3 193 416	2 100 106	825 167	185 875	39 443	5363	46
24	7	4 035 965	6 363 543	4 225 826	1 666 753	396 604	75 606	12 689	230
25	7	7 992 094	12 678 222	8 506 329	3 361 015	840 077	147 784	28 017	894
26	8	15 830 224	25 256 133	17 121 427	6 780 185	1 762 060	297 171	58 718	2946
27	8	31 367 217	50 318 652	34 440 697	13 693 052	3 657 862	613 215	118 465	8568
28	8	62 163 303	1.00×10^{8}	69 234 321	27 697 616	7 530 374	1 290 469	233 192	22 820
29	8	1.23×10^{8}	2.00×10^{8}	1.39×10^{8}	56 080 479	15 404 730	2 739 542	454 761	56 236
30	8	2.45×10^{8}	3.98×10^{8}	2.79×10^{8}	1.13×10^{8}	31 329 808	5 803 412	887 715	129 521
31	9	4.85×10^{8}	7.94×10^{8}	5.60×10^{8}	2.30×10^{8}	63 616 039	12 232 414	1 760 793	283 999
32	9	9.64×10^{8}	1.58×10^{9}	1.12×10^{9}	4.65×10^{8}	1.29×10^{8}	25 579 607	3 565 554	598 677
33	9	1.91×10^{9}	3.16×10^{9}	2.25×10^{9}	9.40×10^{8}	2.62×10^{8}	53 108 556	7 367 039	1 228 836
34	9	3.80×10^{9}	6.30×10^{9}	4.52×10^{9}	1.90×10^{9}	5.32×10^{8}	1.09×10^{8}	15 416 636	2 473 356
35	9	7.55×10^{9}	1.26×10^{10}	9.06×10^{9}	3.83×10^{9}	1.08×10^{9}	2.24×10^{8}	32 498 453	4 948 093
36	10	1.50×10^{10}	2.51×10^{10}	1.82×10^{10}	7.73×10^{9}	2.20×10^{9}	4.59×10^{8}	68 545 789	9 907 571
37	10	2.98×10^{10}	5.00×10^{10}	3.64×10^{10}	1.56×10^{10}	4.47×10^{9}	9.36×10^{8}	1.44×10^{8}	19 982 844
38	10	5.93×10^{10}	9.99×10^{10}	7.30×10^{10}	3.14×10^{10}	9.07×10^{9}	1.91×10^{9}	3.01×10^{8}	40 742 931
39	10	1.18×10^{11}	1.99×10^{11}	1.46×10^{11}	6.33×10^{10}	1.84×10^{10}	3.89×10^{9}	6.26×10^{8}	83 930 709
40	10	2.35×10^{11}	3.98×10^{11}	2.93×10^{11}	1.28×10^{11}	3.74×10^{10}	7.93×10^{9}	1.29×10^{9}	1.74×10^{8}
41	11	4.66×10^{11}	7.93×10^{11}	5.87×10^{11}	2.57×10^{11}	7.57×10^{10}	1.62×10^{10}	2.66×10^{9}	3.64×10^{8}
42	11	9.28×10^{11}	1.58×10^{12}	1.18×10^{12}	5.17×10^{11}	1.53×10^{11}	3.29×10^{10}	5.46×10^{9}	7.61×10^{8}
43	11	1.85×10^{12}	3.16×10^{12}	2.36×10^{12}	1.04×10^{12}	3.11×10^{11}	6.71×10^{10}	1.12×10^{10}	1.59×10^{9}
44	11	3.67×10^{12}	6.31×10^{12}	4.72×10^{12}	2.10×10^{12}	6.29×10^{11}	1.37×10^{11}	2.29×10^{10}	3.32×10^{9}
45	11	7.31×10^{12}	1.26×10^{13}	9.46×10^{12}	4.22×10^{12}	1.27×10^{12}	2.79×10^{11}	4.68×10^{10}	6.91×10^{9}

Table 2: Selected gcd counts $\mathrm{gcd}_d = \#\{\text{odd } s \in [1, 2^{\ell+1}] : (s, n) = d\}$ and auxiliary count $x_3 = \#\{\text{odd } s \in [1, 2^{\ell+1}] : 3 \mid s \text{ or } \omega((s, n)) \geq 3\}$, where n is the product of the first ℓ odd primes; large values are rounded up. Only values needed in the proof are shown; blank entries are not necessarily zero.

ℓ	gcd_{105}	gcd_{15}	gcd_{21}	gcd_3	x_3	gcd_5
3				2		
4				3		
5		2		5		
6		3		10		
7		5		21		
8		9		42		
9		17		86		
10		36		166		
11		76		315		
12		152		604		
13		300		1164		
14		590		2256		
15		1139		4416		
16		2218		8682		
17		4314		17 139		
18		8453		33 877		
19		16 639		66 979		
20		32 846		132 281		
21		64 979		261 372		130 677
22		128 676		516 379		258 258
23	42 293	254 834	169 795	1 020 848		510 604
24	83 729	504 881	336 514	2 019 785	6 334 949	1 010 179
25	166 004	1 000 144	666 745	3 998 146	12 706 706	1 999 526
26	329 275	1 980 869	1 320 714	7 916 785	25 481 743	3 958 891
27	653 169	3 923 935	2 616 309	15 683 688	51 096 769	7 842 216
28	1 295 341	7 773 941	5 183 268	31 078 505	1.02×10^8	15 539 148
29	2 568 728	15 406 877	10 272 217	61 607 914	2.06×10^8	30 803 093
30	5 097 322	30 566 423	20 378 662	1.22×10^8	4.12×10^8	61 123 344
31	10 115 856	60 661 064	40 441 466	2.43×10^8	8.26×10^8	1.21×10^8
32	20 080 727	1.20×10^8	80 288 746	4.82×10^8	1.66×10^9	2.41×10^8
33	39 866 096	2.39×10^8	1.59×10^8	9.57×10^8	3.32×10^9	4.78×10^8
34	79 187 622	4.75×10^8	3.17×10^8	1.90×10^9	6.66×10^9	9.50×10^8
35	1.57×10^8	9.44×10^8	6.29×10^8	3.78×10^9	1.34×10^{10}	1.89×10^9
36	3.13×10^8	1.88×10^9	1.25×10^9	7.50×10^9	2.68×10^{10}	3.75×10^9
37	6.21×10^8	3.73×10^9	2.49×10^9	1.49×10^{10}	5.36×10^{10}	7.46×10^9
38	1.24×10^9	7.41×10^9	4.94×10^9	2.97×10^{10}	1.08×10^{11}	1.48×10^{10}
39	2.46×10^9	1.47×10^{10}	9.83×10^9	5.90×10^{10}	2.15×10^{11}	2.95×10^{10}
40	4.89×10^9	2.93×10^{10}	1.95×10^{10}	1.17×10^{11}	4.32×10^{11}	5.86×10^{10}
41	9.72×10^9	5.83×10^{10}	3.89×10^{10}	2.33×10^{11}	8.65×10^{11}	1.17×10^{11}
42	1.93×10^{10}	1.16×10^{11}	7.73×10^{10}	4.64×10^{11}	1.73×10^{12}	2.32×10^{11}
43	3.85×10^{10}	2.31×10^{11}	1.54×10^{11}	9.23×10^{11}	3.47×10^{12}	4.62×10^{11}
44	7.65×10^{10}	4.59×10^{11}	3.06×10^{11}	1.84×10^{12}	6.96×10^{12}	9.18×10^{11}
45	1.52×10^{11}	9.14×10^{11}	6.09×10^{11}	3.66×10^{12}	1.39×10^{13}	1.83×10^{12}

Table 3: Census of values of $\omega((s,n))$ for $s \in [1, 2^{\ell}]$ where n is the product of the first ℓ odd primes; large values are rounded up.

ℓ	$\omega_{\max}$	c_0	c_1	c_2	c_3	c_4	c_5	c_6	$c_{\geq 7}$
3	1	4	4						
4	2	6	9	1					
5	2	11	18	3					
6	2	22	30	12					
7	3	44	51	32	1				
8	3	87	91	72	6				
9	3	171	180	138	23				
10	3	328	375	251	70				
11	4	626	793	451	174	4			
12	4	1200	1646	847	381	22			
13	4	2316	3359	1653	785	79			
14	5	4510	6717	3407	1507	242	1		
15	5	8832	13 321	7145	2823	639	8		
16	5	17 400	26 245	15 033	5318	1494	46		
17	5	34 338	51 657	31 407	10 240	3247	183		
18	6	67 840	101 977	64 647	20 478	6606	595	1	
19	6	134 032	202 022	131 428	42 198	12 911	1687	10	
20	6	264 639	401 506	264 780	88 485	24 820	4281	65	
21	6	522 702	799 799	530 538	186 141	47 696	9993	283	
22	6	1 032 593	1 595 114	1 060 794	389 735	93 243	21 796	1029	
23	7	2 041 220	3 182 621	2 121 272	808 354	186 821	45 105	3209	6
24	7	4 038 813	6 350 266	4 246 629	1 659 760	382 912	89 884	8900	52
25	7	7 995 366	12 665 960	8 515 547	3 381 389	797 705	175 613	22 568	284
26	7	15 832 644	25 252 891	17 098 634	6 854 406	1 673 928	341 895	53 269	1197
27	8	31 366 915	50 335 662	34 359 252	13 851 260	3 510 598	671 330	118 470	4241
28	8	62 157 666	1.00×10^{8}	69 065 837	27 961 797	7 329 469	1 341 027	251 722	13 225
29	8	1.23×10^{8}	2.00×10^{8}	1.39×10^{8}	56 440 743	15 199 047	2 730 221	515 906	37 449
30	8	2.45×10^{8}	3.99×10^{8}	2.79×10^{8}	1.14×10^{8}	31 245 966	5 636 269	1 027 549	97 383
31	8	4.85×10^{8}	7.95×10^{8}	5.60×10^{8}	2.30×10^{8}	63 883 643	11 755 588	2 023 659	237 081
32	9	9.64×10^{8}	1.58×10^{9}	1.12×10^{9}	4.64×10^{8}	1.30×10^{8}	24 606 119	3 981 833	546 323
33	9	1.91×10^{9}	3.16×10^{9}	2.25×10^{9}	9.38×10^{8}	2.64×10^{8}	51 472 118	7 908 459	1 205 696
34	9	3.80×10^{9}	6.30×10^{9}	4.52×10^{9}	1.90×10^{9}	5.36×10^{8}	1.07×10^{8}	15 904 855	2 563 508
35	9	7.55×10^{9}	1.26×10^{10}	9.06×10^{9}	3.83×10^{9}	1.09×10^{9}	2.22×10^{8}	32 484 581	5 314 501
36	9	1.50×10^{10}	2.51×10^{10}	1.82×10^{10}	7.73×10^{9}	2.21×10^{9}	4.57×10^{8}	67 141 601	10 821 108
37	10	2.98×10^{10}	5.00×10^{10}	3.64×10^{10}	1.56×10^{10}	4.47×10^{9}	9.38×10^{8}	1.40×10^{8}	21 818 440
38	10	5.93×10^{10}	9.98×10^{10}	7.30×10^{10}	3.14×10^{10}	9.07×10^{9}	1.92×10^{9}	2.92×10^{8}	43 904 888
39	10	1.18×10^{11}	1.99×10^{11}	1.46×10^{11}	6.33×10^{10}	1.84×10^{10}	3.91×10^{9}	6.10×10^{8}	88 614 835
40	10	2.35×10^{11}	3.98×10^{11}	2.93×10^{11}	1.28×10^{11}	3.73×10^{10}	7.97×10^{9}	1.27×10^{9}	1.80×10^{8}
41	10	4.66×10^{11}	7.93×10^{11}	5.87×10^{11}	2.57×10^{11}	7.57×10^{10}	1.62×10^{10}	2.63×10^{9}	3.69×10^{8}
42	11	9.28×10^{11}	1.58×10^{12}	1.18×10^{12}	5.18×10^{11}	1.53×10^{11}	3.30×10^{10}	5.44×10^{9}	7.60×10^{8}
43	11	1.85×10^{12}	3.16×10^{12}	2.36×10^{12}	1.04×10^{12}	3.10×10^{11}	6.72×10^{10}	1.12×10^{10}	1.57×10^{9}
44	11	3.67×10^{12}	6.31×10^{12}	4.72×10^{12}	2.10×10^{12}	6.28×10^{11}	1.37×10^{11}	2.30×10^{10}	3.27×10^{9}
45	11	7.31×10^{12}	1.26×10^{13}	9.46×10^{12}	4.22×10^{12}	1.27×10^{12}	2.79×10^{11}	4.70×10^{10}	6.81×10^{9}

Table 4: Selected gcd counts $\mathrm{gcd}_d = \#\{s \in [1, 2^\ell] : (s, n) = d\}$ and auxiliary count $x_3 = \#\{s \in [1, 2^\ell] : 3 \mid s \text{ or } \omega((s, n)) \geq 3\}$, where n is the product of the first ℓ odd primes; large values are rounded up. Only values needed in the proof are shown; blank entries are not necessarily zero.

ℓ	gcd_{105}	gcd_{15}	gcd_{21}	gcd_3	x_3	gcd_5
3				2		
4				4		
5				7		
6				11		
7		7		19		
8		12		37		
9		20		75		
10		35		154		
11		68		310		
12		138		612		
13		280		1195		
14		566		2320		
15		1135		4504		
16		2241		8783		
17		4400		17 182		
18		8607		33 788		
19		16 846		66 658		
20		33 048		131 710		
21		65 061		260 517		130 105
22		128 425		515 466		257 532
23		254 093	169 296	1 020 164		509 903
24	84 153	503 484	335 445	2 020 025	6 320 878	1 009 955
25	166 227	998 109	665 082	4 000 127	12 695 486	2 000 257
26	328 933	1 978 571	1 318 640	7 921 325	25 484 436	3 961 321
27	651 867	3 922 276	2 614 483	15 691 232	51 130 089	7 846 881
28	1 292 724	7 774 396	5 182 819	31 088 550	1.03×10^8	15 546 187
29	2 564 674	15 411 506	10 274 822	61 618 338	2.06×10^8	30 811 418
30	5 092 434	30 577 232	20 386 319	1.22×10^8	4.12×10^8	61 130 339
31	10 111 570	60 679 387	40 455 703	2.43×10^8	8.27×10^8	1.21×10^8
32	20 079 847	1.20×10^8	80 309 360	4.82×10^8	1.66×10^9	2.41×10^8
33	39 872 679	2.39×10^8	1.59×10^8	9.57×10^8	3.32×10^9	4.78×10^8
34	79 206 028	4.75×10^8	3.17×10^8	1.90×10^9	6.66×10^9	9.50×10^8
35	1.57×10^8	9.44×10^8	6.29×10^8	3.78×10^9	1.33×10^{10}	1.89×10^9
36	3.13×10^8	1.88×10^9	1.25×10^9	7.50×10^9	2.68×10^{10}	3.75×10^9
37	6.21×10^8	3.73×10^9	2.49×10^9	1.49×10^{10}	5.36×10^{10}	7.46×10^9
38	1.24×10^9	7.41×10^9	4.94×10^9	2.97×10^{10}	1.08×10^{11}	1.48×10^{10}
39	2.46×10^9	1.47×10^{10}	9.83×10^9	5.90×10^{10}	2.15×10^{11}	2.95×10^{10}
40	4.89×10^9	2.93×10^{10}	1.95×10^{10}	1.17×10^{11}	4.32×10^{11}	5.86×10^{10}
41	9.72×10^9	5.83×10^{10}	3.89×10^{10}	2.33×10^{11}	8.65×10^{11}	1.17×10^{11}
42	1.93×10^{10}	1.16×10^{11}	7.73×10^{10}	4.64×10^{11}	1.73×10^{12}	2.32×10^{11}
43	3.85×10^{10}	2.31×10^{11}	1.54×10^{11}	9.23×10^{11}	3.47×10^{12}	4.62×10^{11}
44	7.65×10^{10}	4.59×10^{11}	3.06×10^{11}	1.84×10^{12}	6.96×10^{12}	9.18×10^{11}
45	1.52×10^{11}	9.14×10^{11}	6.09×10^{11}	3.66×10^{12}	1.39×10^{13}	1.83×10^{12}
46	3.03×10^{11}	1.82×10^{12}	1.21×10^{12}	7.28×10^{12}	2.79×10^{13}	3.64×10^{12}

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