Matchable numbers

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For a natural number n let

$$D(n) = \{d > 0 : d | n\}, \qquad \tau(n) = \#D(n).$$

Say n is *matchable* if $D(n)$ can be matched with $\{1, 2, \ldots, \tau(n)\}$ where each match is a coprime pair.

For example, $n = 10$:

$$\begin{array}{rcl} 1 & \longleftrightarrow & 2 \\ 2 & \longleftrightarrow & 3 \\ 5 & \longleftrightarrow & 4 \\ 10 & \longleftrightarrow & 1 \end{array}$$

Non-matchable numbers

In a matching for n, its even divisors must be paired with odd numbers in $\{1, 2, \ldots, \tau(n)\}$.

Consider $n = 8$: There are 3 even numbers in $D(8) = \{1, 2, 4, 8\}$ but only 2 odd numbers in $\{1, 2, 3, 4\}$. So, 8 is not matchable.

This generalizes: Every number $4m$ has at least 2/3 of $D(4m)$ even, but $\{1, 2, \ldots, t\}$ has $< 2/3$ of its elements odd, once $t \geq 4$. Thus $4m$ is non-matchable provided $m > 1$.

So, the non-matchable numbers contain a subset of asymptotic density 1/4, namely the multiples of 4 except for 4 itself.

Non-matchable numbers, continued

Now consider $n = 27m = 3^3m$. Then $D(27m)$ has ≥75% of its members divisible by 3. In a matching these must be sent to non-multiples of 3. If $t \geq 9$, then $\{1, 2, \ldots, t\}$ has $<$ 75% non-multiples of 3, so if $27m$ is to be matchable, then $\tau(27m) < 9$. One can check that $27q$ is matchable for every prime q, but not for other multiples of 27.

But from a density aspect, we see that 0% of the multiples of 27 are matchable.

What about multiples of 5^5? 7^7? In general, if $p^p \,|\, n$ and $\tau(n) \geq p^2$, then n is not matchable.

Theorem: Among the set of numbers divisible by some p^p, with p prime, the matchable ones comprise a set of density 0.

M-numbers

Say n is an "M-number" if it is not divisible by any p^p with p prime. So, all squarefree numbers are M-numbers. We've seen that matchable numbers that are *not* M-numbers are sparse.

Note that the set of M-numbers has asymptotic density

$$\alpha := \prod_p (1 - 1/p^p) = 0.72199\ldots.$$

Conjecture: *Every* M-*number is matchable.*

Theorem (McNew, P): *The set of non-matchable* M-*numbers comprise a set of asymptotic density* 0.

Corollary: *The set of matchable numbers has density* α.

A possible way to prove the conjecture that every M-number is matchable:

Let S be a finite set of primes and let

$$n_S = \prod_{p \in S} p^{p-1}.$$

We prove that n_S is matchable: Note that $\tau(n_S) = \prod_{p \in S} p$. For $j \le \tau(n_S)$, send j to $\prod_{p \in S} p^{j \bmod p}$. The exponent on p is 0 if and only if $p | j$, so that j gets sent to a number that is coprime to j. Also, the exponent is $\le p - 1$, so j gets sent to a divisor of n_S. And the Chinese remainder theorem shows the map is one-to-one.

So, to prove the conjecture, one needs only show that all divisors of matchable numbers are themselves matchable, an observation of Joachim König. In fact, one need only prove that if n is matchable and $p^2 | n$, then n/p is matchable.

Another way to show a number is matchable: the König–Hall theorem (these are Dénes König and Philip Hall):

Suppose we have a bipartite graph from finite set A to B and for each $S \subset A$ with neighboring set $T \subset B$ (here T is the set of elements of B connected to some member of A), we have $\#S \le \#T$. Then the graph contains a matching from A into B. (So, if $\#B = \#A$, it is a perfect matching.)

In our case we have $A = \{1, 2, \ldots, \tau(n)\}$, $B = D(n)$, and the graph connects each $a \in A$ to all $b \in B$ with $(a, b) = 1$.

A simple consequence

Let $\omega(n) = \sum_{p\,|\,n} 1$ and $f(n) = \sum_{p\,|\,n} 1/p$.

Lemma. *Suppose n is squarefree and $f(n) \le 1/2$. Then n is matchable.*

Proof. Let $A \subset \{1, 2, \ldots, \tau(n)\}$ and let B be the set of divisors of n coprime to at least one member of A. Choose $a \in A$ with $k = \omega((a, n))$ minimal over all $a \in A$. Then B has at least $\tau(n)/2^k$ elements, namely all of those divisors of n coprime to a (using n squarefree). But, using the minimality of k,

$$\#A \;\le \sum_{\substack{a \le \tau(n) \\ \omega((a,n)) \ge k}} 1 \;\le\; \sum_{\substack{d\,|\,n \\ \omega(d)=k}} \frac{\tau(n)}{d} \;\le\; \tau(n)\frac{f(n)^k}{k!} \;\le\; \frac{\tau(n)}{2^k k!} \;\le\; \frac{\tau(n)}{2^k}.$$

This is $\le \#B$, so by König–Hall the Lemma is proved. $\square$

As an aside, lets look at the step above:

$$\sum_{\substack{a \le \tau(n) \\ \omega((a,n)) \ge k}} 1 \le \sum_{\substack{d | n \\ \omega(d) = k}} \frac{\tau(n)}{d}.$$

To justify this note that if $\omega((a,n)) \ge k$, then a is divisible by some $d | n$ with $\omega(d) \ge k$ and so is divisible by some $d | n$ with $\omega(d) = k$. And the number of multiples of d that are in $\{1, 2, \ldots, \tau(n)\}$ is $\le \tau(n)/d$.

Later we'll generalize this Lemma where we don't have an initial interval $\{1, 2, \ldots, \tau(n)\}$ but we are concerned about the multiples of d in the set we do have.

But for now, what does the Lemma get us?

A positive proportion is matchable

We can show a positive proportion of integers satisfy the two hypotheses: n is squarefree and $f(n) \le 1/2$ (where recall that $f(n) = \sum_{p \mid n} 1/p$). This is done by averaging $f(n)$ over squarefrees, this is $< 1/3$, so a positive proportion of squarefrees have $f(n) \le 1/2$.

However to make further progress we should at least try to show that asymptotically *all* squarefrees are matchable, and then see if we can tackle M-numbers. As mentioned, we conjecture that all M-numbers (which includes all squarefrees) are matchable.

Filling in

Note that the Lemma is not very useful if n is even. By slightly strengthening the Lemma we can get around this.

Suppose n is odd and squarefree with $f(n) \le 1/2$. Then we can not only coprimely match the divisors of n with $\{1, 2, \ldots, \tau(n)\}$, but we can coprimely match them with the first $\tau(n)$ odd numbers $\{1, 3, \ldots, 2\tau(n) - 1\}$ (using a slightly stronger Lemma). These two matchings show that $2n$ is matchable. Indeed, the matching of $D(n)$ to the odds can be used to match $2D(n)$ with the odds in $[1, \tau(2n)] = [1, 2\tau(n)]$. And the matching of $D(n)$ to the numbers up to $\tau(n)$ can be used to match the odd divisors of $2n$ with $\{2, 4, \ldots, 2\tau(n)\}$, since $D(n) = D(2n) \setminus 2D(n)$.

Filling in, continued

We can use a similar plan to fill in an odd prime, say 3. So suppose n is squarefree with $f(n) \le 1/2$ and $3 \nmid n$. We try to show that $3n$ is matchable. We partition $D(3n)$ into $D(n)$ and $3D(n)$. And we partition $\{1, 2, \ldots, \tau(3n)\} = \{1, 2, \ldots, 2\tau(n)\}$ into the multiples of 3 and the non-multiples. The non-multiples of 3 are the set difference of two AP's. We match the set $3D(n)$ into the least $\tau(n)$ of these non-multiples of 3. The remaining $\tau(n)$ numbers in the interval consist of 2 parts, all of the multiples of 3 and some of the non-multiples of 3. We match $D(n)$ into these remaining numbers.

We then show that we can fill in with many small primes. What's needed is a stronger Lemma that allows us to do this.

Stronger Lemma. *Suppose n is squarefree, $f(n) \le 1/2$, $\sqrt{\log n} \le \tau(n) \le \log n$, and the number of primes $p|n$ with $p \le (\log n)^2$ is at most $\sqrt{\log\log n}$. Suppose $A \subset [1, (\log n)^2] \cap \mathbb{N}$ with $\#A = \tau(n)$, and $0 \le \kappa \le (\log n)^{1/3}$ is such that for each $d \mid n$, the number of members of A divisible by d is within κ of $\tau(n)/d$. There is an absolute constant N_0 such that if $n \ge N_0$, there is a one-to-one correspondence between $D(n)$ and A such that corresponding numbers are relatively prime.*

Note too that all numbers n but for a set of density 0 have $\sqrt{\log n} \le \tau(n) \le \log n$.

A positive proportion of squarefree numbers u have $f(u) > 1/2$, so the trick is to write $u = mn$ where m has the small primes in u, say up to $(\log u)^2$, and $n = u/m$. Then we have $f(n) = o(1)$, so we may assume $f(n) \le 1/2$.

We then use the filling-in plan to show that all squarefree numbers, but for a possible exceptional set of asymptotic density 0, are matchable.

And by generalizing the filling-in algorithm to prime powers p^j where $j \le p - 1$, we show the same for M-numbers. (Recall that n is an M-number if it is not divisible by p^p for any prime p.)

Theorem (McNew, P). *All* M*-numbers, but for a possible exceptional set of asymptotic density* 0*, are matchable.*

As mentioned, we have shown that the set of matchable non-M-numbers has asymptotic density 0. As a corollary we have that the density of the set of matchable numbers exists and is equal to

$$\prod_p (1 - 1/p^p) = 0.72199\ldots.$$

We conjecture that every M-number is matchable. In fact, we even conjecture that $D(n)$ for an M-number n can be coprimely matched into *any* interval of $\tau(n)$ consecutive integers.

This is not true for non-M-numbers. Namely if n is a non-M-number then there is an interval of $\tau(n)$ consecutive integers that has no coprime matching with $D(n)$.

Take the matchable numbers 4, 27, and 135 as examples.
Consider $n = 4$ and the interval $\{2, 3, 4\}$.
Or $n = 27$ and the interval $\{3, 4, 5, 6\}$.
Or $n = 135$ and the interval $\{3, 4, 5, 6, 7, 8, 9, 10\}$.

Thank you