MAA MathFest—Winkler Session

August 7, 2026

Matching Numbers

Carl Pomerance, Dartmouth College

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A simple answer: No.

For example, $I = \{4\}$, $J = \{6\}$.

Or $I = \{3, 4\}$, $J = \{5, 6\}$.

Or $I = \{4, 5, 6\}$, $J = \{12, 13, 14\}$.

In the first two examples, $\{4\}$, $\{6\}$ and $\{3,4\}$, $\{5,6\}$, one set contains a number divisible by a prime divisor of each number in the other set. Namely, "6" in both cases.

The third example, $\{4,5,6\}$, $\{12,13,14\}$, has a strict majority of even numbers in both sets.

There are other "monsters" too, like

$$I = \{10,11,12,13\},\ J = \{15,16,17,18\}.$$

(Both 10 and 12 match only to 17.)

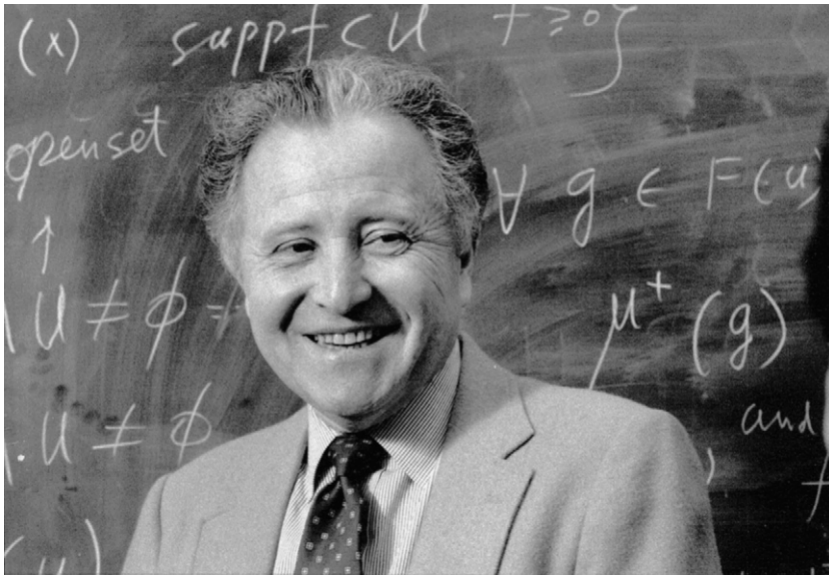
Around 1960, **D. J. Newman** conjectured that in the special case that

$I = [n] = \{1, 2, \ldots, n\}$, J is any interval of n consecutive integers,

there must be a coprime matching. (That is, there is a 1-1 correspondence with corresponding numbers coprime.)

In a lecture in 1962 at the University of Reading, **Paul Erdős** offered £5 for a proof of the weaker conjecture where $I = [n]$ and $J = \{n+1, \ldots, 2n\}$. A year later, two Reading professors, **D. E. Daykin** and **M. J. Baines** proved this weaker conjecture. Mike Baines tells me they collected £2.5 each.

In 1971, **Vašek Chvátal** proved the full Newman conjecture for $n \leq 1000$.



D. J. Newman

Vašek Chvátal

In 1979 I attended a conference in Carbondale, Illinois, meeting **John Selfridge** who told me about Newman's conjecture, and described an algorithm that, if correct, would give a coprime matching.

We worked on this for a few months, and ended up with a proof of Newman's conjecture, published in Mathematika in 1980.

John Selfridge

Fast forward 41 years to 2021: **Tom Bohman** and **Fei Peng** posted a paper to arXiv, proving the following:

Bohman, Peng: Suppose n is even and I, J are intervals of n consecutive integers contained in $[N]$. There is a positive constant c such that if $n > e^{c(\log\log N)^2}$ then there is coprime matching from I to J.

They used this result to prove a weak form of the "lonely runner conjecture" (more on this shortly). I was intrigued, having worked on this conjecture and coprime matchings, and I was able to improve this:

P: The same, but we only require that $n > c(\log N)^2$.

Tom Bohman

Fei Peng

The lonely runner conjecture: Suppose $v_1, \ldots, v_k$ are distinct positive integers. There is some real number t such that the fractional parts $\{v_1 t\}, \ldots, \{v_k t\}$ are all in $[1/(k+1), 1-1/(k+1)]$.

One thinks of k runners on a circular track of length 1, with the v_i being their velocities. The special time t here makes a $(k+1)$st runner with speed 0 lonely. This was proved for $k=4$ by **Tom Cusick** and me in 1984, for $k=5$ by **Bohman, Holzman, & Kleitman** in 2001, and $k=6$ by **Barajas & Serra** in 2008.

Terry Tao showed it in the general case when all velocities are $\leq 1.2k$ and the new results on coprime matchings show it holds when the velocities are $\leq (2-\epsilon)k$. The connection, shown by **Bohman, Peng**, is not at all obvious. (My result gets a slightly smaller ϵ than the **Bohman, Peng** result.)

Erdős and **Tao**

One can also ask for a coprime matching from $\{1, 2, \ldots, n\}$ to itself. These would be **coprime permutations**, and it is interesting to count them.

Let $C(n)$ denote the number of coprime permutations of $\{1, 2, \ldots, n\}$.

Here's a proof that $C(4) = 4$. It's an even–odd thing. The numbers 2, 4 must be sent to 1, 3 in some order, and vice versa.

I asked **Sergi Elizalde** if he knew anything about this problem. He computed the first few values and then checked OEIS, finding that **David Jackson** had computed $C(n)$ for $n \leq 24$ in 1977.

Jackson's view of the problem: Take the $n \times n$ matrix M where the i, j entry is 1 if $\gcd(i, j) = 1$ and is 0 otherwise (the adjacency matrix for the coprime graph on $[n]$). Then $C(n)$ is the **permanent** of M.

Using some of the techniques from my paper with **Selfridge** on Newman's conjecture, I was able to show that for all large n,

$$\frac{n!}{3.73^n} \leq C(n) \leq \frac{n!}{2.5^n}.$$

Then, **McNew** identified a constant c where it seemed likely that $C(n) = n!/(c + o(1))^n$, and then **Sah** and **Sawhney**, grad students at MIT at the time, proved it.

Ashwin Sah

Mehtaab Sawhney

Nathan McNew

MATCHABLE NUMBERS

NATHAN MCNEW AND CARL POMERANCE

ABSTRACT. We say a natural number n is matchable if there is a bijection from the set of $\tau(n)$ divisors of n to the set $\{1, 2, \ldots, \tau(n)\}$, where corresponding numbers are relatively prime. We show that the set of matchable numbers has an asymptotic density, which we compute, and we show that every squarefree number is matchable. We also present some related unsolved problems.

(Just appeared in Mathematika)

Let $\tau(n)$ denote the divisor function (it returns the number of divisors of n). Is there a **coprime matching** between the set of $\tau(n)$ divisors of n and $\{1, 2, \ldots, \tau(n)\}$? If so, we say n is matchable. This concept was introduced by Bernardo Recamán on mathoverflow in 2022. He asked if more numbers are matchable than not.

For example, 6 is matchable:

$$1 \iff 4, \quad 2 \iff 3, \quad 3 \iff 2, \quad 6 \iff 1.$$

And 8 is not matchable: The divisors are $\{1, 2, 4, 8\}$ which contains 3 even numbers, but $\{1, 2, 3, 4\}$ contains only 2 odd numbers. Similarly all the multiples of 4, starting with 8, are not matchable.

If m is composite, then $27m$ is not matchable. (Use a similar argument with multiples of 3: there are too many of them among the divisors of $27m$.)

So the upper density of the set of matchable numbers is $\leq (1-2^{-2})(1-3^{-3})$.

This generalizes to all primes, and we get the upper density of the matchable numbers is at most

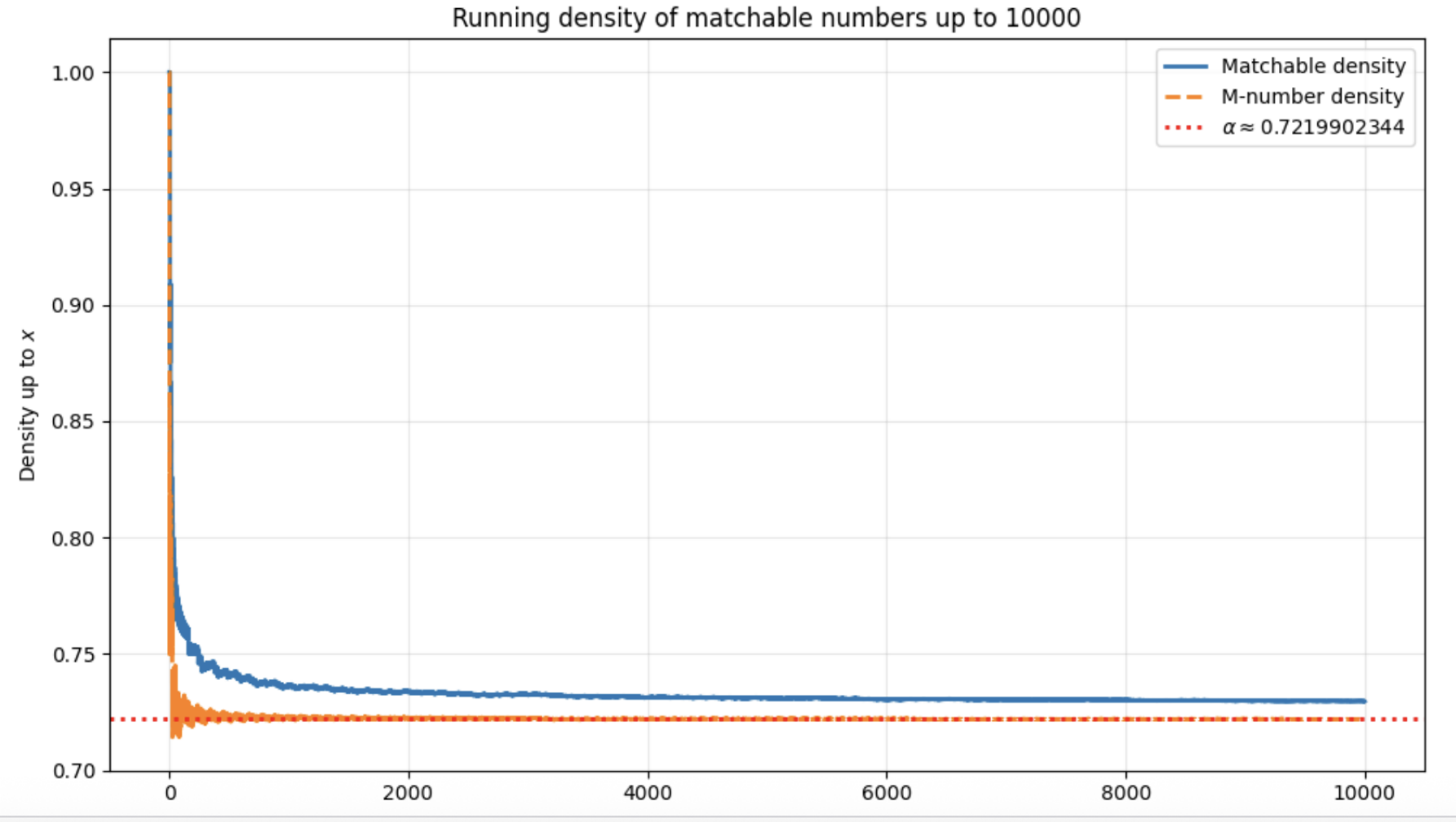
$$\alpha := \prod_p (1-p^{-p}) = 0.72199023441955\ldots.$$

Theorem. (**McNew** and **Pomerance**) The asymptotic density of the set of matchable numbers exists and is α.

On the way to proving this we show that every squarefree number is matchable.

Say an integer n is an **M-number** if it is not divisible by any p^p with p prime. So the asymptotic density of the set of M-numbers is α. We prove the theorem by showing but for a possible exceptional set of density 0, every M-number is matchable, and that the set of non-M-numbers that are matchable (like 4) has asymptotic density 0. We conjecture that the first of these density-0 sets is empty.

Say n is **strongly matchable** if there is a coprime matching between the $\tau(n)$ divisors of n and every coprime arithmetic progression of length $\tau(n)$. For example, though 4 is matchable, it is not strongly matchable: Try matching $\{1,2,4\}$ with $\{2,3,4\}$. Every strongly matchable number is an M-number, and we conjecture the converse. We are able to prove that a positive proportion of the M-numbers are strongly matchable.



Graph courtesy of Scott Kominers

Now for the opposite of the coprime problem . . .

The **divisor graph** on $\mathbb{N}$, where there is an edge from a to b when $a \neq b$ and $a \mid b$ or $b \mid a$, is complementary to the coprime graph, and there are many attractive problems here as well.

For example, how long is the longest chain in the divisor graph restricted to $[n]$? This has been worked on by me, **Pollington**, **Saias**, **Tenenbaum**, **Weingartner**, and **McNew**.

Another example: How many permutations σ of $[n]$ have the property that for each $j \in [n]$, $j \mid \sigma(j)$ or $\sigma(j) \mid j$? See papers of me and **McNew**.

Erdős and **P** (1980): Let $f(n)$ be the least number such that in every interval $[m+1, m+f(n)]$ there are distinct integers $a_1, a_2, \ldots, a_n$ with $i \mid a_i$ for $i = 1, 2, \ldots, n$.

The problem of estimating $f(n)$ has seen a flurry of activity in the last few days, some of it even by humans (in separate papers): **van Doorn**, **Kominers**, **Lebowitz-Lockard**, and **Chen**.

Even more!

We can make a digraph out of the divisor graph on $[n]$ by having $a \to b$ when $b \mid a$. Say these arrows are independently reversed with probability $\rho \in (0,1)$. Very recently **Jihyung Kim** and **Tristan Phillips** showed that the expected size of the largest strongly connected component has size $\sim n$ as $n \to \infty$.

There is also the concept of an anti-coprime permutation. This is a permutation σ of $\{2,3,\ldots,n\}$ where each $\gcd(j,\sigma(j)) > 1$. I have a result about how many of these there are, and it was also brought up in the talk of **Fan Chung Graham**.

Thank You!

And Happy Birthday Peter!