Sets of monotonicity for Euler's function

Carl Pomerance, Dartmouth College
Hanover, New Hampshire, USA

West Coast Number Theory Conference
December 16–20, 2011

Based on joint work with
Paul Pollack & Enrique Treviño

Euler's φ-function is quite chaotic:

n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$
1	1	11	10	21	12	31	30
2	1	12	4	22	10	32	16
3	2	13	12	23	22	33	20
4	2	14	6	24	8	34	16
5	4	15	8	25	20	35	24
6	2	16	8	26	12	36	12
7	6	17	16	27	18	37	36
8	4	18	6	28	12	38	18
9	6	19	18	29	28	39	24
10	4	20	8	30	8	40	16

But already we can see some patterns:

n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$
1	1	11	**10**	21	12	31	**30**
2	**1**	12	4	22	10	32	16
3	**2**	13	**12**	23	**22**	33	20
4	2	14	6	24	8	34	16
5	**4**	15	8	25	20	35	24
6	2	16	8	26	12	36	12
7	**6**	17	**16**	27	18	37	**36**
8	4	18	6	28	12	38	18
9	6	19	**18**	29	**28**	39	24
10	4	20	8	30	8	40	16

The φ-function is monotone on the primes.

And it is trivially monotone where it is constant:

n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$
1	1	11	10	21	**12**	31	30
2	1	12	4	22	10	32	16
3	2	13	**12**	23	22	33	20
4	2	14	6	24	**8**	34	16
5	4	15	**8**	25	20	35	24
6	2	16	**8**	26	**12**	36	**12**
7	6	17	16	27	18	37	36
8	4	18	6	28	**12**	38	18
9	6	19	18	29	28	39	24
10	4	20	**8**	30	**8**	40	16

Define $M^{\uparrow}(n)$ to be the size of the largest subset S of $\{1, 2, \ldots, n\}$ on which φ is nondecreasing.

Similarly define $M^{\downarrow}(n)$ to be the size of the largest subset S of $\{1, 2, \ldots, n\}$ on which φ is nonincreasing.

Then $M^{\downarrow}(40) = 6$, since we can take the five 8's, which start at $\varphi(15) = 8$, preceded either by $\varphi(11) = 10$ or $\varphi(13) = 12$.

Another down sequence has n starting with 4 prime powers: $23, 25, 27, 32, 34, 40$. And it can be continued to a 7th term at 42.

We can do better than the 12 primes for $M^{\uparrow}(40) \ldots$

In fact $M^{\uparrow}(40) = 19$:

n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$	n	$\varphi(n)$
1	**1**	11	10	21	**12**	31	30
2	**1**	12	**4**	22	10	32	**16**
3	**2**	13	12	23	22	33	**20**
4	**2**	14	**6**	24	8	34	16
5	**4**	15	**8**	25	20	35	**24**
6	2	16	**8**	26	**12**	36	12
7	6	17	16	27	18	37	**36**
8	**4**	18	6	28	**12**	38	18
9	6	19	18	29	28	39	24
10	**4**	20	**8**	30	8	40	16

Two years ago at this conference, I posed several problems:

Do we have $M^{\uparrow}(n) = o(n)$ as $n \to \infty$?

Do we have $M^{\uparrow}(n) - \pi(n)$ unbounded?

Do we have $M^{\downarrow}(n) = o(n)$ as $n \to \infty$?

Let $C(n)$ denote the size of the largest subset of $\{1, 2, \ldots, n\}$ on which φ is constant. We know that $C(n) > n^{0.7}$ for all large n, and $C(n) > n^{1-\epsilon}$ is conjectured. Here's a new question:

Do we have $M^{\downarrow}(n) - C(n)$ unbounded?

We can answer 3 of these 4 questions:

Pollack, P, & Treviño (2011): For each fixed $\epsilon > 0$ and n sufficiently large, $M^{\uparrow}(n) < n/(\log n)^{1-\epsilon}$.

Pollack, P, & Treviño (2011): For each fixed $\epsilon > 0$ and n sufficiently large, $M^{\downarrow}(n) < n/\exp\left((\frac{1}{2} - \epsilon)\sqrt{\log n \log\log n}\right)$.

Pollack, P, & Treviño (2011): For n sufficiently large, $M^{\downarrow}(n) - C(n) > n^{0.13}$.

Still unsolved:

Is $M^{\uparrow}(n) - \pi(n)$ unbounded?

We have some numerical evidence that perhaps for all large n we have $M^{\uparrow}(n) - \pi(n) = 64$. It is true for $n = 31{,}957$ and it continues to hold for all larger values up to 10,000,000.

A related problem which may be easier: Is there is a constant c such that if φ is monotone on $S \subset [1, n]$, then

$$\sum_{n \in S} \frac{1}{n} \leq \log\log n + c\,?$$

We cannot do this one either.

Some ideas involved in the proofs.

Let $W(n)$ denote the size of the set $\{\varphi(1), \varphi(2), \ldots, \varphi(n)\}$. That is, $W(n)$ is the number of distinct values of φ restricted to $[1, n]$. It has been known since Erdős in 1935 that $W(n) \le n/(\log n)^{1-\epsilon}$ for n sufficiently large. Thus, if $M^{\uparrow}(n)$ is considerably bigger than $W(n)$, there would be many solution pairs a, k to the equation

$$\varphi(a) = \varphi(a+k), \quad \text{with} \quad a+k \le n, \quad k < \log n. \tag{1}$$

In a paper from 1999 of Graham, Holt, & P it is shown that $\varphi(a) = \varphi(a+k)$ has few solutions in $[1, n]$ for fixed k. What we had to do was to make this result uniform in k up to $\log n$ so that we could bound the number of solutions in (1).

This proof works for $M^{\downarrow}(n)$ as well, but we can do better by a simpler argument: Get a good estimate for the number of integers in a down sequence with a given number of prime factors, starting from the observation that the case of 1 prime factor is very easy, and using induction. (E.g., note that the down sequence $23, 25, 27, 32$ of prime powers has increasing exponents.)

We remark that while sets where φ is constant also count as monotone nondecreasing, they don't compete with the primes. We know from work of Erdős, as improved by P, that

$$C(n) \le n^{1-(1-\epsilon)\log\log\log n/\log\log n},$$

so that $C(n)$ is tiny compared to $\pi(n)$. Though we can prove that $M^{\downarrow}(n) - C(n)$ tends to infinity, it may be that $M^{\downarrow}(n) \sim C(n)$ as $n \to \infty$.

I close with one final result from our paper: The maximum size of a set of *consecutive* integers in $[1, n]$ for which φ is nondecreasing is

$$\frac{\log_3 n}{\log_6 n} + (c + o(1))\frac{\log_3 n}{(\log_6 n)^2}$$

as $n \to \infty$, where $\log_k$ is the k-fold iterated logarithm, and $c = 0.0028428289\ldots$ is a constant. The same holds for nonincreasing.

The proof borrows from a similar result of Erdős. The details are in the paper, plus several other results and problems.

Pollack, P, & Treviño, *Sets of monotonicity for Euler's totient function,* submitted for publication, available at:
`www.math.dartmouth.edu/~carlp`

Thank You!