# A BOUND FOR THE MĄKOWSKI–SCHINZEL CONJECTURE CONCERNING THE ARITHMETIC FUNCTIONS $\sigma$ AND $\varphi$

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Abstract. With $\sigma$ the sum-of-divisors function and $\varphi$ Euler's function, it was conjectured by Mąkowski and Schinzel that we always have $\sigma(\varphi(n)) \geq n/2$. We show that the conjecture holds for every number $n$ divisible only by primes at most $10^6$, and also every $n \leq 10^{60}$.

*In memory of Florian Luca, 1969–2026*

## 1. Introduction

Two of the most common number theoretic functions are $\varphi$ and $\sigma$, Euler's function and the sum-of-divisors function, respectively. In some sense one behaves inversely to the other with $\sigma(n)/n$ the same order of magnitude as $n/\varphi(n)$. In fact, we always have

$$\frac{6n}{\pi^2\varphi(n)} < \frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)}.$$

The Mąkowski–Schinzel conjecture [7] from 1964 asserts that for every natural number $n$ we have

$$\alpha(n) := \sigma(\varphi(n))/n \geq 1/2.$$

In this paper we show that the conjecture holds for all $n \leq 10^{60}$.

**Theorem 1.** *If $n \leq 10^{60}$, then $\alpha(n) \geq 1/2$.*

Let $P(n)$ denote the largest prime factor of $n$ (and let $P(1) = 1$). We also show the following result.

**Theorem 2.** *If $P(n) \leq 10^6$, then $\alpha(n) \geq 1/2$.*

Theorem 2 improves a result of Cohen [3] that the Mąkowski–Schinzel conjecture holds for those $n$ with $P(n) < 1780$.

It was shown in 1989, see [8], that there is a positive constant $\kappa$ such that $\alpha(n) \geq \kappa$ for all $n$, and in 2002 Ford [4] showed this with $\kappa = 1/39.4$.

We know that $\alpha(n)$ can be large, see [1, 2]. In fact in [6] it is shown that $\alpha(n) \to \infty$ on a set of integers $n$ of asymptotic density 1. Previously, in [5], it was shown that $\alpha(n) \geq 1/2$ on a set of positive density.

Let $\mathrm{rad}(n)$ denote the largest squarefree divisor of $n$. For every $n$ we have $\alpha(\mathrm{rad}(n)) \leq \alpha(n)$. In particular, it suffices for our theorem to consider only squarefree numbers. Some further notation: We let $v_p(n)$ denote the number of factors of the prime $p$ in $n$ and we let $\omega(n)$ denote the number of primes $p$ with $v_p(n) > 0$.

*Date*: November 22, 2026.

2010 *Mathematics Subject Classification.* 11A05, 11N45.

## 2. Preliminaries

**Lemma 1.** *Suppose $p = 2$ or $p$ is a Fermat prime and let $n$ be a squarefree integer with $p \nmid n$ and $p - 1 \mid \varphi(n)$. If $n$ has a prime factor $q > 2p$, then $\alpha(pn/q) < \alpha(n)$.*

*Proof.* Let $k = v_2(\varphi(n))$, $j = v_2(p-1)$ (so that $p = 2^j + 1$) and $i = v_2(q-1)$. Further, let $m, m'$ be such that $\varphi(n) = 2^k m$ and $\varphi(pn/q) = 2^{k+j-i}m'$. Note that $m, m'$ are odd and that $m' \mid m$. The latter fact implies that $\sigma(m)/m \geq \sigma(m')/m'$. We have

$$\alpha(pn/q) = \frac{\sigma(\varphi(pn/q))}{\varphi(pn/q)} \cdot \frac{\varphi(pn/q)}{pn/q} = \frac{2^{k+j-i+1}-1}{2^{k+j-i}} \cdot \frac{\sigma(m')}{m'} \cdot \frac{1-1/p}{1-1/q} \cdot \frac{\varphi(n)}{n}$$

and

$$\alpha(n) = \frac{2^{k+1}-1}{2^k} \cdot \frac{\sigma(m)}{m} \cdot \frac{\varphi(n)}{n},$$

so that

$$\frac{\alpha(pn/q)}{\alpha(n)} = \frac{2^{k+j-i+1}-1}{2^{j-i}(2^{k+1}-1)} \cdot \frac{\sigma(m')/m'}{\sigma(m)/m} \cdot \frac{1-1/p}{1-1/q} < \frac{2^{k+1}}{2^{k+1}-1} \cdot \frac{1-1/p}{1-1/q}.$$

We deduce that

$$\frac{\alpha(pn/q)}{\alpha(n)} < \left(1 + \frac{1}{2^{k+1}-1}\right)\left(1 - \frac{1}{2p-1}\right),$$

using the hypothesis that $q > 2p$. Since $p - 1 \mid \varphi(n)$ we have $p \leq 2^k$, so we have $\alpha(pn/q)/\alpha(n) < 1$, which is what we wanted to show. □

For an odd prime $r$ and a positive integer $n$, let $n_r$ denote the largest divisor of $n$ supported on primes that are $1 \pmod r$.

**Lemma 2.** *Let $r$ be an odd prime. Let $n$ be a squarefree positive integer with $n_r > 1$ and let $k = \omega(n_r)$. If*

$$\sigma(r^k)/r^k \geq n_r/\varphi(n_r), \tag{1}$$

*then $\alpha(n/n_r) < \alpha(n)$.*

*Proof.* Replacing $n$ with $n/n_r$ removes all of the factors of $r$ in $\varphi(n)$ and it removes some other prime factors as well. Since $k \leq v_r(\varphi(n))$, we have

$$\frac{\sigma(\varphi(n))}{\varphi(n)} > \frac{\sigma(r^k)}{r^k} \cdot \frac{\sigma(\varphi(n/n_r))}{\varphi(n/n_r)}.$$

Further,

$$\varphi(n)/n = \frac{\varphi(n_r)}{n_r} \cdot \frac{\varphi(n/n_r)}{n/n_r},$$

so that multiplying these two displays we get that

$$\alpha(n) > \frac{\sigma(r^k)}{r^k} \cdot \frac{\varphi(n_r)}{n_r} \cdot \alpha(n/n_r).$$

Our claim follows. □

We remark that a similar argument appears in Cohen [3].

We say a positive integer $n$ is *primitive* if $\alpha(n) < 1/2$ and for all proper divisors $d$ of $n$, we have $\alpha(d) \geq 1/2$. Note that if $n$ is primitive, then (1) does *not* hold.

**Corollary 1.** *Let $r$ be an odd prime and let $q_1 < q_2 < \dots$ be the sequence of primes $q_i \equiv 1 \pmod r$. Suppose that $n$ is primitive and that $r \mid \varphi(n)$. If $k$ is such that*

$$\frac{\sigma(r^j)}{r^j} \prod_{i \le j} \frac{q_i - 1}{q_i} \ge 1$$

*for $j = 1, 2, \dots, k$, then $\omega(n_r) \ge k+1$ and so $n_r \ge q_1 q_2 \dots q_{k+1}$.*

*Proof.* Let $n, r$ be as given. Suppose that $j = \omega(n_r) \le k$. Then

$$\frac{\sigma(r^j)}{r^j} \frac{\varphi(n_r)}{n_r} \ \ge \ \frac{\sigma(r^j)}{r^j} \left(1 - \frac{1}{q_1}\right) \dots \left(1 - \frac{1}{q_j}\right).$$

If the latter expression is $\ge 1$, so is the former expression, contradicting Lemma 2 and that $n$ is primitive. Thus, the hypothesis of the Corollary implies that $\omega(n_r) \ge k+1$ and the conclusion follows. □

**Corollary 2.** *Suppose that $n$ is primitive and $P(n) \le 10^6$. For each odd prime $r \le 10^6$, let $Q(r) = \{q_1 < q_2 < \dots < q_k\}$ denote the set of primes $q \le 10^6$ with $q \equiv 1 \pmod r$. Suppose that*

$$\frac{r - r^{-j}}{r-1} \prod_{i \le j} \left(1 - \frac{1}{q_i}\right) \ge 1$$

*for each $j \le k$. Then $r \nmid \varphi(n)$.*

*Proof.* Using that $\sigma(r^j)/r^j = (r - r^{-j})/(r-1)$, this follows immediately from Corollary 1 with $k = \#Q(r)$. □

## 3. A large prime must appear

In this section we prove Theorem 2. It suffices to show that if $P(n) \le 10^6$, then $n$ is not primitive. We suppose that $n$ is primitive and that $P(n) \le 10^6$, and will derive a contradiction.

Using only Corollary 2, we have verified that $\varphi(n)$ must be supported on the primes in the set

$$\begin{aligned} S_0 = \{&2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 37, 41, 43, 47, 53, 73, 83, 89, 97, 113, 131, 173, 179, 191, 233, \\ &239, 251, 277, 281, 293, 431, 443, 491, 509, 593, 641, 653, 659, 719, 761, 809, 1013, 1103, \\ &1229, 1289, 1409, 1451, 1559, 1811, 1931, 1973, 2741, 2753, 3593, 3761, 4391, 5231, 6101\}. \end{aligned}$$

Using that no prime $q \mid n$ can be $\equiv 1 \pmod r$ for the various small primes $r$ *not* in $S_0$ (such as, 31, 59, 61, 67, 71, etc.) and running the check again, we are left with a somewhat smaller set. This process can be continued until it stabilizes, with this set:

$$\begin{aligned} S_1 = \{&2, 3, 5, 7, 11, 13, 17, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 233, 239, 293, 443, 509, \\ &761, 1013, 1289, 1811, 1973, 2741, 2753, 3593, 3761, 4391, 5231\}. \end{aligned}$$

We now use a new filter, namely, corresponding to each odd prime $r \in S_1$, we only consider primes $q \mid n$ with $q \equiv 1 \pmod r$ and $P((q-1)/r) \in S_1$, and then iterate this procedure arriving at smaller and smaller sets of primes. The process comes to a halt at

$$S_2 = \{2, 3, 5, 11, 131\}.$$

That is, we have that if $n$ exists, then $\varphi(n)$ is supported on the primes in $S_2$.

We complete the argument as follows. First we find all of the primes $q \le 10^6$ with $q \equiv 1 \pmod{11}$ with $q-1$ supported on $S_2$, but $q \not\equiv 1 \pmod 5$. Using Corollary 2 we find that if $11 \mid \varphi(n)$, then also $5 \mid \varphi(n)$. Namely, if these primes are $q_1 < q_2 < \cdots < q_m$, then

$$\frac{11 - 11^{-k}}{10} \prod_{j \le k} \left(1 - \frac{1}{q_j}\right) > 1$$

for all $k \le m$. As in Corollary 2, $n$ is not primitive since $\alpha(n/(q_1 \dots q_m)) < \alpha(n)$. Thus, if $11 \mid \varphi(n)$, then also $5 \mid \varphi(n)$.

Similarly, we find that if $5 \mid \varphi(n)$, then $11 \mid \varphi(n)$. If neither 5 nor 11 divide $\varphi(n)$, we find that $3 \nmid \varphi(n)$ and also $131 \nmid \varphi(n)$, so that $\varphi(n)$ is a power of 2. It is easy to see, and was shown in [3], that such a number $n$ is not a counter-example. We conclude that both 5 and 11 divide $\varphi(n)$. If 3 also divides $\varphi(n)$ we find that $n$ is not primitive. That is, letting $u$ be the product of the primes $q \le 10^6$ with $q \mid n$, $(q-1, 3 \cdot 5 \cdot 11) > 1$ and $q-1$ supported on $S_2$, then $\alpha(n/u) < \alpha(n)$ as in Corollary 2. Thus, we have $3 \nmid \varphi(n)$. However as with the 5 and 11 argument, we have that if $5 \mid \varphi(n)$, then also $3 \mid \varphi(n)$ a contradiction. The only possibility left is that $\varphi(n)$ is a power of 2. However, as shown in [3], no such $n$ is a counter-example to the Mąkowski–Schinzel conjecture. This completes the proof of Theorem 2.

## 4. No small counter-examples

In this section we prove Theorem 1. First, we note a corollary of the bound $10^6$ for $P(n)$.

**Corollary 3.** *If $n$ is the least number with $\alpha(n) < 1/2$, then $n$ is divisible by $2^{33} - 2$.*

*Proof.* Note that $2^{33} - 2 = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$ is the product of all of the known primes $p$ with $p - 1$ a power of 2. Since $P(n) > 10^6$, we have $\varphi(n)$ even. Lemma 1 first implies that $2 \cdot 3 \mid n$. Since $P(n) > 10^6$, we then have $4 \mid \varphi(n)$, so that $5 \mid n$. Then $16 \mid \varphi(n)$, so we must have $17 \mid n$. We continue to 257 and 65537. The process stops here since $2^{32} + 1$ is composite. □

**Proposition 1.** *If $n$ is the least number with $\alpha(n) < 1/2$ and $r$ is a prime factor of $\varphi(n)$ then either $2r + 1$ is prime or $n > 10^{60}$.*

*Proof.* Suppose that $n, r$ are as given and $2r + 1$ is composite. We first consider some small cases. For $r = 7$, note that the primes $q \equiv 1 \pmod 7$ with $q < 1000$ are

$$29, 43, 71, 113, 127, 197, 211, 239, 281, 337, 379, 421, 449, 463, 491, 547, 617, 631, 659, 673, 701,$$
$$743, 757, 827, 883, 911, 953, 967.$$

By the criterion in Corollary 1, we have $n_r$ greater than the product $M$ of these primes. Using $P(n) > 10^6$ from Section 3 and that $2^{33} - 2 \mid n$ from Corollary 3, we have $n > (2^{33} - 2)10^6 M > 10^{80}$. A similar ploy works with the first 30 primes that are 1 (mod 13), the first 30 primes that are 1 (mod 17), and the first 25 primes that are 1 (mod 19). For the remaining primes $r \le 79$ with $2r + 1$ composite we consider the first 20 primes $\equiv 1 \pmod r$ and verify in each case that the proposition holds for $n$. In each of these cases we have the stronger assertion that either $n$ does not exist or $n > 10^{80}$. We thus may assume that $r \ge 97$.

We first consider the case when $r \equiv 1 \pmod 3$. Let $c_1, c_2, \dots$ be the sequence of positive integers $\equiv 0, 4 \pmod 6$. If $n_r$ has the prime factorization $q_1 q_2 \dots q_k$ with $q_1 < q_2 < \cdots < q_k$,

then $q_i \geq c_i r + 1$ for $i \leq k$. Let

$$f_j(r) = \prod_{i \leq j} \left(1 - \frac{1}{c_i r + 1}\right) \cdot \frac{\sigma(r^j)}{r^j} = \prod_{i \leq j} \left(1 - \frac{1}{c_i r + 1}\right) \cdot \frac{r - r^{-j}}{r - 1}.$$

A computer calculation shows that $f_j(r) > 1$ for all $j \leq 14$ and all $r \geq 7$, so that if $r \mid \varphi(n)$ and $r \equiv 1 \pmod 3$, Corollary 1 implies that $\omega(n_r) \geq 15$. Using $P(n) > 10^6$ and that $2^{33} - 2 \mid n$ we have

$$n > (2^{33} - 2)(10^6) \prod_{i \leq 14} (c_i r + 1) > 6.49 \times 10^{33} r^{14} > 10^{60}$$

when $r \equiv 1 \pmod 3$ and $r \geq 97$.

We next assume that $r \equiv 2 \pmod 3$, $r \geq 101$, and that $2r + 1$ is composite. Now the sequence $c_1, c_2, \ldots$ comprises the numbers $c \equiv 0, 2 \pmod 6$, starting with 6. These $c$'s are greater than the corresponding $c$'s in the prior case, so the various inequalities shown continue to hold. □

**Proposition 2.** *Suppose that $n$ is the least number with $\alpha(n) < 1/2$ and $r > 3$ is a prime factor of $\varphi(n)$ with $2r + 1$ prime. Then either $n > 10^{60}$ or both $6r + 1$, $8r + 1$ are prime.*

*Proof.* Since $2r+1$ is prime we have $r \equiv 2 \pmod 3$, so that $cr+1$ is composite when $c \not\equiv 0, 2 \pmod 6$. Assume that $8r+1$ is composite. Our $c$-sequence is $2, 6, 12, 14, \ldots$. However, since we may assume via Proposition 1 that every prime $r' \mid \varphi(n)$ has $2r'+1$ prime, our $c$-sequence is

$$2, 6, 12, 18, 20, 24, 30, 32, 36, 44, 48, 50, 54, 60, 66, 72, 78, 80, 90, \ldots.$$

A further reduction can be made since if $cr + 1$ is prime it is not divisible by 5, so we can remove one nonzero residue class mod 5. The least favorable cases are 3, 4 (mod 5), so it suffices to deal with the sequence

$$2, 6, 12, 20, 24, 30, 32, 36, 44, 50, 54, 60, 66, 72, 80, 90, \ldots$$

and the sequence

$$2, 6, 12, 18, 20, 30, 32, 36, 48, 50, 60, 66, 72, 78, 80, 90, \ldots$$

With the first sequence we can take the first 11 terms and see that $n > 10^{60}$ when $r > 1000$, and with the second sequence we can take the first 10 terms and see that $n > 10^{60}$ when $r > 3200$. The values of $r$ in $(3, 3200)$ not covered (namely, $r$ and $2r+1$ are prime and $8r+1$ is composite) are

$$23, 41, 53, 83, 89, 113, 173, 191, 233, 251, 281, 293, 359, 419, 443, 593, 641, 653, 683, 719, 743,$$
$$911, 953, 1013, 1019, 1031, 1049, 1103, 1223, 1439, 1451, 1481, 1499, 1511, 1583, 1733, 1889,$$
$$1901, 1931, 1973, 2003, 2039, 2063, 2141, 2273, 2351, 2393, 2399, 2459, 2543, 2693, 2699, 2753,$$
$$2819, 2903, 2939, 2963, 3023.$$

For each $r$ in this list we compute the first $k$ primes $q$ with $q \equiv 1 \pmod r$ and with $q - 1$ not divisible by 7, 13, 17, and 19. Here $k = 16$ for $r = 23$, $k = 14$ for $r = 41$, $k = 13$ for the next 4 primes, $k = 12$ for the next 6 primes, $k = 11$ for the next 12 primes, and $k = 10$ for the remaining primes. In each case we get a strict lower bound for $n_r$, and the bound times $(2^{33} - 2)10^6$ exceeds $10^{60}$.

So now assume that $6r+1$ is composite. Our two choices for the $c$-sequence are the same as above, but with 8 replacing 6. The least choice for $r$ is 29, and similar calculations lead to acceptable lower bounds for $n_r$ and $n$. □

**Proposition 3.** *Let $n$ be the least number with $\alpha(n) < 1/2$. Then either $n > 10^{60}$ or $\varphi(n)$ is supported on the primes in*

$$\{2, 3, 5, 11, 131, 761\}. \tag{2}$$

*Proof.* Assume that $r > 3$ and $r$ is a prime factor of $\varphi(n)$. As in the previous propositions, we look at the $c$-sequence for $r$, which consists of numbers $c$ with $cr+1$ possibly a prime factor of $n$. We also remove any $c$ divisible by a prime $r'$ dealt with in Proposition 1, giving us

$$2, 6, 8, 12, 18, 20, 24, 30, 32, 36, 44, 48, 50, 54, 60, 66, 72, 80, 90, \ldots.$$

The least favorable residue class mod 5 to remove is 4, and then the first 7 elements are $2, 6, 8, 12, 18, 20, 30$, labeled $c_1, \ldots, c_7$. Let

$$g_j(r) = \prod_{i \le j} \left(1 - \frac{1}{c_i r + 1}\right) \frac{r - r^{-j}}{r - 1}.$$

We see that $g_j(r) > 1$ for $j \le 6$ and all $r \ge 5$, so that $\omega(n_r) \ge 7$ and

$$n > (2^{33} - 2)10^6 \prod_{j \le 6} (c_j r + 1) > 3.56 \times 10^{21} r^6.$$

Thus, we may assume that $r < 2.56 \times 10^6$. There are over 200 primes $r$ less than this bound with $2r+1, 6r+1, 8r+1$ all prime, so we consider the two sub-cases: $12r+1$ prime and $12r+1$ composite. In the second case we change the $c$-sequence to $2, 6, 8, 18, 20, 30, 32, 36$ and alter $g_j(r)$ accordingly. We find that we may assume in this case that $r < 24{,}000$. Together with the 2,6,8,12 primes, the remaining values of $r > 3$ to consider are in the set

$$\{5, 11, 131, 761, 1811, 2741, 3761, 4391, 5231, 6101, 13001, 19301, 21011, 22481, 42611, 70061,$$
$$112691, 196331, 198221, 224831, 305741, 399131, 435881, 521831, 542831, 565661, 649661,$$
$$691661, 937571, 1174451, 1206581, 1235831, 1321031, 1401461, 1581071, 1952921, 2102651,$$
$$2132891, 2230121, 2380271, 2529161\}.$$

The primes $r \ge 196331$ here all have $\omega(n_r) > 7$ if $r \mid \varphi(n)$, and in each case the product of the least 8 primes $q \equiv 1 \pmod r$ (and $q-1$ not divisible by a prime $r'$ with $2r'+1$ composite), multiplied by $2^{33} - 2$, exceeds $10^{60}$. The next largest 6 primes $r$ have $\omega(n_r) > 8$, with $(2^{33} - 2)n_r > 10^{60}$. The next 4 largest primes $r$ have $\omega(n_r) > 9$ and $(2^{33} - 2)n_r > 10^{60}$. The next 3 largest primes $r$ have $\omega(n_r) > 10$ and $(2^{33} - 2)n_r > 10^{60}$.

Thus, if $r$ is a prime factor of $\varphi(n)$ and $r > 3$, then $r \in \{5, 11, 131, 761\}$. This completes the proof of the Proposition. □

**Proposition 4.** *If $n$ is the least number with $\alpha(n) < 1/2$ and $\varphi(n)$ is divisible by an odd prime $r$ in the set in (2), then $n > 10^{60}$.*

*Proof.* For each $r$ we consider the case that $r = P(\varphi(n))$. If $r = 761$ we compute that $\omega(n_r) > 16$ and that $(2^{33} - 2)n_r > 10^{89}$. For $r = 131$, we find that $\omega(n_r) > 18$ and that $(2^{33} - 2)n_r > 10^{83}$. With $r = 11$ we use that if $q \mid n$ then $P(q-1) \le 11$ and $7 \nmid q - 1$. In this case we find that $\omega(n_r) > 24$ and that $(2^{33} - 2)n_r > 10^{84}$. The cases for 5 and 3 are huge wins; we find in each case that $\omega(n_r) > 60$. □

We remark that with a little extra work one can show that if $\mathrm{rad}(\varphi(n)) \mid 30$, then $\alpha(n) \geq 1/2$. That is, the Mąkowski–Schinzel conjecture holds for those $n$ with $P(\varphi(n)) \leq 5$. It is likely that this type of result can be further extended.

We now summarize what we have proved. The various Propositions show that if $n$ is the least counter-example to the Mąkowski–Schinzel conjecture, then either $n > 10^{60}$ or $\varphi(n)$ is a power of 2. Using the result from [3] that if $\varphi(m)$ is a power of 2 then $\alpha(m) \geq 1/2$ we have thus proved Theorem 1.

## Acknowledgments

I am grateful to Steve Fan for his interest and help with this paper. I also thank the anonymous referee for some helpful stylistic suggestions and for identifying some typos.

## Dedication

This paper is dedicated to the memory of Florian Luca who passed away at age 57 on July 28, 2026. The news was shocking and completely unexpected. Florian and I have 24 joint papers, and I am sure there would have been more. Our first joint paper is [6] and it is why I picked the topic for this note. Florian will be missed by his many friends the world over.

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