# PROOF OF D. J. NEWMAN'S COPRIME MAPPING CONJECTURE

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§1. *Introduction.* In this paper we prove

THEOREM 1. *If $N$ is a natural number and $I$ is an interval of $N$ consecutive integers, then there is a* 1–1 *correspondence* $f\colon \{1, 2, \ldots, N\} \to I$ *such that* $(i, f(i)) = 1$ *for* $1 \leqslant i \leqslant N$.

We call the function $f$ described in the theorem a *coprime mapping*. Theorem 1 settles in the affirmative a conjecture of D. J. Newman. The special case when $I = \{N+1, N+2, \ldots, 2N\}$ was proved by D. E. Daykin and M. J. Baines [**2**]. V. Chvátal [**1**] established Newman's conjecture for each $N \leqslant 1002$. We prove Theorem 1 constructively by giving an algorithm for the construction of a coprime mapping $f$. This algorithm will be discussed in §2.

If $u$ is a real number and $n$ is a natural number, let $D(u, n)$ denote the number of odd integers $l$, $1 \leqslant l \leqslant 2n-1$, with $\phi(l)/l \leqslant u$, where $\phi$ denotes Euler's function. If also $k$ is a natural number, let $E(k, n)$ denote the maximal number of integers coprime to $k$ that can be found in every set of $n$ consecutive integers. Thus, for example, $E(3, 4) = 2$, since in every set of 4 consecutive integers there are at least 2 integers coprime to 3 and the set $\{0, 1, 2, 3\}$ has exactly 2 integers coprime to 3. If $n > 1$, let $p_1(n)$ denote the largest prime not exceeding $2n-1$. We shall prove

THEOREM 2. *If $n$ is a natural number and $k$ is odd with $1 < k \leqslant 2n-1$ and $k \neq p_1(n)$, then $D(\phi(k)/k, n) < E(k, n)$.*

In §2 we shall show that Theorem 1 is a fairly simple corollary of Theorem 2. The remainder of the paper then will take up the proof of Theorem 2.

Note that the function $D(u, n)$ is related to the well-known distribution function for $\phi$:

$$D_\phi(u) = \lim_{x \to \infty} \frac{1}{x} \sum_{\substack{l \leqslant x \\ \phi(l)/l \leqslant u}} 1 .$$

In fact, if we let $D(u) = \lim_{n \to \infty} D(u, n)/n$, then it can be seen that, like $D_\phi(u)$, $D(u)$ exists for all $u$, $D(u)$ is strictly monotone on $[0, 1]$, $D(u)$ is continuous, and $D(u)$ is singular on $[0, 1]$. We have $D_\phi(u) = \big(D(u) + D(2u)\big)/2$.

For $k$ fixed it is easy to see that $E(k, n) \sim \big(\phi(k)/k\big)n$ as $n \to \infty$. We thus have the following consequence of Theorem 2.

COROLLARY. *For every $u \geqslant 0$, $D(u) \leqslant u$.*

[MATHEMATIKA, 27 (1980), 69–83]

We do not use computers in the proof, although many calculations were performed out of convenience on a hand calculator. However, the proof relies quite heavily on the computer-assisted results of Rosser and Schoenfeld [**4**].

We take this opportunity to thank Paul Erdős for calling our attention to Newman's conjecture. We should also remark that a key step in our proof—namely, the use of the sets $J_t(P)$ in §7—is suggested by an old argument of Erdős (Theorem 3 in [**3**]). We also wish to thank M. Mendes France for informing us of [**1**].

§2. *The algorithm.* In this section we shall inductively describe an algorithm for the construction of a coprime mapping $f$. The algorithm will make use of Theorem 2, which shall be assumed as true in this section. Thus this section will show that Theorem 2 implies Theorem 1.

If $N = 1$, then there is only one mapping $f: \{1\} \to I$, and certainly $f$ is a coprime mapping. If $N = 2$, then let $f(1)$ be the even member of $I$ and let $f(2)$ be the odd member of $I$. This $f$ is a coprime mapping.

Say now $N \geqslant 3$ and we have given algorithms for the construction of coprime mappings for all $M < N$. Let $I$ be a set of $N$ consecutive integers. We first describe $f$ at the odd members of $\{1, 2, ..., N\}$ and then use our induction hypothesis to describe $f$ at even arguments.

Let $n = N-[N/2]$ denote the number of odd numbers in $\{1, 2, ..., N\}$. Then $n \geqslant 2$. Label these odd numbers $k_1, k_2, ..., k_n$ where $\phi(k_i)/k_i \leqslant \phi(k_{i+1})/k_{i+1}$ for $1 \leqslant i < n$. Note that $k_{n-1} = p_1(n)$ and $k_n = 1$. We first describe how to choose $f(k_1), ..., f(k_{n-2})$.

Now $I$ has either $n$ or $n-1$ even numbers. If they are each divided by 2, then we obtain a string of consecutive integers. Thus for each $i$, there are at least $E(k_i, n-1)$ even integers in $I$ that are coprime to $k_i$.

By Theorem 2, if $n \geqslant 3$,

$$E(k_1, n-1) \geqslant E(k_1, n)-1 \geqslant D(\phi(k_1)/k_1, n) \geqslant 1\,,$$

so that there are even numbers in $I$ coprime to $k_1$. Let $f(k_1)$ be the least such number. Say $i \leqslant n-2$ and $f(k_1), ..., f(k_{i-1})$ have already been defined. By Theorem 2,

$$E(k_i, n-1) \geqslant E(k_i, n)-1 \geqslant D(\phi(k_i)/k_i, n) \geqslant i\,,$$

so that there are at least $i$ even numbers in $I$ coprime to $k_i$. Thus we may let $f(k_i)$ be the least even number in $I$ coprime to $k_i$ and not equal to $f(k_1), ..., f(k_{i-1})$.

We now define $f(k_{n-1})$. If $I$ has $n$ even numbers, there are 2 of them left which have not yet been used as a value of $f$. Moreover, $k_{n-1} = p_1(n)$ does not divide both of them since half of their difference is at most $n-1$, while by Bertrand's Postulate, $p_1(n) > n$. Thus we let $f(k_{n-1})$ be the least of the remaining even numbers of $I$ which is not divisible by $p_1(n)$. If $I$ has only $n-1$ even numbers, then both endpoints of $I$ are odd and not both are divisible by $p_1(n)$ (half of their difference is $n-1$). So we let $f(k_{n-1})$ be the least endpoint of $I$ which is not divisible by $p_1(n)$.

In either case, we have exactly one even number left in $I$ which has not yet been used as a value of $f$. We let $f(k_n)$ be this number.

Now we describe how to define $f$ at even arguments, using essentially the same idea as Theorem 2 in [**1**]. Let $J$ be the set of remaining members of $I$ not used already as values of $f$. Then $J$ is an interval of $[N/2]$ consecutive odd numbers. Let $m$ equal the product of the odd primes not exceeding $[N/2]$. Let $I' = \{(j+m)/2 : j \in J\}$. Then $I'$ is an interval of $[N/2]$ consecutive integers. By our induction hypothesis there is a construction for a coprime mapping $g : \{1, 2, \ldots, [N/2]\} \to I'$. If $1 \leqslant k \leqslant [N/2]$, let $f(2k) = 2g(k) - m$. We have thus described a 1–1 correspondence from $\{2, 4, \ldots, 2[N/2]\}$ to $J$ and for each $k$, $1 \leqslant k \leqslant [N/2]$,

$$(2k, f(2k)) = (2k, 2g(k) - m) = 1 .$$

This completes the construction of the coprime mapping $f$.

Before we proceed to the proof of Theorem 2, we describe a simpler algorithm for the construction of a coprime mapping. We do not have a proof that this simpler algorithm is always successful, but we feel that a proof probably could be provided using the methods of this paper. As in the above algorithm, we may assume $N \geqslant 3$. Relabel $\{1, \ldots, N\}$ as $b_1, \ldots, b_N$ where $\phi(b_i)/b_i \leqslant \phi(b_{i+1})/b_{i+1}$ for $1 \leqslant i < N$. If it is not the case that both of $I$'s endpoints are odd, inductively define $f(b_i)$ as the least number in $I$ coprime to $b_i$ and not equal to $f(b_1), \ldots, f(b_{i-1})$. If both of $I$'s endpoints are odd, first define $f(b_{N-1})$ as one of the endpoints, and then inductively define $f(b_i)$ for $i \neq N-1$ as the least number in $I - \{f(b_{N-1})\}$ coprime to $b_i$ and not equal to $f(b_1), \ldots, f(b_{i-1})$. The need for the special treatment of $b_{N-1}$ in the latter case can be seen from the example $N = 9$, $I = \{-1, 0, \ldots, 7\}$.

§3. *Preliminaries.* If $k$ is a natural number, let $\omega(k)$ denote the number of distinct prime factors of $k$.

LEMMA 1. *If $k > 1$, then*

$$E(k, n) > \frac{\phi(k)}{k} n - 2^{\omega(k)} + 1 .$$

*Proof.* If $k_0$ is the largest square-free divisor of $k$, then $E(k_0, n) = E(k, n)$, $\phi(k_0)/k_0 = \phi(k)/k$, and $\omega(k_0) = \omega(k)$. Thus we may assume $k$ itself is square-free. If $m$ is any integer and if $j > 1$ is an integer, then

$$\left| \sum_{\substack{m < i \leqslant m+n \\ j \mid i}} 1 - \frac{n}{j} \right| < 1 .$$

Thus

$$\begin{aligned} \sum_{\substack{m < i \leqslant m+n \\ (i,k) = 1}} 1 &= n + \sum_{\substack{j \mid k \\ j > 1}} (-1)^{\omega(j)} \sum_{\substack{m < i \leqslant m+n \\ j \mid i}} 1 \\ &> n + \sum_{\substack{j \mid k \\ j > 1}} (-1)^{\omega(j)} \frac{n}{j} - \sum_{\substack{j \mid k \\ j > 1}} 1 \\ &= \frac{\phi(k)}{k} n - 2^{\omega(k)} + 1 . \end{aligned}$$

LEMMA 2. *If $1 < k \leqslant 2n-1$ and if $p$ is the largest prime factor of $k$, then,*

$$E(k,n) > \left(\frac{\phi(k)}{k}\cdot\frac{p-2}{p-1}-\frac{1}{p}\right)n-1\,.$$

*Proof.* Write $k = p^i m$ where $p \nmid m$. Thus $pm < 2n$, so that

$$1-m/n > 1-2/p\,.$$

Thus

$$\begin{aligned} E(k,n) &\geqslant E(m,n)-([n/p]+1) \geqslant \phi(m)[n/m]-n/p-1 \\ &> \phi(m)(n/m-1)-n/p-1 = \frac{\phi(m)}{m}\left(1-\frac{m}{n}\right)n-\frac{n}{p}-1 \\ &> \frac{\phi(m)}{m}\left(1-\frac{2}{p}\right)n-\frac{n}{p}-1 = \left(\frac{\phi(k)}{k}\cdot\frac{p-2}{p-1}-\frac{1}{p}\right)n-1\,. \end{aligned}$$

LEMMA 3. *If $m > 1$ is odd, and $n$ is a positive integer,*

$$\left|\sum_{\substack{1 \leqslant k \leqslant 2n-1 \\ 2 \nmid k,\, m \mid k}} 1-\frac{n}{m}\right| \leqslant \frac{1}{2}-\frac{1}{2m}\,.$$

*Proof.* The quantity in the absolute value signs has its minimal value when $2n-1 = m-2$; the value is $-1/2+1/(2m)$. The maximal value is attained when $2n-1 = m$; the value is $1/2-1/(2m)$.

LEMMA 4. *If $m$ is odd, let*

$$F(m,n) = \sum_{\substack{1 \leqslant k \leqslant 2n-1 \\ 2 \nmid k,\, (m,k) > 1}} 1\,.$$

*Then*

$$F(m,n) < \bigl(1-\phi(m)/m\bigr)n+2^{\omega(m)-1}\,,$$

$$F(15,n) \leqslant \tfrac{7}{15}n+\tfrac{2}{3}\,, \qquad F(105,n) \leqslant \tfrac{19}{35}n+\tfrac{9}{7}\,.$$

*Proof.* The maximum value of $F(m,n)-\bigl(1-\phi(m)/m\bigr)n$ for all $n$ can be computed by evaluating this difference for any $m$ consecutive choices for $n$. This is how we established the latter two assertions. To see the first statement, note that by Lemma 3,

$$\begin{aligned} F(m,n) &= \sum_{j \mid m,\, j > 1} (-1)^{\omega(j)+1} \sum_{\substack{1 \leqslant k \leqslant 2n-1 \\ 2 \nmid k,\, j \mid k}} 1 \\ &\leqslant \sum_{j \mid m,\, j > 1} (-1)^{\omega(j)+1}\frac{n}{j} + \sum_{j \mid m,\, j > 1}\left(\frac{1}{2}-\frac{1}{2j}\right) \end{aligned}$$

$$< \left(1 - \frac{\phi(m)}{m}\right) n + \tfrac{1}{2}(2^{\omega(m)} - 1)$$

$$< \left(1 - \frac{\phi(m)}{m}\right) n + 2^{\omega(m)-1},$$

where, as in the proof of Lemma 1, we assume $m$ is square-free.

§4. *The proof of Theorem 2 for $n \leqslant 1000$.* We first note that for any fixed $n$, Theorem 2 may be numerically checked as follows. Order the odd numbers $k$, $1 \leqslant k \leqslant 2n-1$, as $k_1, k_2, \ldots, k_n$ where $\phi(k_i)/k_i \leqslant \phi(k_{i+1})/k_{i+1}$ for $1 \leqslant i < n$. The numbers $E(k_i, n)$ are each evaluated. We then check if $i < E(k_i, n)$ for each $i \leqslant n-2$. If so, Theorem 2 has been established for $n$. Indeed, if $\phi(k)/k = \phi(l)/l$, then $k$ and $l$ must have the same set of prime factors, so that $E(k, n) = E(l, n)$. Thus if $1 \leqslant i \leqslant n-2$ and

$$\phi(k_i)/k_i = \phi(k_j)/k_j < \phi(k_{j+1})/k_{j+1},$$

then

$$D(\phi(k_i)/k_i, n) = j < E(k_j, n) = E(k_i, n).$$

We follow the above procedure for $n \leqslant 16$. Of course there is nothing to prove for $n = 1, 2$. Moreover, one can easily see that if Theorem 2 is true for $n-1$ and $2n-1$ is prime, then Theorem 2 is true for $n$ as well. Thus the above procedure need only be carried out for $n = 5, 8, 11, 13$, and 14. These calculations are displayed in Table 1.

We now follow a different procedure to prove Theorem 2 for $n$ in the interval $17 \leqslant n \leqslant 1000$.

PROPOSITION 1. *Theorem 2 is true if*

$$\phi(k)/k \in \left(1 - (2n-1)^{-1/2}, 1\right]. \tag{1}$$

*Table* 1

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $k_i$ | 3 | 9 | 5 |
| $E(k_i, 5)$ | 3 | 3 | 4 |

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $k_i$ | 15 | 3 | 9 | 5 | 7 | 11 |
| $E(k_i, 8)$ | 3 | 5 | 5 | 6 | 6 | 7 |

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $k_i$ | 15 | 21 | 3 | 9 | 5 | 7 | 11 | 13 | 17 |
| $E(k_i, 11)$ | 5 | 5 | 7 | 7 | 8 | 9 | 10 | 10 | 10 |

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_i$ | 15 | 21 | 3 | 9 | 5 | 25 | 7 | 11 | 13 | 17 | 19 |
| $E(k_i, 13)$ | 6 | 6 | 8 | 8 | 10 | 10 | 11 | 11 | 12 | 12 | 12 |

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_i$ | 15 | 21 | 3 | 9 | 27 | 5 | 25 | 7 | 11 | 13 | 17 | 19 |
| $E(k_i, 14)$ | 7 | 7 | 9 | 9 | 9 | 11 | 11 | 12 | 12 | 12 | 13 | 13 |

*Proof.* Let $n \geqslant 17$, $k$ odd, $1 < k \leqslant 2n-1$, $k \neq p_1(n)$, and assume (1). Thus $k$ is divisible by no prime $p \leqslant \sqrt{(2n-1)}$, so that $k$ is itself a prime and $k > \sqrt{(2n-1)}$. Denote $k$ by $p_i$ where $p_i$ is the $i$-th largest prime not exceeding $2n-1$. Thus

$$D(\phi(k)/k, n) = D(1-1/p_i, n) = n-i, \tag{2}$$

for the only odd $l \leqslant 2n-1$ with $\phi(l)/l > 1-1/p_i$ are $p_{i-1}, p_{i-2}, \ldots, p_1, 1$. Now by Lemma 1,

$$E(k, n) = E(p_i, n) > n(1-1/p_i)-1 .$$

Thus to show $D(\phi(k)/k, n) < E(k, n)$, we need only show that

$$n/p_i < i-1 . \tag{3}$$

Now (3) is certainly true for $n < p_i < p_1$. By Rosser and Schoenfeld [3], the largest $i_0$ with $n < p_{i_0} < p_1$ is exactly

$$\pi(2n)-\pi(n) > n/(2\log n) \quad \text{for all} \quad n \geqslant 5{\cdot}5 .$$

Thus if $\sqrt{(2n-1)} < p_i \leqslant n$, we have

$$n/p_i < n/\sqrt{(2n-1)} < n/(2\log n) < i_0 \leqslant i-1$$

which holds for all $n \geqslant 16$. Thus (3) holds for all $i > 1$.

For future reference we note that (2) implies for all $n > 1$,

$$D\big(1-(2n-1)^{-1/2}, n\big) = n+\pi\big(\sqrt{(2n-1)}\big)-1-\pi(2n) . \tag{4}$$

For the remainder of this section we assume $17 \leqslant n \leqslant 1000$, $k$ is odd, $k \leqslant 2n-1$, and $\phi(k)/k \leqslant 1-(2n-1)^{-1/2}$.

*Case* 1. $\phi(k)/k \geqslant 120/143 = (10/11)(12/13)$. Using (4), a table of primes up to 2000, and a hand calculator, we verify that $D(\phi(k)/k, n) < (0{\cdot}712)n$ (*cf.* Lemma 1 in [1]). If $\omega(k) \geqslant 3$, then $k \geqslant 13 . 17 . 19 > 2n-1$, so we may assume $\omega(k) \leqslant 2$. By Lemma 1, we thus have $E(k, n) > (120/143)n-3$. Thus $D(\phi(k)/k, n) < E(k, n)$ for all $n \geqslant 24$. Assume $17 \leqslant n < 24$. If $\omega(k) \geqslant 2$, then $k \geqslant 11 . 13 > 2n-1$, a contradiction. So assume $\omega(k) = 1$. By Lemma 1, $E(k, n) > (120/143)n-1$. Thus $D(\phi(k)/k, n) < E(k, n)$ for $17 \leqslant n < 24$.

*Case* 2. $120/143 > \phi(k)/k \geqslant 720/1001 = (6/7)(10/11)(12/13)$. Since $11 . 13 . 17 > 2000$, the condition $\phi(k)/k < 120/143$ implies $k$ is divisible by 3, 5, or 7. Moreover, $k$ is not a power of 7. Thus by Lemma 4, $D(\phi(k)/k, n) \leqslant (19/35)n-5/7$. Indeed, this is a direct consequence of Lemma 4 if $n \geqslant 25$, and if $17 \leqslant n < 25$, we note that $E(105, n) < (19/35)n+2/7$. Now the only odd $k < 2000$ with $\omega(k) \geqslant 4$ are

$$3 . 5 . 7 . 11 , \quad 3 . 5 . 7 . 13 , \quad 3 . 5 . 7 . 17 , \quad 3 . 5 . 7 . 19 , \tag{5}$$

and none of these satisfy the conditions of Case 2. Thus $\omega(k) \leqslant 3$. By Lemma 1, $E(k, n) > (720/1001)n-1$. Thus $D(\phi(k)/k, n) < E(k, n)$ for $36 \leqslant n \leqslant 1000$. So assume $17 \leqslant n < 36$. Then $\omega(k) \leqslant 2$, so that $E(k,n) > (720/1001)n-3$. Thus $D(\phi(k)/k, n) < E(k,n)$ for these $n$ as well.

*Case* 3. $720/1001 > \phi(k)/k \geqslant 48/77 = (4/5)(6/7)(10/11)$. Since $k < 2000$, we see that $3 \mid k$ or $5 \mid k$. Moreover $k$ is not a power of 5. Thus for $17 \leqslant n \leqslant 1000$, Lemma 4 implies $D(\phi(k)/k, n) \leqslant (7/15)n-4/3$. Since none of the integers in (5) satisfy the condition of Case 3, we have $\omega(k) \leqslant 3$. Thus $E(k,n) > (48/77)n-7$, so that $D(\phi(k)/k, n) < E(k,n)$ if $n \geqslant 37$. But if $17 \leqslant n < 37$, then $\omega(k) \leqslant 2$, so that $E(k,n) > (48/77)n-3$ and $D(\phi(k)/k, n) < E(k,n)$.

*Case* 4. $48/77 > \phi(k)/k$. Since $5 \,.\, 7 \,.\, 11 \,.\, 13 > 2000$, we have $3 \mid k$ and $k$ is not a power of 3 so that Lemma 3, implies $D(\phi(k)/k, n) \leqslant (1/3)n-8/3$. Since $\omega(k) \leqslant 4$, we have $\phi(k)/k \geqslant (2/3)(4/5)(6/7)(10/11) = 32/77$. Thus Lemma 1 implies $E(k,n) > (32/77)n-15$ and we see that $D(\phi(k)/k, n) < E(k,n)$ if $n \geqslant 150$. So assume $n < 150$. Then $\omega(k) \leqslant 3$ and $\phi(k)/k \geqslant (2/3)(4/5)(6/7) = 16/35$. We have $E(k,n) > (16/35)n-7$ and $D(\phi(k)/k, n) < E(k,n)$ if $n \geqslant 36$. So assume $17 \leqslant n < 36$. Then $\omega(k) \leqslant 2$, $E(k,n) \geqslant (8/15)n-3$ (note that $(2/3)(4/5) = 8/15$), and $D(\phi(k)/k, n) < E(k,n)$.

§5. *The proof of Theorem* 2 *for* $n > 1000$: *the range* $\phi(k)/k \geqslant 1-(\log n)^{-1}$.

PROPOSITION 2. *Theorem* 2 *is true for*

$$\phi(k)/k \in [1-(\log n)^{-1},\ 1-(2n-1)^{-1/2}]. \tag{6}$$

*Proof.* Let $n > 1000$, $1 < k \leqslant 2n-1$, $k$ odd, and assume (6). We distinguish the two cases $\omega(k) \leqslant 5$, $\omega(k) \geqslant 6$.

If $\omega(k) \leqslant 5$, then by Lemma 1,

$$E(k,n) > \left(1-(\log n)^{-1}\right)n-31\,.$$

But by (4) and Rosser and Schoenfeld [**4**],

$$D(\phi(k)/k, n) < n+\sqrt{(2n-1)}-2n/\left(\log(2n)-1/2\right). \tag{7}$$

Thus $D(\phi(k)/k, n) < E(k,n)$.

So we now assume $\omega(k) \geqslant 6$. Then $p$, the largest prime factor of $k$, is at least $5\log n+1$. For if $p < 5\log n+1$, then

$$\frac{\phi(k)}{k} < \left(1-\frac{1}{1+5\log n}\right)\left(1-\frac{1}{-1+5\log n}\right)^5 < \left(1-\frac{1}{5\log n}\right)^6$$

$$< 1-\frac{6}{5\log n}+\frac{15}{25\log^2 n} < 1-\frac{1}{\log n},$$

a contradiction. Thus by Lemma 2,

$$E(k,n) > \left(\frac{\phi(k)}{k}\frac{p-2}{p-1}-\frac{1}{p}\right)n-1$$

$$> \left(\left(1-\frac{1}{\log n}\right)\left(1-\frac{1}{5\log n}\right)-\frac{1}{5\log n}\right)n-1$$

$$> \left(1-\frac{7}{5\log n}\right)n-1\,.$$

Thus by (7), $D(\phi(k)/k, n) < E(k,n)$.

We now note that Proposition 2 in conjunction with Proposition 1 prove Theorem 2 for all $k$ in the range $\phi(k)/k \geqslant 1-(\log n)^{-1}$.

§6. *The range* $\phi(k)/k \leqslant 0{\cdot}623$.

PROPOSITION 3. *For all* $n > 1000$,

$$\sum_{\substack{k<2n\\ 2\nmid k}} k/\phi(k) < (1{\cdot}3)n\,.$$

*Proof.* Throughout the proof, the letters $d, j, k, m$, will stand for positive odd integers. Let $h$ be the multiplicative function such that $h(p) = 1/(p-1)$ for primes $p$ and $h(p^i) = 0$ for $i \geqslant 2$. Then $n/\phi(n) = \sum_{l\mid n} h(l)$. Thus

$$\sum_{k<2n} k/\phi(k) = \sum_{k<2n}\sum_{d\mid k} h(d) = \sum_{d<2n} h(d) \sum_{m<2n/d} 1$$

$$= \sum_{d<2n} h(d)\left[\frac{2n-1}{2d}+\frac{1}{2}\right]$$

$$\leqslant (1/2)\sum_{d<2n} h(d) + (n-1/2)\sum_{d<2n} h(d)/d\,. \tag{8}$$

Now

$$\sum_{d<2n} h(d)/d < \sum_{d} h(d)/d = \prod_{p>2}\left(1+\frac{1}{p(p-1)}\right) = \frac{2}{3}\cdot\frac{\zeta(2)\zeta(3)}{\zeta(6)} < 1{\cdot}2958\,, \tag{9}$$

where $\zeta$ is Riemann's function.

Also let $H$ denote the multiplicative function with $H(p) = p^{-1}(p-1)^{-1}$, $H(p^2) = -H(p)$, and $H(p^i) = 0$ for $i \geqslant 3$. Then $h(n) = \sum_{l\mid n} H(l)l/n$. Thus

$$\sum_{d<2n} h(d) = \left|\sum_{d<2n}\sum_{j\mid d} H(j)\,j/d\right| = \left|\sum_{j<2n} H(j)\sum_{m<2n/j} 1/m\right|$$

$$\leqslant \sum_{j<2n} |H(j)|\left(1+\tfrac{1}{2}\log\left(\frac{2n-1}{j}\right)\right) < \left(1+\tfrac{1}{2}\log(2n)\right)\sum_{j} |H(j)|$$

$$= \left(1+\tfrac{1}{2}\log(2n)\right)\prod_{p>2}\left(1+\frac{2}{p(p-1)}\right) < \left(1+\tfrac{1}{2}\log(2n)\right)\prod_{p>2}\left(1+\frac{1}{p(p-1)}\right)^2$$

$$= \left(1+\tfrac{1}{2}\log(2n)\right)\left(\frac{2}{3}\cdot\frac{\zeta(2)\zeta(3)}{\zeta(6)}\right)^2 < 1{\cdot}6791 + (0{\cdot}8396)\log(2n)\,. \tag{10}$$

From (8), (9), (10), we have

$$\sum_{k<2n} k/\phi(k) < (1{\cdot}2958)n - 0{\cdot}6478 + 0{\cdot}8396 + (0{\cdot}4198)\log(2n)$$

$$< (1{\cdot}3)n.$$

PROPOSITION 4. *For all $n > 1000$ and $0 < u < 1$, $D(u,n) < (0{\cdot}3)un/(1-u)$.*

*Proof.* Throughout the proof, the letter $k$ will stand for a positive odd integer parameter. Let $n > 1000$, $0 < u < 1$ be fixed. Let $c$ be such that $D(u,n) = cn$. Then, using Proposition 3,

$$cn = \sum_{\substack{k<2n\\ \phi(k)/k \leqslant u}} 1 \leqslant u \sum_{\substack{k<2n\\ \phi(k)/k \leqslant u}} k/\phi(k) = u\sum_{k<2n} k/\phi(k) - u \sum_{\substack{k<2n\\ \phi(k)/k > u}} k/\phi(k)$$

$$< (1{\cdot}3)un - u \sum_{\substack{k<2n\\ \phi(k)/k > u}} 1 = (1{\cdot}3)un - u(1-c)n,$$

so that $c < (0{\cdot}3)u + cu$ and our conclusion follows.

PROPOSITION 5. *Theorem 2 is true if $\phi(k)/k \leqslant 0{\cdot}623$.*

*Proof.* Let $n > 1000$, $1 < k \leqslant 2n-1$, $k$ odd, $\phi(k)/k \leqslant 0{\cdot}623$. We distinguish two cases: $\omega(k) \leqslant 7$, $\omega(k) \geqslant 8$.

Say $\omega(k) \leqslant 7$. By Lemma 1, $E(k,n) > (\phi(k)/k)n - 127$. Thus using Proposition 4, $D(\phi(k)/k, n) < E(k,n)$ if (with $u = \phi(k)/k$)

$$un - 127 > (0{\cdot}3)un/(1-u);$$

that is,

$$n > 127(1-u)/u(0{\cdot}7-u). \tag{11}$$

Now the maximal value of $u$ is $0{\cdot}623$ and the minimal value is

$$\prod_{3 \leqslant p \leqslant 19} (1-1/p).$$

But the maximal value of the right side of (11) for the stated range of $u$ is below 1000, so (11) holds.

Now say $\omega(k) \geqslant 8$. Let $p$ denote the largest prime factor of $k$ and write $k = p^i m$, $p \nmid m$. Then $p \geqslant 23$. Let $u = \phi(k)/k$. We have

$$p^{-1} = (p-1)^{-1} u\, m/\phi(m) \leqslant (p-1)^{-1} u \prod_{2<q<p} q/(q-1)$$

$$\leqslant (u/22) \prod_{2<q<23} q/(q-1) < (0{\cdot}1329)u, \tag{12}$$

where we use the fact that the function

$$(p-1)^{-1} \prod_{2<q<p} q/(q-1)$$

defined for odd prime arguments $p$ is decreasing. From (12) and Lemma 2 we have

$$E(k,n) > (21/22-0{\cdot}1329)un-1 > (0{\cdot}8216)un-1\,.$$

Thus by Proposition 4, $D(\phi(k)/k, n) < E(k,n)$ if

$$(0{\cdot}8216)un-1 > (0{\cdot}3)un/(1-u)\,;$$

that is,

$$n > \frac{1}{u}\left(\frac{1-u}{0{\cdot}5216-(0{\cdot}8216)u}\right). \tag{13}$$

Now the maximal value for $0 < u \leqslant 0{\cdot}623$ of the right factor in (13) is less than 40. Thus the proof will be complete if we can show $n > 40/u$.

From Rosser and Schoenfeld [4]

$$1/u = k/\phi(k) < (1/2)e^{\gamma}\operatorname{loglog} k + 1{\cdot}26/\operatorname{loglog} k \quad \text{for} \quad k \geqslant 3, 2 \nmid k\,.$$

Thus it only remains to note that for all $n > 1000$,

$$n > 40\big((1/2)e^{\gamma}\operatorname{loglog}(2n) + 1{\cdot}26/\operatorname{loglog}(2n)\big) > 40/u\,.$$

§7. *The range* $0{\cdot}623 < \phi(k)/k < 1-(\log n)^{-1}$. If $P$ is a prime, $P \geqslant 7$, let $S(P,n)$ denote the set of odd $k$, $1 \leqslant k \leqslant 2n-1$, with $k$ divisible by at least $t+1$ distinct primes in some set defined by

$$J_t = J_t(P) = \begin{cases} [0, P), & \text{if } t = 0\,, \\ [3^{t-1}P, 3^t P), & \text{if } t \geqslant 1\,. \end{cases}$$

If $k$ is odd, $1 \leqslant k \leqslant 2n-1$, and $k \notin S(P,n)$, then

$$\phi(k)/k > \prod_{t=1}^{\infty} \phi(a_t)/a_t\,,$$

where $a_t = a_t(P)$ is defined to be the product of the first $t$ primes in $J_t$ (if $J_t$ does not have $t$ primes, then $a_t$ is the product of all the primes in $J_t$). Write

$$u(P) = \prod_{t=1}^{\infty} \phi(a_t)/a_t\,.$$

Then

$$D\big(u(P), n\big) \leqslant |S(P,n)|\,.$$

Let $D(P, t, n)$ denote the number of odd $k$, $1 \leqslant k \leqslant 2n-1$, such that $k$ is divisible by at least $t+1$ distinct primes in $J_t$. Thus

$$D(u(p), n) \leqslant \sum_{t=0}^{\infty} D(P, t, n). \tag{14}$$

We now proceed to define the column headings in Table 2.

*Definition of $v(P)$.* We define $v(P)$ as a positive quantity satisfying the inequality $v(P) < u(P)$ according to the following scheme. Let

$$v(7) = 0{\cdot}704 < \tfrac{6}{7}(1-\tfrac{3}{7}(\tfrac{3}{4}-\tfrac{1}{3})) = \tfrac{6}{7}\left(1-\sum_{t=2}^{\infty}\frac{t}{3^{t-1}7}\right)$$

$$< \frac{\phi(a_1)}{a_1}\prod_{t=2}^{\infty}\left(1-\frac{1}{3^{t-1}7}\right)^t < u(7).$$

For $P \geqslant 11$, let

$$v(P) = 1-\frac{2{\cdot}25}{P} < \frac{P-1}{P}\left(1-\frac{3}{P}(\tfrac{3}{4}-\tfrac{1}{3})\right) < u(P).$$

*Definition of $w(P)$.* We shall define $w(P)$ as a positive quantity, such that if $P > 7$, then $w(P) \leqslant v(P')$ where $P'$ is the prime just before $P$, according to the following scheme.

$$w(7) = 0{\cdot}623.$$

For $11 \leqslant P \leqslant 37$, let

$$w(P) = v(P').$$

By Rosser and Schoenfeld [4], if $P \geqslant 41$, then $P' > 7P/8$. So we may take

$$w(P) = 1-\frac{18}{7P} < 1-\frac{2{\cdot}25}{P'} = v(P').$$

*Definitions of $d(P)$ and $c(P)$.* We shall define positive quantities $d(P)$, $c(P)$ so that for all $n \geqslant 1$,

$$D(v(P), n) \leqslant D(u(P), n) < d(P)n + c(P). \tag{15}$$

We let

$$d(7) = 0{\cdot}543, \qquad c(7) = 6.$$

It remains to show that (15) holds for $P = 7$. First we note that Lemma 4 implies

$$D(7, 0, n) \leqslant (7/15)n + 2/3.$$

Since $D(7, 1, n)$ is the number of odd $k$, $1 \leqslant k \leqslant 2n-1$, divisible by two distinct primes in $\{7, 11, 13, 17, 19\}$, Lemma 3 implies

$$D(7, 1, n) < (0{\cdot}069)n + 5.$$

We shall use the following estimate (see Rosser and Schoenfeld [4]):

$$\sum_{T \leqslant p < 3T} 1/p < \log\left(1+\frac{\log 3}{\log T}\right)+\frac{1}{\log^2(3T)}+\frac{1}{2\log^2 T} = \alpha(T), \tag{16}$$

say, a result that is valid for all $T > 1$. So

$$\begin{aligned}\sum_{t=2}^{\infty} D(7,t,n) &< 2n\sum_{t=2}^{\infty}\left(\sum_{p\in J_t} 1/p\right)^{t+1}/(t+1)! \\ &< (n/3)\left(\sum_{p\in J_2} 1/p\right)^3 + 2n\sum_{t=3}^{\infty}\alpha(63)^{t+1}/(t+1)! \\ &= (n/3)\left(\sum_{p\in J_2} 1/p\right)^3 + 2n\left(e^{\alpha(63)}-1-\alpha(63)-\tfrac{1}{2}\alpha(63)^2-\tfrac{1}{6}\alpha(63)^3\right) \\ &< (0{\cdot}007)n\,.\end{aligned}$$

Thus by (14), $D(u(7), n) < (0{\cdot}543)n+6$.

We let

$$d(11) = 0{\cdot}625\,, \qquad c(11) = 12\,.$$

To show (15) holds, we note that Lemmas 4 and 3 imply

$$D(11,0,n) \leqslant (19/35)n+9/7\,, \qquad D(11,1,n) < (0{\cdot}064)n+10{\cdot}5\,.$$

Furthermore, by (16),

$$\begin{aligned}\sum_{t=2}^{\infty} D(11,t,n) &< 2n\sum_{t=2}^{\infty}\left(\sum_{p\in J_t} 1/p\right)^{t+1}/(t+1)! \\ &< 2n\left(e^{\alpha(33)}-1-\alpha(33)-\tfrac{1}{2}\alpha(33)^2\right) < (0{\cdot}018)n\,.\end{aligned}$$

Thus $D(u(11), n) < (0{\cdot}625)n+12$.

For $13 \leqslant P \leqslant 37$, we let (*cf.* (16))

$$d(P) = 1-\prod_{2<p<P}(1-1/p)+\left(\sum_{p\in J_1} 1/p\right)^2+2\left(e^{\alpha(3P)}-1-\alpha(3P)-\tfrac{1}{2}\alpha(3P)^2\right),$$

$$c(P) = 2^{\pi(P)-3}\,;$$

and for $P \geqslant 41$, we let

$$d(P) = 1-2e^{-\gamma}/\log P+2e^{-\gamma}/\log^3 P+2\left(e^{\alpha(P)}-1-\alpha(P)\right),$$

$$c(P) = 2^{P/(\log P-1{\cdot}5)-3}\,.$$

It remains to show that (15) holds. Now Lemma 4 implies

$$D(P,0,n) < \left(1 - \prod_{2<p<P} (1-1/p)\right) n + 2^{\pi(P)-3}$$

$$< \left(1 - \frac{2e^{-\gamma}}{\log P} + \frac{2e^{-\gamma}}{\log^3 P}\right) n + 2^{P/(\log P - 1.5) - 3},$$

where the last inequality is valid for all $P \geqslant 5$ from the estimates of Rosser and Schoenfeld [4]. Also by (16),

$$\sum_{t=1}^{\infty} D(P,t,n) < 2n \sum_{t=1}^{\infty} \left(\sum_{p \in J_t} 1/p\right)^{t+1} /(t+1)!$$

$$< \begin{cases} n\left(\sum_{p \in J_1} 1/p\right)^2 + 2n \sum_{t=2}^{\infty} \alpha(3P)^{t+1}/(t+1)!\,, & P < 41 \\ 2n \sum_{t=1}^{\infty} \alpha(P)^{t+1}/(t+1)!\,, & P \geqslant 41 \end{cases}$$

$$= \begin{cases} n\left(\sum_{p \in J_1} 1/p\right)^2 + 2n\left(e^{\alpha(3P)} - 1 - \alpha(3P) - \tfrac{1}{2}\alpha(3P)^2\right), & P < 41 \\ 2n\left(e^{\alpha(P)} - 1 - \alpha(P)\right), & P \geqslant 41\,. \end{cases}$$

Thus (14) implies (15).

*Definition of* pr $(P)$. For $7 \leqslant P \leqslant 37$, we define pr $(P)$ as the least prime $p$ so that there is a $k$ with 6 distinct prime factors, the largest of which is $p$, and $\phi(k)/k \geqslant w(P)$. For $P \geqslant 41$, we let

$$\operatorname{pr}(P) = 35P/18 = 5(1-w(P))^{-1}.$$

We now note that if $k$ has at least 6 distinct prime factors and $\phi(k)/k \geqslant w(P)$, then one of these primes is at least as big as pr $(P)$. For if not,

$$\phi(k)/k < (1 - 1/\operatorname{pr}(P))^6 = (1 - 18/(35P))^6$$

$$< 1 - 6(18/(35P)) + 15(18/(35P))^2$$

$$< 1 - 5(18/(35P)) = w(P).$$

*Definition of* $x(P)$. We shall define $x(P)$ so that if $k$ is divisible by at least 6 distinct primes and $\phi(k)/k \geqslant w(P)$, then

$$E(k,n) > x(P)n - 1\,. \tag{17}$$

For $7 \leqslant P \leqslant 37$, we let

$$x(P) = w(P)\Big(1 - 1/(\mathrm{pr}\,(P)-1)\Big) - 1/\mathrm{pr}\,(P),$$

so that (17) follows from Lemma 2. For $P \geqslant 41$, we let

$$x(P) = 1 - 3{\cdot}6/P < \big(1-18/(7P)\big)\big(1-18/(35\,P)\big) - 18/(35\,P)$$

$$< w(P)\Big(1 - 1/(\mathrm{pr}\,(P)-1)\Big) - 1/\mathrm{pr}\,(P),$$

so again (17) follows from Lemma 2.

Proposition 6. *If $k$ is odd, $1 < k \leqslant 2n-1$, $w(P) \leqslant \phi(k)/k \leqslant v(P)$, and $d(P) < x(P)$, then $D\big(\phi(k)/k, n\big) < E(k,n)$ for all $n > M_1(P)$, with*

$$M_1(P) = \max\left\{\frac{c(P)+31}{w(P)-d(P)}, \frac{c(P)+1}{x(P)-d(P)}\right\}. \qquad (1\text{!}$$

*Proof.* From (15), we have

$$D\big(\phi(k)/k, n\big) < d(P)n + c(P).$$

Say $\omega(k) \leqslant 5$. By Lemma 1,

$$E(k,n) > \big(\phi(k)/k\big)n - 31 \geqslant w(P)n - 31.$$

The assumption $d(P) < x(P)$ implies $d(P) < w(P)$. Thus $D\big(\phi(k)/k, n\big) < E(k,n)$ for a $n > \big(c(P)+31\big)/\big(w(P)-d(P)\big)$.

Now assume $\omega(k) \geqslant 6$. By (17), $D\big(\phi(k)/k, n\big) < E(k,n)$ for a $n > \big(c(P)+1\big)/\big(x(P)-d(P)\big)$.

Proposition 7. *Theorem 2 is true if* $0{\cdot}623 < \phi(k)/k < 1-(\log n)^{-1}$.

*Proof.* Let $n > 1000$, $k$ odd, $1 < k \leqslant 2n-1$, $0{\cdot}623 < \phi(k)/k < 1-(\log n)^{-}$ There exists a prime $P \geqslant 7$ such that $w(P) \leqslant \phi(k)/k \leqslant v(P)$. Thu $w(P) < 1-(\log n)^{-1}$, so that

$$n > \max\{1000, e^{(1-w(P))^{-1}}\} = M_2(P), \qquad (1\text{!}$$

say. By an examination of Table 2, we see that the condition (19) implies tl condition (18) for all $P \geqslant 7$. Thus by Proposition 6, $D\big(\phi(k)/k, n\big) < E(k,n)$.

In comparing the last two columns of Table 2 for $P \geqslant 41$, it is helpful to noı that $M_2(P) = e^{7P/18}$ and since $\alpha(P) < 1{\cdot}316/\log P$,

$$M_1(P) < 6(\log P)(2^{P/(\log P - 1\cdot 5)} + 31).$$

*Table* 2

| $P$ | $v(P)$ | $w(P)$ | $d(P)$ | $c(P)$ | pr $(P)$ | $x(P)$ | $M_1(P)$ | $M_2(P)$ |
|---|---|---|---|---|---|---|---|---|
| 7 | ·704 | ·623 | ·543 | 6 | 29 | ·566 | 463 | 1000 |
| 11 | ·795 | ·704 | ·625 | 12 | 31 | ·648 | 566 | 1000 |
| 13 | ·826 | ·795 | ·706 | 8 | 41 | ·750 | 439 | 1000 |
| 17 | ·867 | ·826 | ·729 | 16 | 43 | ·783 | 485 | 1000 |
| 19 | ·881 | ·867 | ·727 | 32 | 59 | ·835 | 450 | 1842 |
| 23 | ·902 | ·881 | ·742 | 64 | 61 | ·849 | 684 | 4462 |
| 29 | ·922 | ·902 | ·761 | 128 | 73 | ·875 | 1132 | 27013 |
| 31 | ·927 | ·922 | ·759 | 256 | 89 | ·900 | 1823 | 369724 |
| 37 | ·939 | ·927 | ·777 | 512 | 97 | ·907 | 3947 | 889691 |
| $\geqslant 41$ | (a) | (b) | (c) | (d) | (e) | (f) | | |

(a) $1-2{\cdot}25/P$

(b) $1-18/(7P)$

(c) $1-2e^{-\gamma}/\log P-2e^{-\gamma}/\log^3 P+2(e^{\alpha(P)}-1-\alpha(P))$
where $\alpha(P)=\log\big(1+(\log 3)/\log P\big)+1/\log^2(3P)+1/(2\log^2 P)$

(d) $2^{P/(\log P-1\cdot 5)-3}$

(e) $35P/18$

(f) $1-3{\cdot}6/P$

*Round-off notes*
Rounded down: $v(P)$, $w(P)$, $x(P)$, $M_2(P)$
Rounded up: $d(P)$, $M_1(P)$

The entries in the last two columns (see (18), (19) for definitions) are computed from the rounded numbers appearing in the other columns of the table.

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