PROOF OF D. J. NEWMAN'S COPRIME MAPPING CONJECTURE

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§1. Introduction. In this paper we prove

THEOREM 1. If \( N \) is a natural number and \( I \) is an interval of \( N \) consecutive integers, then there is a 1-1 correspondence \( f: \{1, 2, \ldots, N\} \to I \) such that \((i, f(i)) = 1\) for \( 1 \leq i \leq N \).

We call the function \( f \) described in the theorem a coprime mapping. Theorem 1 settles in the affirmative a conjecture of D. J. Newman. The special case when \( I = \{N+1, N+2, \ldots, 2N\} \) was proved by D. E. Daykin and M. J. Baines [2]. V. Chvátal [1] established Newman's conjecture for each \( N \leq 1002 \). We prove Theorem 1 constructively by giving an algorithm for the construction of a coprime mapping \( f \). This algorithm will be discussed in §2.

If \( u \) is a real number and \( n \) is a natural number, let \( D(u, n) \) denote the number of odd integers \( l, 1 \leq l \leq 2n - 1 \), with \( \phi(l)/l \leq u \), where \( \phi \) denotes Euler's function. If also \( k \) is a natural number, let \( E(k, n) \) denote the maximal number of integers coprime to \( k \) that can be found in every set of \( n \) consecutive integers. Thus, for example, \( E(3, 4) = 2 \), since in every set of 4 consecutive integers there are at least 2 integers coprime to 3 and the set \( \{0, 1, 2, 3\} \) has exactly 2 integers coprime to 3. If \( n > 1 \), let \( p_1(n) \) denote the largest prime not exceeding \( 2n - 1 \). We shall prove

THEOREM 2. If \( n \) is a natural number and \( k \) is odd with \( 1 < k \leq 2n - 1 \) and \( k \neq p_1(n) \), then \( D(\phi(k)/k, n) < E(k, n) \).

In §2 we shall show that Theorem 1 is a fairly simple corollary of Theorem 2. The remainder of the paper then will take up the proof of Theorem 2.

Note that the function \( D(u, n) \) is related to the well-known distribution function for \( \phi \):

\[
D_\phi(u) = \lim_{x \to \infty} \frac{1}{x} \sum_{\phi(d) \leq u} 1.
\]

In fact, if we let \( D(u) = \lim_{n \to \infty} D(u, n)/n \), then it can be seen that, like \( D_\phi(u) \), \( D(u) \) exists for all \( u \), \( D(u) \) is strictly monotone on \([0,1]\), \( D(u) \) is continuous, and \( D(u) \) is singular on \([0,1]\). We have \( D_\phi(u) = (D(u) + D(2u))/2 \).

For \( k \) fixed it is easy to see that \( E(k, n) \sim (\phi(k)/k)n \) as \( n \to \infty \). We thus have the following consequence of Theorem 2.

COROLLARY. For every \( u \geq 0 \), \( D(u) \leq u \).
We do not use computers in the proof, although many calculations were performed out of convenience on a hand calculator. However, the proof relies quite heavily on the computer-assisted results of Rosser and Schoenfeld [4]. We take this opportunity to thank Paul Erdős for calling our attention to Newman's conjecture. We should also remark that a key step in our proof—namely, the use of the sets $J_i(P)$ in §7—is suggested by an old argument of Erdős (Theorem 3 in [3]). We also wish to thank M. Mendes France for informing us of [1].

§2. The algorithm. In this section we shall inductively describe an algorithm for the construction of a coprime mapping $f$. The algorithm will make use of Theorem 2, which shall be assumed as true in this section. Thus this section will show that Theorem 2 implies Theorem 1.

If $N = 1$, then there is only one mapping $f: \{1\} \to I$, and certainly $f$ is a coprime mapping. If $N = 2$, then let $f(1)$ be the even member of $I$ and let $f(2)$ be the odd member of $I$. This $f$ is a coprime mapping.

Say now $N \geq 3$ and we have given algorithms for the construction of coprime mappings for all $M < N$. Let $I$ be a set of $N$ consecutive integers. We first describe $f$ at the odd members of $\{1, 2, ..., N\}$ and then use our induction hypothesis to describe $f$ at even arguments.

Let $n = N - \lfloor N/2 \rfloor$ denote the number of odd numbers in $\{1, 2, ..., N\}$. Then $n \geq 2$. Label these odd numbers $k_1, k_2, ..., k_n$ where $\phi(k_i)/k_i \leq \phi(k_{i+1})/k_{i+1}$ for $1 \leq i < n$. Note that $k_{n-1} = p_1(n)$ and $k_n = 1$. We first describe how to choose $f(k_1), ..., f(k_{n-2})$.

Now $I$ has either $n$ or $n-1$ even numbers. If they are each divided by 2, then we obtain a string of consecutive integers. Thus for each $i$, there are at least $E(k_i, n-1)$ even integers in $I$ that are coprime to $k_i$.

By Theorem 2, if $n \geq 3$,

$$E(k_i, n-1) \geq E(k_i, n) - 1 \geq D(\phi(k_i)/k_i, n) \geq 1,$$

so that there are even numbers in $I$ coprime to $k_i$. Let $f(k_i)$ be the least such number. Say $i \leq n-2$ and $f(k_1), ..., f(k_{i-1})$ have already been defined. By Theorem 2,

$$E(k_i, n-1) \geq E(k_i, n) - 1 \geq D(\phi(k_i)/k_i, n) \geq i,$$

so that there are at least $i$ even numbers in $I$ coprime to $k_i$. Thus we may let $f(k_i)$ be the least even number in $I$ coprime to $k_i$ and not equal to $f(k_1), ..., f(k_{i-1})$.

We now define $f(k_{n-1})$. If $I$ has $n$ even numbers, there are 2 of them left which have not yet been used as a value of $f$. Moreover, $k_{n-1} = p_1(n)$ does not divide both of them since half of their difference is at most $n-1$, while by Bertrand's Postulate, $p_1(n) > n$. Thus we let $f(k_{n-1})$ be the least of the remaining even numbers of $I$ which is not divisible by $p_1(n)$. If $I$ has only $n-1$ even numbers, then both endpoints of $I$ are odd and not both are divisible by $p_1(n)$ (half of their difference is $n-1$). So we let $f(k_{n-1})$ be the least endpoint of $I$ which is not divisible by $p_1(n)$.

In either case, we have exactly one even number left in $I$ which has not yet been used as a value of $f$. We let $f(k_n)$ be this number.

Now we describe how to define $f$ at even arguments, using essentially the same idea as Theorem 2 in [1]. Let $J$ be the set of remaining members of $I$ not used.
already as values of \( f \). Then \( J \) is an interval of \([N/2]\) consecutive odd numbers. Let \( m \) equal the product of the odd primes not exceeding \([N/2]\). Let \( I' = \{(j+m)/2 : j \in J\} \). Then \( I' \) is an interval of \([N/2]\) consecutive integers. By our induction hypothesis there is a construction for a coprime mapping \( g: \{1,2,\ldots,[N/2]\} \rightarrow I' \). If \( 1 \leq k \leq [N/2] \), let \( f(2k) = 2g(k) - m \). We have thus described a 1-1 correspondence from \( \{2,4,\ldots,2[N/2]\} \) to \( J \) and for each \( k \), \( 1 \leq k \leq [N/2] \),

\[
(2k, f(2k)) = (2k, 2g(k) - m) = 1.
\]

This completes the construction of the coprime mapping \( f \).

Before we proceed to the proof of Theorem 2, we describe a simpler algorithm for the construction of a coprime mapping. We do not have a proof that this simpler algorithm is always successful, but we feel that a proof probably could be provided using the methods of this paper. As in the above algorithm, we may assume \( N \geq 3 \).

Relabel \( \{1, \ldots, N\} \) as \( b_1, \ldots, b_N \) where \( f(b_j)/b_j \leq f(b_{j+1})/b_{j+1} \) for \( 1 \leq i < N \). If it is not the case that both of \( I \)'s endpoints are odd, inductively define \( f(b_i) \) as the least number in \( I \) coprime to \( b_i \) and not equal to \( f(b_1), \ldots, f(b_{i-1}) \). If both of \( I \)'s endpoints are odd, first define \( f(b_{N-1}) \) as one of the endpoints, and then inductively define \( f(b_i) \) for \( i = N-1 \) as the least number in \( I - \{f(b_{N-1})\} \) coprime to \( b_i \) and not equal to \( f(b_1), \ldots, f(b_{i-1}) \). The need for the special treatment of \( b_{N-1} \) in the latter case can be seen from the example \( N = 9, I = \{-1, 0, \ldots, 7\} \).

§3. Preliminaries. If \( k \) is a natural number, let \( \omega(k) \) denote the number of distinct prime factors of \( k \).

**Lemma 1.** If \( k > 1 \), then

\[
E(k, n) > \frac{\phi(k)}{k} n - 2^{\omega(k)} + 1.
\]

**Proof.** If \( k_0 \) is the largest square-free divisor of \( k \), then \( E(k_0, n) = E(k, n) \), \( \phi(k_0)/k_0 = \phi(k)/k \), and \( \omega(k_0) = \omega(k) \). Thus we may assume \( k \) itself is square-free. If \( m \) is any integer and if \( j > 1 \) is an integer, then

\[
\left| \sum_{\substack{m < i < m+n \\{i, k\} = 1 \\mid j \}} 1 - \frac{n}{j} \right| < 1.
\]

Thus

\[
\sum_{\substack{m < i < m+n \\{i, k\} = 1 \\mid j \}} 1 = n + \sum_{\substack{j > 1 \\mid j \}} (-1)^{\omega(j)} \sum_{\substack{m < i < m+n \\mid j \}} 1
\]

\[
> n + \sum_{\substack{j > 1 \\mid j \}} (-1)^{\omega(j)} \frac{n}{j} - \sum_{\substack{j > 1 \\mid j \}} 1
\]

\[
= \frac{\phi(k)}{k} n - 2^{\omega(k)} + 1.
\]
Lemma 2. If $1 < k \leq 2n - 1$ and if $p$ is the largest prime factor of $k$, then,

$$E(k, n) > \left( \frac{\phi(k)}{k} \cdot \frac{p - 2}{p - 1} - \frac{1}{p} \right) n - 1.$$ 

Proof. Write $k = p^m m$ where $p \nmid m$. Thus $pm < 2n$, so that

$$1 - \frac{m}{n} > 1 - \frac{2}{p}.$$ 

Thus

$$E(k, n) \geq E(m, n) - (\lfloor n/p \rfloor + 1) \geq \phi(m)[n/m] - n/p - 1$$ 

$$> \phi(m)(n/m - 1) - n/p - 1 = \frac{\phi(m)}{m} \left( 1 - \frac{m}{n} \right) n - \frac{n}{p} - 1$$ 

$$> \frac{\phi(m)}{m} \left( 1 - \frac{2}{p} \right) n - \frac{n}{p} - 1 = \left( \frac{\phi(k)}{k} \cdot \frac{p - 2}{p - 1} - \frac{1}{p} \right) n - 1.$$ 

Lemma 3. If $m > 1$ is odd, and $n$ is a positive integer,

$$\left| \sum_{1 \leq k \leq 2n - 1 \atop 2 \nmid k, m \nmid k} 1 - \frac{n}{m} \right| \leq \frac{1}{2} - \frac{1}{2m}.$$ 

Proof. The quantity in the absolute value signs has its minimal value when $2n - 1 = m - 2$; the value is $-1/2 + 1/(2m)$. The maximal value is attained when $2n - 1 = m$; the value is $1/2 - 1/(2m)$. 

Lemma 4. If $m$ is odd, let

$$F(m, n) = \sum_{1 \leq k \leq 2n - 1 \atop 2 \nmid k, (m, k) > 1} 1.$$ 

Then

$$F(m, n) < (1 - \phi(m)/m)n + 2^{\omega(m) - 1},$$ 

$$F(15, n) \leq \frac{7}{15}n + \frac{3}{5}, \quad F(105, n) \leq \frac{13}{33}n + \frac{9}{13}.$$ 

Proof. The maximum value of $F(m, n) - (1 - \phi(m)/m)n$ for all $n$ can be computed by evaluating this difference for any $m$ consecutive choices for $n$. This is how we established the latter two assertions. To see the first statement, note that by Lemma 3,

$$F(m, n) = \sum_{j \mid m, j > 1} (-1)^{\omega(j)+1} \sum_{1 \leq k \leq 2n - 1 \atop 2 \nmid k, j \mid k} 1$$ 

$$\leq \sum_{j \mid m, j > 1} (-1)^{\omega(j)+1} \sum_{j \mid m, j > 1} 1$$ 

$$\leq \sum_{j \mid m, j > 1} (-1)^{\omega(j)+1} \frac{n}{j} + \sum_{j \mid m, j > 1} \left( \frac{1}{2} - \frac{1}{2j} \right).$$
where, as in the proof of Lemma 1, we assume $m$ is square-free.

§4. The proof of Theorem 2 for $n \leq 1000$. We first note that for any fixed $n$, Theorem 2 may be numerically checked as follows. Order the odd numbers $k$, $1 \leq k \leq 2n - 1$, as $k_1, k_2, \ldots, k_n$ where $\phi(k_i)/k_i \leq \phi(k_{i+1})/k_{i+1}$ for $1 \leq i < n$. The numbers $E(k_i, n)$ are each evaluated. We then check if $i < E(k_i, n)$ for each $i \leq n - 2$. If so, Theorem 2 has been established for $n$. Indeed, if $\phi(k)/k = \phi(l)/l$, then $k$ and $l$ must have the same set of prime factors, so that $E(k, n) = E(l, n)$. Thus if $1 \leq i \leq n - 2$ and

$$\phi(k_i)/k_i = \phi(k_j)/k_j < \phi(k_{j+1})/k_{j+1},$$

then

$$D(\phi(k_i)/k_i, n) = j < E(k_j, n) = E(k_i, n).$$

We follow the above procedure for $n \leq 16$. Of course there is nothing to prove for $n = 1, 2$. Moreover, one can easily see that if Theorem 2 is true for $n - 1$ and $2n - 1$ is prime, then Theorem 2 is true for $n$ as well. Thus the above procedure need only be carried out for $n = 5, 8, 11, 13, \text{and } 14$. These calculations are displayed in Table 1.

We now follow a different procedure to prove Theorem 2 for $n$ in the interval $17 \leq n \leq 1000$.

**Proposition 1.** Theorem 2 is true if

$$\phi(k)/k \in (1-(2n-1)^{-1/2}, 1].$$

(1)

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Proof. Let \( n \geq 17 \), \( k \) odd, \( 1 < k \leq 2n - 1 \), \( k \neq p_i(n) \), and assume (1). Thus \( k \) is divisible by no prime \( p \leq \sqrt{2n-1} \), so that \( k \) is itself a prime and \( k > \sqrt{2n-1} \). Denote \( k \) by \( p_i \) where \( p_i \) is the \( i \)-th largest prime not exceeding \( 2n - 1 \). Thus

\[
D(\phi(k)/k, n) = D(1 - 1/p_i, n) = n - i ,
\]

(2)

for the only odd \( l \leq 2n - 1 \) with \( \phi(l)/l > 1 - 1/p_i \) are \( p_{i-1}, p_{i-2}, \ldots, p_1, 1 \). Now by Lemma 1,

\[
E(k, n) = E(p_i, n) > n(1 - 1/p_i) - 1 .
\]

Thus to show \( D(\phi(k)/k, n) < E(k, n) \), we need only show that

\[
n/p_i < i - 1 .
\]

(3)

Now (3) is certainly true for \( n < p_i < p_1 \). By Rosser and Schoenfeld [3], the largest \( i_0 \) with \( n < p_{i_0} < p_1 \) is exactly

\[
\pi(2n) - \pi(n) > n/(2 \log n) \quad \text{for all} \quad n \geq 5 \cdot 5 .
\]

Thus if \( \sqrt{2n-1} < p_i \leq n \), we have

\[
n/p_i < n/\sqrt{2n-1} < n/(2 \log n) < i_0 \leq i - 1
\]

which holds for all \( n \geq 16 \). Thus (3) holds for all \( i > 1 \).

For future reference we note that (2) implies for all \( n > 1 \),

\[
D(1 - (2n - 1)^{-1/2}, n) = n + \pi(\sqrt{2n-1}) - 1 - \pi(2n) .
\]

(4)

For the remainder of this section we assume \( 17 \leq n \leq 1000 \), \( k \) odd, \( k \leq 2n - 1 \), and \( \phi(k)/k \leq 1 - (2n - 1)^{-1/2} \).

Case 1. \( \phi(k)/k \geq 120/143 = (10/11)(12/13) \). Using (4), a table of primes up to 2000, and a hand calculator, we verify that \( D(\phi(k)/k, n) < 0.712n \) (cf. Lemma 1 in [1]). If \( \omega(k) \geq 3 \), then \( k \geq 13 \cdot 17 \cdot 19 > 2n - 1 \), so we may assume \( \omega(k) \leq 2 \). By Lemma 1, we thus have \( E(k, n) > (120/143)n - 3 \). Thus \( D(\phi(k)/k, n) < E(k, n) \) for all \( n \geq 24 \). Assume \( 17 \leq n < 24 \). If \( \omega(k) \geq 2 \), then \( k \geq 11 \cdot 13 > 2n - 1 \), a contradiction. So assume \( \omega(k) = 1 \). By Lemma 1, \( E(k, n) > (120/143)n - 1 \). Thus \( D(\phi(k)/k, n) < E(k, n) \) for \( 17 \leq n < 24 \).

Case 2. \( 120/143 > \phi(k)/k \geq 720/1001 = (6/7)(10/11)(12/13) \). Since \( 11 \cdot 13 \cdot 17 > 2000 \), the condition \( \phi(k)/k < 120/143 \) implies \( k \) is divisible by 3, 5, or 7. Moreover, \( k \) is not a power of 7. Thus by Lemma 4, \( D(\phi(k)/k, n) \leq (19/35)n - 5/7 \). Indeed, this is a direct consequence of Lemma 4 if \( n \geq 25 \), and if \( 17 \leq n < 25 \), we note that \( E(105, n) < (19/35)n + 2/7 \). Now the only odd \( k < 2000 \) with \( \omega(k) \geq 4 \) are

\[
3 \cdot 5 \cdot 7 \cdot 11, \quad 3 \cdot 5 \cdot 7 \cdot 13, \quad 3 \cdot 5 \cdot 7 \cdot 17, \quad 3 \cdot 5 \cdot 7 \cdot 19,
\]

(5)

and none of these satisfy the conditions of Case 2. Thus \( \omega(k) \leq 3 \). By Lemma 1, \( E(k, n) > (720/1001)n - 1 \). Thus \( D(\phi(k)/k, n) < E(k, n) \) for \( 36 \leq n \leq 1000 \). So
assume $17 \leq n < 36$. Then $\omega(k) \leq 2$, so that $E(k, n) > (720/1001)n - 3$. Thus $D(\phi(k)/k, n) < E(k, n)$ for these $n$ as well.

**Case 3.** $720/1001 > \phi(k)/k \geq 48/77 = (4/5)(6/7)(10/11)$. Since $k < 2000$, we see that $3 \mid k$ or $5 \mid k$. Moreover $k$ is not a power of 5. Thus for $17 \leq n \leq 1000$, Lemma 4 implies $D(\phi(k)/k, n) \leq (7/15)n - 4/3$. Since none of the integers in (5) satisfy the condition of Case 3, we have $\omega(k) \leq 3$. Thus $E(k, n) > (48/77)n - 7$, so that $D(\phi(k)/k, n) < E(k, n)$ if $n \geq 37$. But if $17 \leq n < 37$, then $\omega(k) \leq 2$, so that $E(k, n) > (48/77)n - 3$ and $D(\phi(k)/k, n) < E(k, n)$.

**Case 4.** $48/77 > \phi(k)/k$. Since $5 \cdot 7 \cdot 11 \cdot 13 > 2000$, we have $3 \mid k$ and $k$ is not a power of 3 so that Lemma 3, implies $D(\phi(k)/k, n) \leq (1/3)n - 8/3$. Since $\omega(k) \leq 4$, we have $\phi(k)/k \leq (2/3)(4/5)(6/7)(10/11) = 32/77$. Thus Lemma 1 implies $E(k, n) > (32/77)n - 15$ and we see that $D(\phi(k)/k, n) < E(k, n)$ if $n \geq 150$. So assume $n < 150$. Then $\omega(k) \leq 3$ and $\phi(k)/k \leq (2/3)(4/5)(6/7) = 16/35$. We have $E(k, n) > (16/35)n - 7$ and $D(\phi(k)/k, n) < E(k, n)$ if $n \geq 36$. So assume $17 \leq n < 36$. Then $\omega(k) \leq 2$, $E(k, n) \geq (8/15)n - 3$ (note that $(2/3)(4/5) = 8/15)$, and $D(\phi(k)/k, n) < E(k, n)$.

§5. The proof of Theorem 2 for $n > 1000$: the range $\phi(k)/k \geq 1 - (\log n)^{-1}$.

**Proposition 2.** Theorem 2 is true for

$$\phi(k)/k \in [1 - (\log n)^{-1}, 1 - (2n - 1)^{-1/2}] .$$

(6)

**Proof.** Let $n > 1000$, $1 < k < 2n - 1$, $k$ odd, and assume (6). We distinguish the two cases $\omega(k) \leq 5$, $\omega(k) \geq 6$.

If $\omega(k) \leq 5$, then by Lemma 1,

$$E(k, n) > (1 - (\log n)^{-1})n - 31 .$$

But by (4) and Rosser and Schoenfeld [4],

$$D(\phi(k)/k, n) < n + \sqrt{(2n - 1) - 2n/(\log (2n) - 1/2)} .$$

(7)

Thus $D(\phi(k)/k, n) < E(k, n)$.

So we now assume $\omega(k) \geq 6$. Then $p$, the largest prime factor of $k$, is at least $5 \log n + 1$. For if $p < 5 \log n + 1$, then

$$\frac{\phi(k)}{k} < \left(1 - \frac{1}{1 + 5 \log n}\right)\left(1 - \frac{1}{-1 + 5 \log n}\right)^5 < \left(1 - \frac{1}{5 \log n}\right)^6$$

$$< 1 - \frac{6}{5 \log n} + \frac{15}{25 \log^2 n} < 1 - \frac{1}{\log n} ,$$

a contradiction. Thus by Lemma 2,

$$E(k, n) > \left(\frac{\phi(k)}{k} \frac{p - 2}{p - 1} - \frac{1}{p}\right)n - 1.$$
\[
> \left(1 - \frac{1}{\log n}\left(1 - \frac{1}{5 \log n}\right) - \frac{1}{5 \log n}\right) n - 1 \\
> \left(1 - \frac{7}{5 \log n}\right) n - 1.
\]
Thus by (7), \( D(\phi(k)/k, n) < E(k, n) \).

We now note that Proposition 2 in conjunction with Proposition 1 prove Theorem 2 for all \( k \) in the range \( \phi(k)/k \geq 1 - (\log n)^{-1} \).

§6. The range \( \phi(k)/k \leq 0.623 \).

**Proposition 3.** For all \( n > 1000 \),

\[
\sum_{k < 2n} k/\phi(k) < (1.3)n.
\]

**Proof.** Throughout the proof, the letters \( d, j, k, m \), will stand for positive odd integers. Let \( h \) be the multiplicative function such that \( h(p) = 1/(p - 1) \) for primes \( p \) and \( h(p^i) = 0 \) for \( i \geq 2 \). Then \( n/\phi(n) = \sum_{l|n} h(l) \). Thus

\[
\sum_{k < 2n} k/\phi(k) = \sum_{k < 2n} \sum_{d|k} h(d) = \sum_{d < 2n} h(d) \sum_{m < 2n/d} 1 \\
= \sum_{d < 2n} h(d) \left[ \frac{2n - 1}{2d} + \frac{1}{2} \right] \\
\leq (1/2) \sum_{d < 2n} h(d) + (n - 1/2) \sum_{d < 2n} h(d)/d.
\]

Now

\[
\sum_{d < 2n} h(d)/d < \sum_{d} h(d)/d = \prod_{p \geq 2} \left(1 + \frac{1}{p(p - 1)}\right) = \frac{2}{3} \cdot \zeta(2) \zeta(3) < 1.2958,
\]

where \( \zeta \) is Riemann's function.

Also let \( H \) denote the multiplicative function with \( H(p) = p^{-1}(p - 1)^{-1} \), \( H(p^2) = -H(p) \), and \( H(p^i) = 0 \) for \( i \geq 3 \). Then \( h(n) = \sum_{l|n} H(l)/l \). Thus

\[
\sum_{d < 2n} h(d) = \left| \sum_{j < 2n/j} H(j) j/d \right| = \left| \sum_{j < 2n} H(j) \sum_{m < 2n/j} 1/m \right| \\
\leq \sum_{j < 2n} |H(j)| \left(1 + \frac{1}{2} \log \left(\frac{2n - 1}{j}\right)\right) < (1 + \frac{1}{2} \log (2n)) \sum_{j} |H(j)| \\
= (1 + \frac{1}{2} \log (2n)) \prod_{p \geq 2} \left(1 + \frac{2}{p(p - 1)}\right) < (1 + \frac{1}{2} \log (2n)) \prod_{p \geq 2} \left(1 + \frac{1}{p(p - 1)}\right)^2 \\
= (1 + \frac{1}{2} \log (2n)) \left(\frac{2}{3} \cdot \frac{\zeta(2) \zeta(3)}{\zeta(6)}\right)^2 < 1.6791 + (0.8396) \log (2n).
\]

PROOF OF D. J. NEWMAN'S COPRIME MAPPING CONJECTURE

From (8), (9), (10), we have

\[ \sum_{k < 2n} k/\phi(k) < (1.2958)n - 0.6478 + 0.8396 + (0.4198) \log(2n) \]

\[ < (1.3)n . \]

**Proposition 4.** For all \( n > 1000 \) and \( 0 < u < 1 \), \( D(u, n) < (0.3)u/(1-u) \).

**Proof.** Throughout the proof, the letter \( k \) will stand for a positive odd integer parameter. Let \( n > 1000, 0 < u < 1 \) be fixed. Let \( c \) be such that \( D(u, n) = cn \). Then, using Proposition 3,

\[ cn = \sum_{k < 2n} 1 \leq u \sum_{k < 2n} k/\phi(k) = u \sum_{k < 2n} k/\phi(k) - u \sum_{k < 2n} k/\phi(k) \]

\[ < (1.3)un - u \sum_{k < 2n} 1 = (1.3)un - u(1-c)n , \]

so that \( c < (0.3)u + cu \) and our conclusion follows.

**Proposition 5.** Theorem 2 is true if \( \phi(k)/k < 0.623 \).

**Proof.** Let \( n > 1000, 1 < k \leq 2n-1, k \) odd, \( \phi(k)/k \leq 0.623 \). We distinguish two cases: \( \omega(k) \leq 7, \omega(k) \geq 8 \).

Say \( \omega(k) \leq 7 \). By Lemma 1, \( E(k, n) > (\phi(k)/k)n - 127 \). Thus using Proposition 4, \( D(\phi(k)/k, n) < E(k, n) \) if (with \( u = \phi(k)/k \))

\[ un - 127 > (0.3)un/(1-u) ; \]

that is,

\[ n > 127(1-u)/u(0.7-u) . \] (11)

Now the maximal value of \( u \) is 0.623 and the minimal value is

\[ \prod_{3 \leq p \leq 19} (1 - 1/p) . \]

But the maximal value of the right side of (11) for the stated range of \( u \) is below 1000, so (11) holds.

Now say \( \omega(k) \geq 8 \). Let \( p \) denote the largest prime factor of \( k \) and write \( k = p'm, p \nmid m \). Then \( p \geq 23 \). Let \( u = \phi(k)/k \). We have

\[ p^{-1} = (p-1)^{-1} u m/\phi(m) \leq (p-1)^{-1} u \prod_{2 < q < p} q/(q-1) \]

\[ \leq (u/22) \prod_{2 < q < 23} q/(q-1) < (0.1329)u , \] (12)
where we use the fact that the function
\[
(p-1)^{-1} \prod_{2 \leq q < p} \frac{q}{q-1}
\]
defined for odd prime arguments \(p\) is decreasing. From (12) and Lemma 2 we have
\[
E(k, n) > (21/22 - 0.1329)un - 1 > (0.8216)un - 1.
\]
Thus by Proposition 4, \(D(\phi(k)/k, n) < E(k, n)\) if
\[
(0.8216)un - 1 > (0.3)un/(1 - u);
\]
that is,
\[
n > \frac{1}{u} \left( \frac{1 - u}{0.5216 - (0.8216)u} \right). \tag{13}
\]
Now the maximal value for \(0 < u \leq 0.623\) of the right factor in (13) is less than 40. Thus the proof will be complete if we can show \(n > 40/u\).

From Rosser and Schoenfeld [4]
\[
1/u = k/\phi(k) < (1/2) e^x \log \log k + 1.26/\log \log k
\]
for \(k \geq 3, 2 \nmid k\).

Thus it only remains to note that for all \(n > 1000\),
\[
n > 40((1/2) e^x \log \log (2n) + 1.26/\log \log (2n)) > 40/u.
\]

§7. The range \(0.623 < \phi(k)/k < 1 - (\log n)^{-1}\). If \(P\) is a prime, \(P \geq 7\), let \(S(P, n)\) denote the set of odd \(k, 1 \leq k \leq 2n - 1\), with \(k\) divisible by at least \(t + 1\) distinct primes in some set defined by
\[
J_t = J_t(P) = \begin{cases} 
[0, P), & \text{if } t = 0, \\
[3^t - 1 P, 3^t P), & \text{if } t \geq 1.
\end{cases}
\]
If \(k\) is odd, \(1 \leq k \leq 2n - 1\), and \(k \notin S(P, n)\), then
\[
\phi(k)/k > \prod_{i=1}^{\infty} \phi(a_i)/a_i,
\]
where \(a_t = a_t(P)\) is defined to be the product of the first \(t\) primes in \(J_t\) (if \(J_t\) does not have \(t\) primes, then \(a_t\) is the product of all the primes in \(J_t\)). Write
\[
u(P) = \prod_{t=1}^{\infty} \phi(a_t)/a_t.
\]
Then
\[
D(u(P), n) \leq |S(P, n)|.
\]
Let $D(P, t, n)$ denote the number of odd $k$, $1 \leq k \leq 2n - 1$, such that $k$ is divisible by at least $t + 1$ distinct primes in $J_t$. Thus

$$D(u(p), n) \leq \sum_{t=0}^{\infty} D(P, t, n). \quad (14)$$

We now proceed to define the column headings in Table 2.

**Definition of $v(P)$:** We define $v(P)$ as a positive quantity satisfying the inequality $v(P) < u(P)$ according to the following scheme. Let

$$v(7) = 0.704 < \Phi(1 - 3(\frac{1}{3} - \frac{1}{4})) = \Phi \left(1 - \sum_{t=2}^{\infty} \frac{t}{3^{t-1}} \right)$$

$$< \frac{\Phi(a_1)}{a_1} \prod_{i=2}^{\infty} \left(1 - \frac{1}{3^{i-1}} \right)^t < u(7).$$

For $P \geq 11$, let

$$v(P) = 1 - \frac{2.25}{P} < \frac{P - 1}{P} \left(1 - \frac{3}{P} \left(\frac{1}{3} - \frac{1}{4}\right)\right) < u(P).$$

**Definition of $w(P)$:** We shall define $w(P)$ as a positive quantity, such that if $P > 7$, then $w(P) \leq v(P')$ where $P'$ is the prime just before $P$, according to the allowing scheme. By Rosser and Schoenfeld [4], if $P \geq 41$, then $P' > 7P/8$. So we may take

$$w(7) = 0.623.$$  

For $11 \leq P \leq 37$, let

$$w(P) = v(P').$$

**Definitions of $d(P)$ and $c(P)$:** We shall define positive quantities $d(P)$, $c(P)$ so that for all $n \geq 1$,

$$D(v(P), n) \leq D(u(P), n) < d(P)n + c(P). \quad (15)$$

We let

$$d(7) = 0.543, \quad c(7) = 6.$$

It remains to show that (15) holds for $P = 7$. First we note that Lemma 4 implies

$$D(7, 0, n) \leq (7/15)n + 2/3.$$  

Since $D(7, 1, n)$ is the number of odd $k$, $1 \leq k \leq 2n - 1$, divisible by two distinct primes in $\{7, 11, 13, 17, 19\}$, Lemma 3 implies

$$D(7, 1, n) < (0.069)n + 5.$$
We shall use the following estimate (see Rosser and Schoenfeld [4]):

\[
\sum_{T < p < T^c} \frac{1}{p} < \log \left(1 + \frac{\log 3}{\log T}\right) + \frac{1}{\log^2(3T)} + \frac{1}{2 \log T} = \alpha(T),
\]

say, a result that is valid for all \( T > 1 \). So

\[
\sum_{t=2}^{\infty} D(t, t, n) < 2n \sum_{t=2}^{\infty} \left( \sum_{p \in J_t} \frac{1}{p} \right)^{t+1} / (t+1)!
\]

\[
< \frac{(n/3)}{\left( \sum_{p \neq J_1} \frac{1}{p} \right)^3} + 2n \sum_{t=3}^{\infty} \alpha(63)^{t+1} / (t+1)!
\]

\[
= \frac{(n/3)}{\left( \sum_{p \neq J_1} \frac{1}{p} \right)^3} + 2n(e^{\alpha(63)} - 1 - \alpha(63) - \frac{1}{2} \alpha(63)^2 - \frac{1}{3} \alpha(63)^3)
\]

\[
< (0.007)n.
\]

Thus by (14), \( D(u(7), n) < (0.543)n + 6 \).

We let \( d(11) = 0.625 \), \( c(11) = 12 \).

To show (15) holds, we note that Lemmas 4 and 3 imply

\[ D(11, 0, n) \leq (19/35)n + 9/7, \quad D(11, 1, n) < (0.064)n + 10.5. \]

Furthermore, by (16),

\[
\sum_{t=2}^{\infty} D(11, t, n) < 2n \sum_{t=2}^{\infty} \left( \sum_{p \in J_t} \frac{1}{p} \right)^{t+1} / (t+1)!
\]

\[
< 2n(e^{\alpha(33)} - 1 - \alpha(33) - \frac{1}{2} \alpha(33)^2)(0.018)n.
\]

Thus \( D(u(11), n) < (0.625)n + 12 \).

For \( 13 \leq P \leq 37 \), we let (cf. (16))

\[
d(P) = 1 - \prod_{2 < p < P} (1 - 1/p) + \left( \sum_{p \neq J_1} 1/p \right)^2 + 2(e^{\alpha(P)} - 1 - \alpha(3P) - \frac{1}{2} \alpha(3P)^2),
\]

\[
c(P) = 2^{\alpha(P)-1};
\]

and for \( P \geq 41 \), we let

\[
d(P) = 1 - 2e^{-c}/\log P + 2e^{-c}/\log^3 P + 2(e^{\alpha(P)} - 1 - \alpha(P)),
\]

\[
c(P) = 2^{\alpha(P)-1.5}.
\]
It remains to show that (15) holds. Now Lemma 4 implies

$$D(P, 0, n) < \left(1 - \prod_{2 < p < P} (1 - 1/p)\right)n + 2^{m(n-1)}$$

$$< \left(1 - \frac{2 e^{-\gamma}}{\log P} + \frac{2 e^{-\gamma}}{\log^3 P}\right)n + 2^{m(\log P - 1) - 3},$$

where the last inequality is valid for all $P \geq 5$ from the estimates of Rosser and Schoenfeld [4]. Also by (16),

$$\sum_{t=1}^{\infty} D(P, t, n) < 2n \sum_{i=1}^{\infty} \left(\sum_{p \leq t} 1/p\right)^{i+1} / (t+1)!$$

$$\leq \begin{cases} 
\frac{n}{2} \left(\sum_{p \leq t} 1/p\right)^2 + 2n \sum_{i=2}^{\infty} \frac{\alpha(3P)^{i-1}}{(t+1)!}, & P < 41 \\
2n \sum_{i=1}^{\infty} \frac{\alpha(P)^{i-1}}{(t+1)!}, & P \geq 41
\end{cases}$$

$$= \begin{cases} 
n \left(\sum_{p \leq t} 1/p\right)^2 + 2n(\varepsilon^{3P} - 1 - \alpha(3P) - \frac{1}{2} \alpha(3P)^2), & P < 41 \\
2n(\varepsilon^{3P} - 1 - \alpha(P)), & P \geq 41
\end{cases}.$$

Thus (14) implies (15).

**Definition of $\text{pr}(P)$.** For $7 \leq P \leq 37$, we define $\text{pr}(P)$ as the least prime $p$ so that there is a $k$ with 6 distinct prime factors, the largest of which is $p$, and $\phi(k)/k \geq w(P)$. For $P \geq 41$, we let

$$\text{pr}(P) = 35P/18 = 5(1 - w(P))^{-1}.$$

We now note that if $k$ has at least 6 distinct prime factors and $\phi(k)/k \geq w(P)$, then one of these primes is at least as big as $\text{pr}(P)$. For if not,

$$\phi(k)/k < (1 - 1/\text{pr}(P))^6 = (1 - 18/(35P))^6$$

$$< 1 - 6(18/(35P)) + 15(18/(35P))^2$$

$$< 1 - 5(18/(35P)) = w(P).$$

**Definition of $x(P)$.** We shall define $x(P)$ so that if $k$ is divisible by at least 6 distinct primes and $\phi(k)/k \geq w(P)$, then

$$E(k, n) > x(P)n - 1.$$ (17)
For \(7 \leq P \leq 37\), we let
\[
\alpha(P) = w(P) \left(1 - \frac{1}{\varphi(P) - 1}\right) - \frac{1}{\varphi(P)},
\]
so that (17) follows from Lemma 2. For \(P \geq 41\), we let
\[
\alpha(P) = 1 - \frac{3.6}{P} < \left(1 - \frac{18}{7P}\right)\left(1 - \frac{18}{35P}\right) - \frac{18}{35P},
\]
so again (17) follows from Lemma 2.

** Proposition 6. ** If \(k\) is odd, \(1 < k \leq 2n - 1\), \(w(P) \leq \phi(k)/k \leq \varphi(P)\), and \(d(P) < \alpha(P)\), then \(D(\phi(k)/k, n) < E(k, n)\) for all \(n > M_1(P)\), with
\[
M_1(P) = \max \left\{ \frac{c(P) + 31}{w(P) - d(P)}, \frac{c(P) + 1}{x(P) - d(P)} \right\}.
\]

** Proof. ** From (15), we have
\[
D(\phi(k)/k, n) < d(P)n + c(P).
\]
Say \(\omega(k) < 5\). By Lemma 1,
\[
E(k, n) > \left(\phi(k)/k\right)n - 31 \geq w(P)n - 31.
\]
The assumption \(d(P) < \alpha(P)\) implies \(d(P) < w(P)\). Thus \(D(\phi(k)/k, n) < E(k, n)\) for a \(n > (c(P) + 31)/(w(P) - d(P))\).

Now assume \(\omega(k) \geq 6\). By (17), \(D(\phi(k)/k, n) < E(k, n)\) for a \(n > (c(P) + 1)/(x(P) - d(P))\).

** Proposition 7. ** Theorem 2 is true if \(0.623 < \phi(k)/k < 1 - (\log n)^{-1}\).

** Proof. ** Let \(n > 1000\), \(k\) odd, \(1 < k \leq 2n - 1\), \(0.623 < \phi(k)/k < 1 - (\log n)^{-1}\).

There exists a prime \(P \geq 7\) such that \(w(P) \leq \phi(k)/k \leq \varphi(P)\). Thus \(w(P) < 1 - (\log n)^{-1}\), so that
\[
n > \max \{1000, e^{1 - w(P)}\} = M_2(P),
\]
say. By an examination of Table 2, we see that the condition (19) implies the condition (18) for all \(P \geq 7\). Thus by Proposition 6, \(D(\phi(k)/k, n) < E(k, n)\).

In comparing the last two columns of Table 2 for \(P \geq 41\), it is helpful to note that \(M_2(P) = e^{P/18}\) and since \(\alpha(P) < 1.316/\log P\),
\[
M_1(P) < 6(\log P)(2^{P/\log P^{1.5}} + 31).
\]
Table 2

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<th>P</th>
<th>v(P)</th>
<th>w(P)</th>
<th>d(P)</th>
<th>c(P)</th>
<th>pr(P)</th>
<th>x(P)</th>
<th>M₁(P)</th>
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(a) \(1 - 2.25/P\)
(b) \(1 - 18/(7P)\)
(c) \(1 - 2e^{-y}/\log P - 2e^{-y}/\log 3P + 2(e^{2P} - 1 - a(P))\) where \(a(P) = \log (1 + (\log 3)/\log P) + 1/\log 2 (3P) + 1/(2 \log 2 P)\)
(d) \(2^{\log P - 1 - 1/3}\)
(e) \(35P/18\)
(f) \(1 - 3.6/P\)

Round-off notes

Rounded down: \(v(P), w(P), x(P), M_2(P)\)
Rounded up: \(d(P), M_1(P)\)

The entries in the last two columns (see (18), (19) for definitions) are computed from the rounded numbers appearing in the other columns of the table.

References