

# ORDER AND CHAOS

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## Perfect shuffles

Suppose you take a deck of 52 cards, cut it in half, and perfectly shuffle it (with the bottom card staying on the bottom and the top card staying on the top).

If this is done 8 times, the deck returns to the order it was in before the first shuffle.

But, if you include the 2 jokers, so there are 54 cards, then it takes 52 shuffles, while a deck of 50 cards takes 21 shuffles.

Do you believe me? And what's going on?



Persi Diaconis

Lets try it out for smaller decks. Say 4 cards.

Number the 4 positions in the deck 0, 1, 2, 3, where 0 is the postion for the top card, 1 is the position for the second card, and so on. (This is the way computer scientists count, and the way floors are numbered in Europe.)

And here's one shuffle:

0			0
1	0	2	2
2	1	3	1
3			3

So doing one perfect shuffle on a deck of 4 cards just reverses the two middle cards, so doing it twice would return the deck to its original order.

Lets try 6 cards. Here are two shuffles:

0			0			0
1	0	3	3	0	4	4
2	1	4	1	3	2	3
3	2	5	4	1	5	2
4			2			1
5			5			5

Two shuffles reverse the order of the middle 4 cards, so four shuffles would return this deck to its starting order.

Lets try 8 cards:

0			0			0			0
1	0	4	4	0	2	2	0	1	1
2	1	5	1	4	6	4	2	3	2
3	2	6	5	1	3	6	4	5	3
4	3	7	2	5	7	1	6	7	4
5			6			3			5
6			3			5			6
7			7			7			7

So, with 8 cards, it takes 3 shuffles.

And now lets try to see what's happening with  $2n$  cards. Here's one shuffle:

0	0	$n$	0
1	1	$n + 1$	$n$
2	2	$n + 2$	1
3	3	$n + 3$	$n + 1$
4	4	$n + 4$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n - 2$	$n - 2$	$2n - 2$	$n - 1$
$2n - 1$	$n - 1$	$2n - 1$	$2n - 1$

Is there some simple way to explain in a formula what happens to the card in position  $i$  after one shuffle?

0	0	$n$	0
1	1	$n + 1$	$n$
2	2	$n + 2$	1
3	3	$n + 3$	$n + 1$
4	4	$n + 4$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2n - 2$	$n - 2$	$2n - 2$	$n - 1$
$2n - 1$	$n - 1$	$2n - 1$	$2n - 1$

So, the card in position 0 goes to position 0, the card in position 1 goes to position 2, and so on. For the first half the card in position  $i$  goes to position  $2i$ .

In the second half of the deck: The card in position  $n + i$  goes to position  $2i + 1$ , which we could write as  $2(n + i) - (2n - 1)$ .



Let  $S(i)$  be the position that a card in position  $i$  gets sent to after one perfect shuffle. We have

$$S(i) = \begin{cases} 2i, & i < n \\ 2i - (2n + 1), & n \leq i \leq 2n - 1 \end{cases}$$

That is,

$$S(i) \equiv 2i \pmod{2n - 1}.$$

So, if we do two shuffles, we have

$$S^{(2)}(i) = S(S(i)) \equiv 2^2 i \pmod{2n - 1}$$

and in general after  $k$  shuffles,

$$S^{(k)}(i) \equiv 2^k i \pmod{2n - 1}.$$

We're nearly there: We just need to find the least number  $k$  with

$$2^k \equiv 1 \pmod{2n - 1}.$$

What are the powers of 2 modulo 51? They are

2, 4, 8, 16, 32, 13, 26, 1,

so we have

$$2^8 \equiv 1 \pmod{51}$$

and this explains the 8 perfect shuffles for a deck of 52 cards.

Here's a question: Given a deck of size  $2n$  are we sure there will be *some* number of perfect shuffles to return it to its order?

That is, are we sure that there is *some* positive integer  $k$  with

$$2^k \equiv 1 \pmod{2n - 1} ?$$

We know this from [Lagrange](#), since 2 is in the multiplicative group modulo  $2n - 1$ .

But we knew this earlier.

For a positive integer  $m$ , let  $\varphi(m)$  be the number of integers in  $\{1, 2, \dots, m\}$  that are relatively prime to  $m$ . For example,  $\varphi(3) = 2$ ,  $\varphi(10) = 4$ ,  $\varphi(51) = 32$ .

**Euler:** If the integer  $a$  is relatively prime to  $m$ , then

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$



Leonhard Euler

We are looking at the **order function**. If  $a, m$  are relatively prime, let  $l_a(m)$  denote the order of  $a$  modulo  $m$ , namely the smallest positive integer  $k$  with  $a^k \equiv 1 \pmod{m}$ .

From [Euler](#), we know that  $l_a(m)$  exists, and in fact,  $l_a(m) \mid \varphi(m)$ .

Here are some values with  $a = 2$ , so that it corresponds to shuffling:

$$l_2(47) = 23, l_2(49) = 21, l_2(51) = 8, l_2(53) = 52, \dots$$

$$l_2(123) = 20, l_2(125) = 100, l_2(127) = 7, \dots$$

When a small change in input can produce a large change in output, we are looking at a *chaotic* function. This function  $l_2(m)$  for odd numbers  $m$  appears to be chaotic.

Here's another example. Consider the length of the repeating period for the decimal for  $1/n$ . Let this be denoted  $\text{Peri}(n)$ , so for example,  $\text{Peri}(3) = 1$ ,  $\text{Peri}(7) = 6$ . Here are some values for numbers coprime to 10 starting above 100:

$$\text{Peri}(101) = 4$$

$$\text{Peri}(103) = 34$$

$$\text{Peri}(107) = 53$$

$$\text{Peri}(109) = 108$$

$$\text{Peri}(111) = 3$$

$$\text{Peri}(113) = 112$$

For numbers  $m$  relatively prime to 10,  $\text{Peri}(m) = l_{10}(m)$ , so again we have an order function, and again it is chaotic.



We have seen in these examples that the order function  $l_a(n)$  is chaotic, thus explaining the title of this lecture.

The order function has applications in cryptography and in computing the periods of certain pseudo-random number generators. In fact, the RSA cryptosystem relies for its security on the difficulty in computing the order function, both directly and indirectly.

The **Blum–Blum–Shub** pseudo-random number generator:  
Start with a positive integer  $m$  and a “seed”  $s$ , and let

$$x_j = s^{2^j} \bmod m,$$

for  $j = 0, 1, \dots$ . To go from  $x_j$  to  $x_{j+1}$  one just squares, divides by  $m$ , and takes the remainder. Often this is done with  $m$  the product of two large prime numbers, and one creates a stream of 0's and 1's based on whether  $x_j$  is even or odd.

This is not really random, and in fact it will eventually be periodic. Say the largest odd divisor of  $l_s(m)$  is  $d$ . Then the period length is  $l_2(d)$ .

Here's an exercise in case you're interested:

Show that a black box that can compute multiplicative orders of elements modulo  $n$  can be used to quickly read messages encrypted in the RSA cryptosystem with modulus  $n$ .

In fact, show in addition that this black box can be used to quickly find the prime factorization of  $n$ .

Like computing  $\varphi(m)$ , computing orders is essentially as hard as computing the prime factorization of the modulus  $m$ , and we know no way to routinely factor large numbers. That is, on conventional computers.

Quantum computers theoretically can compute orders very easily. Except it is not so easy to build a quantum computer!

How might one “tame” a chaotic function? One way is to look at it statistically. Lets take as an example, the function  $\omega(n)$ , the number of primes that are divisors of  $n$ .

For example,  $\omega(10) = 2$ ,  $\omega(11) = 1$ ,  $\omega(12) = 2, \dots$  . It does not look very chaotic!

However, there is chaos, just a more gentle variety. Consider for example that

$$\omega(2309) = 1, \quad \omega(2310) = 5, \quad \omega(2311) = 1.$$

It is easy to show that on average,  $\omega(n)$  behaves like  $\log \log n$ . That is,

$$\frac{1}{x} \sum_{n \leq x} \omega(n) = \log \log x + c + o(1).$$

Thus, the average order of  $\omega(n)$  is  $\log \log n$ . This is also the “normal order”: for each  $\epsilon > 0$ , the set of integers  $n$  with

$$(1 - \epsilon) \log \log n < \omega(n) < (1 + \epsilon) \log \log n$$

has asymptotic density 1 ([Hardy & Ramanujan](#)).



G. H. Hardy



S. Ramanujan

We even have the bell curve showing up. From [Erdős & Kac](#), we know that for each real number  $u$ , the asymptotic density of the set of integers  $n$  with

$$\omega(n) \leq \log \log n + u\sqrt{\log \log n}$$

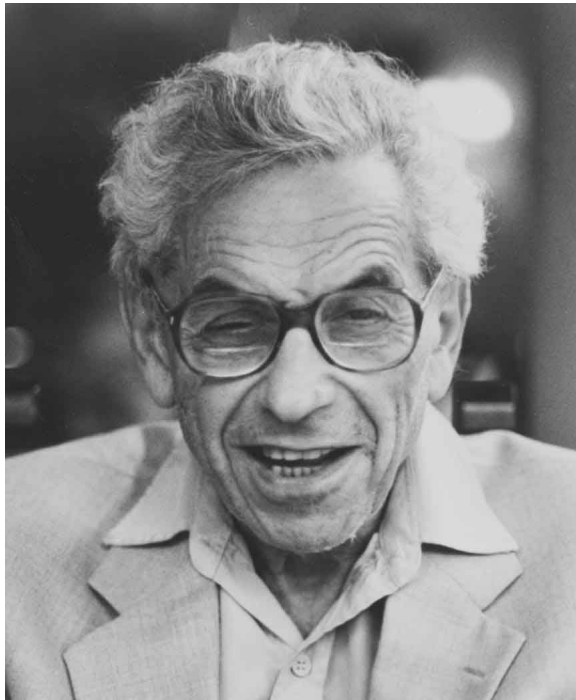
is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-t^2/2} dt,$$

the Gaussian normal distribution.

([Erdős & Kac](#) did *not* remark: ‘[Einstein](#) says that God does not play dice with the universe. Maybe so, but something is going on with the primes.’)





Paul Erdős



Mark Kac

There is some famous work concerning  $l_a(p)$  where  $p$  is a prime not dividing the integer  $a$ . We know that  $l_a(p) \mid \varphi(p)$  and that  $\varphi(p) = p - 1$ . We also know that there are choices for  $a$  where  $l_a(p) = p - 1$ .

For example, with  $a = 2$  and  $p = 53$ . That's why it takes a whopping 52 perfect shuffles for a deck of 54 cards.

Another example is with  $a = 10$  and  $p = 109$ . That's why the length of the repeating period for the decimal expansion of  $1/109$  is a whopping 108.

Over two centuries ago, [Gauss](#) asked if this deal with the decimal for  $1/p$  occurred for infinitely many primes  $p$ . I.e., do we have  $l_{10}(p) = p - 1$  for infinitely many primes  $p$ ?

In the mid twentieth century, [Artin](#) generalized Gauss's conjecture as follows.

Suppose that  $a$  is an integer which is not a square and not  $-1$ . The [Artin](#) conjecture: There is a positive constant  $A(a)$  such that asymptotically the proportion of primes  $p$  with  $l_a(p) = p - 1$  among all primes tends to  $A(a)$ .

This is still not proven, nor even the weaker assertion that there are infinitely many primes  $p$  with  $l_a(p) = p - 1$ . (This is the [Gauss](#) conjecture when  $a = 10$ .)

However, the full [Artin](#) conjecture is known *conditionally* under the assumption of the Generalized [Riemann](#) Hypothesis, a theorem of [Hooley](#).



Carl Friedrich Gauss



Emil Artin

One could ask about analogies for composite numbers. In general, let  $\lambda(n)$  denote the largest possible value of  $l_a(n)$  as  $a$  varies over numbers relatively prime to  $n$ . We always have  $\lambda(n) \mid \varphi(n)$ , and when  $n$  is prime, they are equal. But most of the time  $\lambda(n)$  is much smaller than  $\varphi(n)$ . For example,  $\varphi(91) = 72$  but  $\lambda(91) = 12$ .

A natural generalization of the [Gauss–Artin](#) problem:  
For a fixed integer  $a$  outside of some possible sparse exceptional set, do we have  $l_a(n) = \lambda(n)$  for a positive proportion  $B(a)$  of integers  $n$  relatively prime to  $a$ ?

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In recent work with [Li](#), we showed that under the assumption of the Generalized [Riemann](#) Hypothesis, the density of such integers  $n$  does *not* exist: the limsup of the density is indeed a positive number  $B(a)$ , but the liminf is 0.



Shuguang Li



It is easy to come up with sets of numbers which do not have an asymptotic density. For example, take the numbers with an even number of digits.

It is a bit of a surprise though when oscillations occur in non-artificial situations. Where does the oscillation come from in considering the frequency of numbers  $n$  with  $l_a(n) = \lambda(n)$ ?

Consider a game where you have a chance to win a quarter:

I give you  $n$  quarters, you flip them all, and return to me all that land tails.

You repeat this over and over, but if you get down to a single quarter, you get to keep it. (So, for example, if you have 2 quarters at one point, you flip them, and they both come up tails, you lose.)

What is the probability of winning as  $n \rightarrow \infty$ ? If you work it out numerically it appears to converge to some positive number, but in fact, it does not converge, it oscillates slightly.

When we're faced with hard problems, sometimes a way of getting some partial information is to consider the situation on average. We discussed this already with  $\omega(n)$ , where we understand this function on average, and we also understand it normally.

So, we could instead study the average values of  $l_a(p)$  as  $p$  varies, and also the average value of  $l_a(n)$  as  $n$  varies. One could consider the average as a function of  $a$  or over both variables. For example, [Luca](#) worked out the asymptotic behavior of

$$\sum_{p \leq x} \sum_{a=1}^{p-1} l_a(p)$$

and [Hu](#) did the analogous thing for more general finite fields.



Florian Luca



Yilan Hu

The question of the average order of  $l_a(n)$  for  $a$  fixed was recently discussed by [V. I. Arnold](#).

After some numerical experiments, he concluded that

$$\frac{1}{x} \sum_{n \leq x} l_a(n) \sim C_a \frac{x}{\log x}.$$

He gave a heuristic argument for this based on the physical principle of turbulence. This is in the paper

Number-theoretical turbulence in Fermat–Euler arithmetics and large Young diagrams geometry statistics, *Journal of Fluid Mechanics* **7** (2005), S4–S50.



Vladimir I. Arnold

Arnold writes in the abstract:

*“Many stochastic phenomena in deterministic mathematics had been discovered recently by the experimental way, imitating Kolmogorov’s semi-empirical methods of discovery of the turbulence laws. From the deductive mathematics point of view most of these results are not theorems, being only descriptions of several millions of particular observations. However, I hope that they are even more important than the formal deductions from the formal axioms, providing new points of view on difficult problems where no other approaches are that efficient.”*

And he says that his conjecture is supported by *billions* of experiments.



I think we should be a bit suspicious!

First, even billions of experiments may not be enough to tease out extra factors that may grow more slowly than  $\log x$ .

Second, [Arnold](#) did not seem to investigate any of the literature dealing with  $l_a(n)$ . In fact, there are interesting papers on the subject going back to [Romanoff](#) (who proved that the sum of  $1/(nl_a(n))$  for  $n$  coprime to  $a$  is convergent), with later papers by [Erdős](#), [P](#), [Pappalardi](#), [Li](#), [Kurlberg](#), [Murty](#), [Rosen](#), [Silverman](#), [Saidak](#), [Moree](#), [Luca](#), [Shparlinski](#), and others.

In addition he seemed to be unaware of work done on  $\lambda(n)$ .

For  $l_a(n)$  we could ask first the easier question: What is the average value of  $\lambda(n)$ ? (Recall that we always have  $l_a(n) \mid \lambda(n)$  and often they are equal.)

What this question means is: How does

$$\frac{1}{x} \sum_{n \leq x} \lambda(n)$$

behave as  $x \rightarrow \infty$  ?

**Erdős, P, Schmutz:** As  $x \rightarrow \infty$ ,

$$\frac{1}{x} \sum_{n \leq x} \lambda(n) = \frac{x}{\log x} \exp \left( \frac{(D + o(1)) \log \log x}{\log \log \log x} \right)$$

for a certain explicit positive constant  $D$ .



Eric Schmutz

But...

It's good to have outsiders investigate a field, and if they were expected to first read the literature thoroughly, it might dampen the fresh insight they might bring.

And, his conjecture that the average order of  $l_2(n)$  grows like  $x/\log x$ : it's supported on one side by [Hooley's](#) GRH-conditional proof of [Artin's](#) conjecture. (Assuming the GRH, a positive proportion of primes  $p$  have  $l_2(p) = p - 1$ , so that just the contribution of primes to the sum of  $l_2(n)$  gives an average order of the shape  $x/\log x$ .) And perhaps  $l_a(n)$  is sufficiently small for composite numbers  $n$ , that these do not contribute too much. Further, perhaps the average order of  $\lambda(n)$  is not that relevant, since this average is supported on a thin set of numbers  $n$  with abnormally large  $\lambda$  values, and the behavior for  $l_a(n)$  may be markedly different.

However...

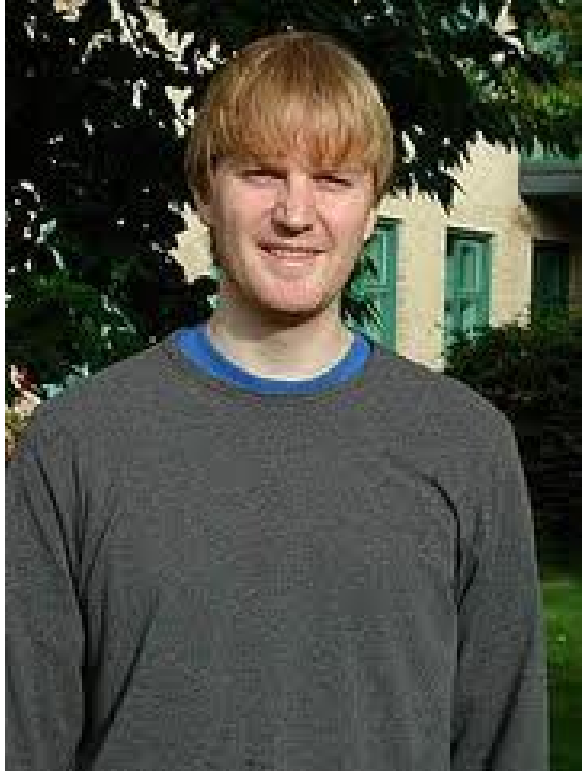
**Kurlberg** and **P**: Let  $|a| > 1$ . Assuming the Generalized Riemann Hypothesis,

$$\frac{1}{x} \sum_{\substack{n \leq x \\ (a,n)=1}} l_a(n) = \frac{x}{\log x} \exp\left(\frac{(D + o(1)) \log \log x}{\log \log \log x}\right).$$

Here “ $D$ ” is the same constant that appears in the average order of  $\lambda(n)$ , namely

$$D = e^{-\gamma} \prod_p \left(1 - \frac{1}{(p-1)^2(p+1)}\right) = 0.345372\dots$$

In particular, the upper bound in the theorem holds unconditionally.



Pär Kurlberg

The proof is a bit intense, borrowing heavily from the structure of the proof in [Erdős, P, & Schmutz](#) of the corresponding result for  $\lambda(n)$ .

Perhaps it is better to end now, and reflect how the innocent problem of perfect shuffles has led all this way.



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**Thank You**

Further reading:

V. I. Arnold, *Number-theoretical turbulence in Fermat–Euler arithmetics and large Young diagrams geometry statistics*, J. Fluid Mechanics **7** (2005), S4–S50.

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P. Erdős, C. Pomerance, and E. Schmutz, *Carmichael's lambda function*, Acta Arith. **58** (1991), 363–385.

C. Hooley, *On Artin's conjecture*, J. Reine Angew. Math. **225** (1967), 209–220.

S. Li and C. Pomerance, *On the Artin–Carmichael primitive root problem on average*, Mathematika **55** (2009), 167–176.