NEARLY PARALLEL VECTORS

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§1. *Introduction.* The first problem in diophantine approximation is for a given real number ξ and positive number x to find a fraction s/t with $t \leqslant x$ which is close to ξ. This problem can be rephrased in geometric terms. Given a vector $\mathbf{v}$ in $\mathbb{R}^2$, find a vector $\mathbf{a} = (a_1, a_2)$ with integer coordinates and $1 \leqslant a_1 \leqslant x$ such that the vectors $\mathbf{a}$ and $\mathbf{v}$ are nearly parallel. Simultaneous approximation of $d-1$ real numbers can be recast in terms of approximation of the angle between two vectors in d-dimensional Euclidean space.

In the geometric form it is natural to replace the condition $1 \leqslant a_1 \leqslant x$ by the symmetric condition that $\mathbf{a}$ lies in a sphere $S(r)$ of radius r centred at the origin. We shall investigate how nearly parallel two vectors in $S(r) \cap \mathbb{Z}^d$ can be.

We define in the usual way the length of a vector and the angle between two vectors in d-dimensional space, $d \geqslant 2$. Let $\theta(r)$ denote the infimum over all dimensions $d \geqslant 2$ of the minimal positive angle between all pairs of nonzero vectors in $S(r) \cap \mathbb{Z}^d$. We shall consider the problem of estimating $\theta(r)$ and characterizing the pairs of vectors for which the minimum is attained.

Initial speculation centred upon angles formed by pairs of vectors such as (3, 5, 8) and (5, 8, 13). The components of these vectors are successive Fibonacci numbers, and hence the ratios of the corresponding components are nearly constant. However, it is easily seen that a significantly smaller angle is provided by the more prosaic pair of vectors (15, 1) and (16, 1). This and some other examples suggested that, with a small number of exceptions, the minimal angles were to be found among a few classes of two dimensional vectors. Precisely, we have

THEOREM 1. *If $r \geqslant \sqrt{10}$ then the minimal positive angle $\theta(r)$ is achieved by pairs of two dimensional vectors belonging only to the following four classes (up to reflections and rotations with respect to the coordinate axes):*

Class	*Vectors*	*Asymptotic Direction*
1	$(n, 1)$ *and* $(n-1, 1)$	$(1, 0)$
2	$(n, n-1)$ *and* $(n+1, n)$	$(1, 1)$
3	$(2n-1, n-1)$ *and* $(2n+1, n)$	$(2, 1-)$
4	$(2n+1, n+1)$ *and* $(2n-1, n)$	$(2, 1+)$

There are four exceptional pairs of vectors making a small angle. The minimal angles achieved by pairs of vectors of small norm are listed in Table 1. Notice that for $\sqrt{34} \leqslant r < \sqrt{37}$ we have two essentially different ways of realizing $\theta(r)$. We take up this phenomenon of ties in §5.

Some computation suggests that the minimizing vectors belong to class 1 for about 3/4 the values of r, to class 2 for about 1/5 the values of r, and to class 3 and class 4 each for about 1/50 the values of r. We ask whether each of the four cases actually provides a minimal angle for a sequence of r's tending to infinity. We answer this question in the

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affirmative by showing that the occurrence of each class has a positive asymptotic density. For $j = 1, 2, 3, 4$ and m Lebesgue measure, let

$$d_j = \lim_{x \to \infty} x^{-1} m\{r \leqslant x : \theta(r) \text{ is achieved in class } j\},$$

provided that these densities exist.

Table 1

r range	vectors	$\theta(r)$ in degrees	$\cot \theta(r)$
$[1, \sqrt{2})$	(1, 0) and (0, 1)	90	0
$[\sqrt{2}, \sqrt{3})$	(1, 1) and (0, 1)	45	1
$[\sqrt{3}, 2)$	(1, 1, 0) and (1, 1, 1)	35·26	$\sqrt{2}$
$[2, \sqrt{5})$	(1, 1, 1, 0) and (1, 1, 1, 1)	30	$\sqrt{3}$
$[\sqrt{5}, 3)$	(2, 1) and (1, 1)	18·43	3
$[3, \sqrt{10})$	(1, 1, 1) and (1, 2, 2)	15·79	$5/\sqrt{2}$
$[\sqrt{10}, \sqrt{13})$	(3, 1) and (2, 1)	8·13	7
$[\sqrt{13}, \sqrt{17})$	(2, 1) and (3, 2)	7·13	8
$[\sqrt{17}, 5)$	(4, 1) and (3, 1)	4·40	13
$[5, \sqrt{26})$	(3, 2) and (4, 3)	3·18	18
$[\sqrt{26}, \sqrt{37})$	(5, 1) and (4, 1)	2·73	21
$[\sqrt{34}, \sqrt{37})$	(5, 3) and (3, 2)	2·73	21

THEOREM 2. *The densities d_1, d_2, d_3, and d_4 exist. Their values are*

$$d_1 = (960-275\sqrt{2}-252\sqrt{5}+54\sqrt{10})/240 \doteq 0{\cdot}743188,$$

$$d_2 = (-180+325\sqrt{2}-72\sqrt{10})/240 \doteq 0{\cdot}216398,$$

$$d_3 = d_4 = (-270-25\sqrt{2}+126\sqrt{5}+9\sqrt{10})/240 \doteq 0{\cdot}020207.$$

We can restate the theorem in more colourful language: for a randomly chosen radius r, the probability that $\theta(r)$ is achieved by a pair of (1, 0) vectors is about 74%, etc.

We have also found the relative frequency of occurrence of the four classes. More precisely, let $N_j(r)$ denote the number of pairs of vectors $\mathbf{a}, \mathbf{b}$ in class j for which $r \geqslant |\mathbf{a}| \geqslant |\mathbf{b}|$ and $\mathbf{a}, \mathbf{b}$ determines the minimal angle for vectors of length at most $|\mathbf{a}|$. We prove

THEOREM 3. *For $1 \leqslant j \leqslant 4$, the relation $N_j(r) \sim c_j r$ holds. The values of the c_j are*

$$c_1 = 1, \qquad c_2 = \tfrac{3}{4}\sqrt{2}-\tfrac{1}{2} \doteq 0{\cdot}560660,$$

and

$$c_3 = c_4 = (-30-15\sqrt{2}+18\sqrt{5}+5\sqrt{10})/40 \doteq 0{\cdot}121185.$$

A corollary of Theorems 2 and 3 is that if the pair $\mathbf{a}, \mathbf{b}$ gives a minimal angle and lies in class j, then the average length of the interval of r's for which the pair gives the minimal angle is d_j/c_j.

We mention a few further problems which we shall not discuss.

(1) *Higher dimensions.* Given $d \geqslant 3$, can one similarly characterize triples of vectors in $S(r) \cap \mathbb{Z}^d$ which make a minimal positive solid angle? Analogous questions can be asked for quadruples of vectors, *etc.*

(2) *Metrics.* Suppose a_2/a_1 and b_2/b_1 are adjacent terms in some Farey sequence in $[0, 1]$ (*cf.* [**1**, Ch. 3]). Their difference is $|a_2/a_1 - b_2/b_1| = 1/(a_1 b_1)$. Another measure of the proximity of a_2/a_1 and b_2/b_1 is provided by the tangent of the positive angle θ between the vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, a relation we shall exploit. We have

$$\tan\theta = \left|\frac{a_2/a_1 - b_2/b_1}{1 + (a_2/a_1)(b_2/b_1)}\right| = \frac{1}{a_1 b_1 + a_2 b_2} = \frac{1}{\mathbf{a} \,.\, \mathbf{b}}.$$

What other measurements of diophantine approximation have a reasonable interpretation and admit analytic treatment?

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§2. *The four classes of vectors: proof of Theorem* 1. Let $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ and let $\theta = \theta(\mathbf{a}, \mathbf{b})$ denote the (non-negative) angle between $\mathbf{a}$ and $\mathbf{b}$. We have

$$\cos^2\theta = \frac{(\mathbf{a} \,.\, \mathbf{b})^2}{|\mathbf{a}|^2 |\mathbf{b}|^2} = 1 - \frac{\delta^2}{|\mathbf{a}|^2 |\mathbf{b}|^2}, \tag{1}$$

where

$$\begin{aligned} \delta = \delta(\mathbf{a}, \mathbf{b}) &= \{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \,.\, \mathbf{b})^2\}^{1/2} \\ &= \Big\{ \sum_{1 \leqslant i < j \leqslant d} (a_i b_j - a_j b_i)^2 \Big\}^{1/2} \\ &= |\mathbf{a}|\,|\mathbf{b}| \sin\theta = \text{area of the parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}. \end{aligned} \tag{2}$$

Equivalent with (1), we can write

$$\cot\theta = \frac{\cos\theta}{\sqrt{(1 - \cos^2\theta)}} = \mathbf{a} \,.\, \mathbf{b}/\delta. \tag{3}$$

Several of our arguments will depend on

LEMMA 1. *Suppose* $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$, $|\mathbf{a}| \geqslant |\mathbf{b}|$, *and* $0 < \theta(\mathbf{a}, \mathbf{b}) \leqslant \pi/2$. *Then*

$$\cot\theta(\mathbf{a}, \mathbf{b}) \leqslant \frac{|\mathbf{a}|^2}{\delta(\mathbf{a}, \mathbf{b})} - \frac{|\mathbf{a}|\,|\mathbf{a} - \mathbf{b}|}{\delta(\mathbf{a}, \mathbf{b})} + 1.$$

Proof. We have by (2) that

$$\delta(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \left| \frac{(\mathbf{a} \,.\, \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2} - \mathbf{b} \right|.$$

The triangle inequality and the inequality $1-[(\mathbf{a}\,.\,\mathbf{b})/|\mathbf{a}|^2] \geqslant 0$ yield

$$|\mathbf{a}-\mathbf{b}| \leqslant \left|\mathbf{a}-\frac{(\mathbf{a}\,.\,\mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}\right| + \left|\frac{(\mathbf{a}\,.\,\mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}-\mathbf{b}\right|$$

$$= |\mathbf{a}| - \frac{(\mathbf{a}\,.\,\mathbf{b})}{|\mathbf{a}|} + \frac{\delta}{|\mathbf{a}|}.$$

Thus,

$$\frac{(\mathbf{a}\,.\,\mathbf{b})}{\delta} \leqslant \frac{|\mathbf{a}|^2}{\delta} - \frac{|\mathbf{a}|\,|\mathbf{a}-\mathbf{b}|}{\delta} + 1$$

and the desired inequality follows from (3).

All vectors in $\mathbb{Z}^d$ have length $\sqrt{N}$, N a non-negative integer. It follows that $\theta(r) = \theta(\sqrt{[r^2]})$, and in the rest of this section we shall assume that r^2 is a positive integer.

LEMMA 2. *If* $r \geqslant 3/2$ *then* $\cot\theta(r) \geqslant r^2-3r+3$.

Proof. Let $n = [\sqrt{(r^2-1)}]$, so that $n^2+1 \leqslant r^2 \leqslant (n+1)^2$. By (3) we have

$$\cot\theta(r) \geqslant \cot\theta\big((n-1,1),\,(n,1)\big)$$

$$= n^2-n+1 = (n+1)^2-3(n+1)+3 \geqslant r^2-3r^2-3r+3\,.$$

Now suppose $\theta(\mathbf{a},\mathbf{b}) = \theta(r)$ where $r \geqslant |\mathbf{a}| \geqslant |\mathbf{b}|$. We show that

$$|\mathbf{a}-\mathbf{b}| < 3 \qquad \text{for all } r\,, \tag{4}$$

$$\delta(\mathbf{a},\mathbf{b}) = 1 \qquad \text{for all } r \geqslant \sqrt{37}\,. \tag{5}$$

To show (4) we may assume that $r \geqslant 3/2$. Thus $\theta(r) < 60^\circ$ so $|\mathbf{a}| > |\mathbf{a}-\mathbf{b}|$. By Lemma 1 we then have

$$\cot\theta(r) \leqslant \frac{|\mathbf{a}|^2}{\delta(\mathbf{a},\mathbf{b})} - \frac{|\mathbf{a}|\,|\mathbf{a}-\mathbf{b}|}{\delta(\mathbf{a},\mathbf{b})} + 1$$

$$\leqslant |\mathbf{a}|^2 - |\mathbf{a}|\,|\mathbf{a}-\mathbf{b}| + 1$$

since $\delta(\mathbf{a},\mathbf{b}) \geqslant 1$. Now Lemma 2 and the fact that $|\mathbf{a}| > \frac{1}{2}|\mathbf{a}-\mathbf{b}|$ give (4).

To show (5) we first note that $\delta(\mathbf{a},\mathbf{b}) = 1$ for all $r > \sqrt{46}$. Indeed, if $\delta > 1$, then $\delta \geqslant \sqrt{2}$ and by Lemmas 1 and 2 we have

$$r^2-3r+3 \leqslant \cot\theta(r) \leqslant |\mathbf{a}|^2/\sqrt{2} - |\mathbf{a}|/\sqrt{2} + 1$$

$$\leqslant r^2/\sqrt{2} - r/\sqrt{2} + 1\,,$$

which fails if $r \geqslant \sqrt{47}$. For $\sqrt{37} \leqslant r \leqslant \sqrt{46}$ we have

$$r^2/\sqrt{2}-r/\sqrt{2}+1 < 29 < 31 = \cot \theta((5,1),\ (6,1)),$$

which shows that $\delta = 1$ for these values of r as well.

Now $\delta = 1$, if, and only if, after a possible reordering of coordinates, we have

$$a_1 b_2 - a_2 b_1 = \pm 1, \tag{6}$$

and for $(i, j) \neq (1, 2)$ or $(2, 1)$

$$a_i b_j - a_j b_i = 0. \tag{7}$$

In particular, if $n \geqslant 3$ and $3 \leqslant j \leqslant d$, then (7) yields

$$a_1 b_j = a_j b_1, \qquad a_j b_2 = a_2 b_j.$$

These equations imply that

$$a_1 b_2 b_j = a_2 b_1 b_j, \qquad a_2 a_j b_1 = a_1 a_j b_2.$$

Combining the last equations with (6) we get that $a_j = b_j = 0$ for $3 \leqslant j \leqslant d$. Thus, if $\delta = 1$, we can assume that the vectors **a** and **b** both lie in the plane $x_3 = x_4 = \ldots = x_d = 0$.

It now follows from (5) that if $r \geqslant \sqrt{37}$, $\theta(\mathbf{a}, \mathbf{b}) = \theta(r)$ and $r \geqslant |\mathbf{a}| \geqslant |\mathbf{b}|$, then **a** and **b** can be assumed to lie in $\mathbb{Z}^2$. We write $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. The condition $\delta(\mathbf{a}, \mathbf{b}) = 1$ implies that (6) holds. We shall assume that both **a** and **b** lie in the first octant, since we can use $(1, 1)$ as a vector for $r \geqslant \sqrt{2}$. Since there can be no lattice points on the interior of the line segment joining **a** and **b**, the coordinates of $\mathbf{a}-\mathbf{b}$ are relatively prime. Thus we have by (4), $|\mathbf{a}-\mathbf{b}| = 1$, $\sqrt{2}$, or $\sqrt{5}$.

Using the preceding remarks and some elementary calculations we obtain the four classes described in Theorem 1. The case $|\mathbf{a}-\mathbf{b}| = 1$ gives class 1, the case $|\mathbf{a}-\mathbf{b}| = \sqrt{2}$ gives class 2, and the case $|\mathbf{a}-\mathbf{b}| = \sqrt{5}$ gives classes 3 and 4.

The evaluation of $\theta(r)$ for $\sqrt{10} \leqslant r < \sqrt{37}$ involves similar but somewhat more technical arguments. In particular, we used a sharper form of Lemma 1. For $r < \sqrt{10}$ we employed a direct finite search. We shall suppress the details of these calculations; the data for these r are given in Table 1.

§3. *Densities of the four classes: proof of Theorem* 2. Given a value $r \geqslant \sqrt{10}$, which of the four classes provides a pair of vectors in $S(r) \cap \mathbb{Z}^d$ of minimal positive angle? We compute the cotangent of the smallest angle in each of the four classes and compare their size. For each class i, let $n_i = n_i(r)$ denote the largest integer such that the class i vectors for n_i have length at most r; let $\theta_i(r)$ denote the corresponding angle. Theorem 1 asserts that for $r \geqslant \sqrt{10}$, $\theta(r) = \min\left(\theta_1(r), \theta_2(r), \theta_3(r), \theta_4(r)\right)$.

Table 2

Class	$n_1(r)$	$\cot\,\theta_i(r)$
1	$n_i = [\sqrt{(r^2-1)}]$	$n_1^2-n_1+1$
2	$n_2 = [\sqrt{(\frac{1}{2}r^2-\frac{1}{4})}-\frac{1}{2}]$	$2n_2^2$
3	$n_3 = [\sqrt{(\frac{1}{5}r^2-\frac{1}{25})}-\frac{2}{5}]$	$5n_3^2-n_3-1$
4	$n_4 = [\sqrt{(\frac{1}{5}r^2-\frac{1}{25})}-\frac{3}{5}]$	$5n_4^2+n_4-1$

Letting $\{y\}$ denote the fractional part of y, define

$$S = \{\sqrt{(r^2-1)}\}, \qquad T = \{\sqrt{(\tfrac{1}{2}r^2-\tfrac{1}{4})}-\tfrac{1}{2}\}, \qquad U = \{\sqrt{(\tfrac{1}{5}r^2-\tfrac{1}{25})}-\tfrac{2}{5}\},$$

and

$$V = \{\sqrt{(\tfrac{1}{5}r^2-\tfrac{1}{25})}-\tfrac{3}{5}\}.$$

In terms of these new variables we write

$$f_1(r) = n_1^2-n_1+1 = r^2-(2S+1)\sqrt{(r^2-1)}+S^2+S$$

$$f_2(r) = 2n_2^2 = r^2-(2\sqrt{2}T+\sqrt{2})\sqrt{(r^2-\tfrac{1}{2})}+T^2+T$$

$$f_3(r) = 5n_3^2-n_3-1 = r^2-(2\sqrt{5}U+\sqrt{5})\sqrt{(r^2-\tfrac{1}{5})}+5U^2+5U$$

$$f_4(r) = 5n_4^2+n_4-1 = r^2-(2\sqrt{5}V+\sqrt{5})\sqrt{(r^2-\tfrac{1}{5})}+5V^2+5V.$$

Each of the f's contains an expression $\sqrt{(r^2-c)}$ which we replace by $r+O(1/r)$. Thus for r outside a negligible set we have

$$f_1(r) > \max\left(f_2(r), f_3(r), f_4(r)\right),$$

provided that

$$2S+1 < \min\,(2\sqrt{2}T+\sqrt{2}, 2\sqrt{5}U+\sqrt{5}, 2\sqrt{5}V+\sqrt{5}).$$

Similar inequalities determine when the other f's are maximal.

We can express the densities $d_1, \ldots, d_4$ in terms of the new variables. For example,

$$d_1 = \lim_{x\to\infty} \frac{1}{x} m\{r \leqslant x : 2S+1 < \min\,(2\sqrt{2}T+\sqrt{2}, 2\sqrt{5}U+\sqrt{5}, 2\sqrt{5}V+\sqrt{5})\}. \quad (8)$$

This calculation is complicated by the fact that S, T, U, and V are not piecewise linear functions of r. However, they are nearly so since $\sqrt{(r^2-c)} = r+o(1)$. If we set $s = \{r\}$, $t = \{(r/\sqrt{2})-\frac{1}{2}\}$, $u = \{(r/\sqrt{5})-\frac{2}{5}\}$ and $v = \{(r/\sqrt{5})-\frac{3}{5}\}$ then (8) holds with the capital letters replaced by the corresponding small letters.

Because of the similar form of the expressions involving u and v, it is practical to keep these expressions together. Let $w = \min\,(u, v)$. For I_1, I_2, and I_3 intervals in $[0, 1)$ let $R = I_1 \times I_2 \times I_3$. We want to show that

$$P(R) \stackrel{\text{def}}{=} \lim_{x\to\infty} \frac{1}{x} m\{r \leqslant x : s \in I_1, t \in I_2, w \in I_3\}$$

exists and calculate its value. We note that $u < v$, if, and only if, $0 \leqslant u < \frac{1}{5}$ and $v < u$, if, and only if, $0 \leqslant v < \frac{4}{5}$. (These are complementary sets!) Thus

$$\begin{aligned} P(R) &= \lim_{x\to\infty} \frac{1}{x} m\{r \leqslant x : s \in I_1, t \in I_2, u \in I_3'\} \\ &\quad + \lim_{x\to\infty} \frac{1}{x} m\{r \leqslant x : s \in I_1, t \in I_2, v \in I_3''\} \\ &= P'(I_1 \times I_2 \times I_3') + P''(I_1 \times I_2 \times I_3''), \end{aligned}$$

where $I_3' = I_3 \cap [0, \frac{1}{5})$ and $I_3'' = I_3 \cap [0, \frac{4}{5})$.

Since $1, 1/\sqrt{2}, 1/\sqrt{5}$ are linearly independent over the rationals

$$P'(R) = \lim_{x\to\infty} \frac{1}{x} m\left\{r \leqslant x : \{r\} \in I_1, \left\{\frac{r}{\sqrt{2}} - \frac{1}{2}\right\} \in I_2, \left\{\frac{r}{\sqrt{5}} - \frac{2}{5}\right\} \in I_3\right\}$$

exists for each rectangle $R = I_1 \times I_2 \times I_3 \subset [0, 1)^3$ and has value $|I_1||I_2||I_3|$ [**3**, Satz 5], [**2**, §1.9]. A similar result holds for P''. If Q denotes the density function of P, then the preceding argument shows that

$$Q(s, t, w) = \begin{cases} 2, & \text{if } 0 < w < \frac{1}{5}, \\ 1, & \text{if } \frac{1}{5} < w < \frac{4}{5}, \\ 0, & \text{if } \frac{4}{5} < w < 1. \end{cases}$$

Let G_1 denote the set of points (s, t, w) in $[0,1)^3$ satisfying

$$2s + 1 < \min(2\sqrt{2}t + \sqrt{2}, 2\sqrt{5}w + \sqrt{5}).$$

Then we have

$$d_1 = \iiint_{G_1} Q(s, t, w)\,ds\,dt\,dw.$$

Evaluating the integral we obtain the stated value of d_1.

The calculation of d_2 proceeds similarly. If G_2 denotes the region in (s, t, w) space where

$$2\sqrt{2}t + \sqrt{2} < \min(2s + 1, 2\sqrt{5}w + \sqrt{5}),$$

then we obtain

$$d_2 = \iiint_{G_2} Q(s, t, w)\,ds\,dt\,dw.$$

The function value $f_3(r)$ is maximal (*i.e.* the vectors of minimal positive angle are in class 3) if

$$2\sqrt{5}u + \sqrt{5} < 2s + 1, \; 2\sqrt{5}u + \sqrt{5} < 2\sqrt{2}t + \sqrt{2}, \text{ and } 2\sqrt{5}u + \sqrt{5} < 2\sqrt{5}v + \sqrt{5}.$$

The last condition is equivalent to $u < v$, which is to say $0 \leqslant u < \frac{1}{5}$. As in the calculation of d_1,

$$P'(I_1 \times I_2 \times I'_3) = |I_1||I_2||I'_3|.$$

Thus the density function of P' is 1 for $0 < u < \frac{1}{5}$ and is 0 for $\frac{1}{5} < u < 1$, and d_3 is the volume of the region in (s, t, u) space defined by

$$0 < u < \min\left\{\frac{s}{\sqrt{5}} - \frac{\sqrt{5}-1}{2\sqrt{5}},\ \sqrt{\tfrac{2}{5}}t - \frac{\sqrt{5}-\sqrt{2}}{2\sqrt{5}},\ \tfrac{1}{5}\right\}.$$

It turns out that the condition $u < \frac{1}{5}$ is satisfied automatically and can be omitted from the last inequality.

The function f_4 is maximal if

$$2\sqrt{5}v+\sqrt{5} < 2s+1,\ 2\sqrt{5}v+\sqrt{5} < 2\sqrt{2}t+\sqrt{2},\text{ and } 2\sqrt{5}v+\sqrt{5} < 2\sqrt{5}u+\sqrt{5}.$$

The last condition is equivalent to $v < u$, which is to say $0 \leqslant v < \frac{4}{5}$. Recalling the discussion of d_3 just above, we see that the constraints are the same in both cases, except that the irrelevant condition $0 \leqslant u < \frac{1}{5}$ has been replaced by the less restrictive condition $0 \leqslant u < \frac{4}{5}$. The last condition is also satisfied automatically, and we have $d_4 = d_3$.

§4. *The frequency of the four classes: proof of Theorem* 3. Recall that for each $j = 1, \ldots, 4$, $N_j(x)$ is the number of pairs $\mathbf{a}, \mathbf{b}$ in class j with $\theta(|\mathbf{a}|) = \theta(\mathbf{a}, \mathbf{b})$ and $x \geqslant |\mathbf{a}| \geqslant |\mathbf{b}|$.

We now show $N_1(x) \sim x$ by showing the stronger result

$$N_1(x) = [\sqrt{(x^2-1)}] \qquad \text{for all } x \geqslant 1. \tag{9}$$

Clearly $N_1(x)$ is at most equal to the number of points $(n, 1)$ with $\sqrt{(n^2+1)} \leqslant x$. Thus $N_1(x) \leqslant [\sqrt{(x^2-1)}]$. To show equality, we show that for every positive integer n,

$$\theta(\sqrt{(n^2+1)}) = \theta((n-1, 1), (n, 1)). \tag{10}$$

That (10) is true for small values of n can be seen by examining Table 1. With $r = \sqrt{(n^2+1)}$, we have in the notation of §3 that $S = 0$ and

$$f_1(r) < \max\{f_2(r), f_3(r), f_4(r)\}$$

for all values of T, U, V (assuming $r \geqslant 5$). Thus (10) and hence (9) are established.

We now establish the asymptotic formula

$$N_2(x) \sim c_2 x. \tag{11}$$

Let $N'_2(x)$ denote the number of positive integers $n \leqslant x/\sqrt{2}$ such that

$$\{\sqrt{2}(n+\tfrac{1}{2})\} > (\sqrt{2}-1)/2.$$

It follows from Weyl's Theorem ([**3**, Satz 2], [**2**, §1.2]) that

$$N_2'(x) \sim \left(1 - \frac{\sqrt{2}-1}{2}\right)\frac{x}{\sqrt{2}} = c_2 x .$$

Hence to show (11), it will suffice to show that $N_2(x) \sim N_2'(x)$. For $r = \sqrt{(2n^2+2n+1)}$, we have $T = 0$. Hence

$$\sqrt{2} = 2\sqrt{2}T + \sqrt{2} < \min\{2\sqrt{5}U + \sqrt{5}, 2\sqrt{5}V + \sqrt{5}\}$$

is satisfied for all values of $U, V \in [0, 1)$, and consequently $f_2(r) > \max(f_3(r), f_4(r))$ provided $r \geqslant 13$. Also, we have

$$\sqrt{2} = 2\sqrt{2}T + \sqrt{2} < 2S + 1 ,$$

if, and only if, $S > (\sqrt{2}-1)/2$. Hence $N_2(x)$ is asymptotic to the number of n for which $\sqrt{(2n^2+2n+1)} \leqslant x$ and $\{\sqrt{(2n^2+2n)}\} > (\sqrt{2}-1)/2$. It follows that $N_2(x) \sim N_2'(x)$, which was to be shown.

Now we show that $N_3(x) \sim c_3 x$. If $r = \sqrt{(5n^2+4n+1)}$ for a positive integer n, then $U = 0$ and so $V = \frac{4}{5}$ and $f_3(r) > f_4(r)$ provided $r \geqslant 3$. Hence $N_3(x)$ is asymptotic to the number of n for which $\sqrt{(5n^2+4n+1)} \leqslant x$ and

$$\sqrt{5} < \min\{2S+1, 2\sqrt{2}T + \sqrt{2}\} .$$

It follows that $N_3(x) \sim N_3'(x)$, where $N_3'(x)$ is the number of $n \leqslant x/\sqrt{5}$ such that

$$\{\sqrt{5}(n+\tfrac{2}{5})\} > \frac{\sqrt{5}-1}{2} \qquad \text{and} \qquad \left\{\frac{\sqrt{5}}{\sqrt{2}}(n+\tfrac{2}{5}) - \tfrac{1}{2}\right\} > \frac{\sqrt{5}-\sqrt{2}}{2\sqrt{2}} .$$

Since 1, $\sqrt{5}$, $\sqrt{\frac{5}{2}}$ are linearly independent over the rationals, it follows from the multidimensional version of Weyl's Theorem ([**3**, Satz 4], [**2**, §1.6]) that

$$N_3'(x) \sim \left(1 - \frac{\sqrt{5}-1}{2}\right)\left(1 - \frac{\sqrt{5}-\sqrt{2}}{2\sqrt{2}}\right)\frac{x}{\sqrt{5}} = c_3 x .$$

The same argument shows that $N_4(x) \sim c_3 x$.

§5. *Occurrence of ties.* We noted in §1 that for $\sqrt{34} \leqslant r < \sqrt{37}$ there are two essentially different pairs of vectors for which $\theta(r)$ is attained, namely (4, 1), (5, 1) and (3, 2), (5, 3). We show that this occurrence is rare. Let T be the set of positive numbers r for which $\theta(r)$ is attained by two incongruent pairs of vectors. Since Theorem 2 gives $d_1 + d_2 + d_3 + d_4 = 1$, we have

$$t(x) \overset{\text{def}}{=} m\{r \leqslant x : r \in T\} = o(x), \tag{12}$$

i.e. the chance of a randomly chosen positive number r being in T is 0. In this section we show that $t(x) = O(\log x)$. Although it appears likely that T is unbounded, *i.e.* that ties occur infinitely often, we cannot prove this.

Which classes can be involved in a tie? Class 2 pairs cannot tie with pairs from any other class because, as Table 2 shows, $\cot\theta$ is even for class 2 pairs but odd for pairs from classes 1, 3, and 4. Next, there are no ties between class 3 and class 4 pairs since the equation

$$5n_3^2 - n_3 - 1 = 5n_4^2 + n_4 - 1$$

has no solution in positive integers n_3 and n_4. Thus the only possibilities are a class 1–class 3 tie and a class 1–class 4 tie.

Suppose that we have a class 1–class 4 tie for the radius r, *i.e.* the pairs

$$(n_1 - 1, 1), (n_1, 1) \quad \text{and} \quad (2n_4 - 1, n_4), (2n_4 + 1, n_4 + 1)$$

determine the same angle $\theta(r)$. Then we have $n_1^2 + 1 < 5n_4^2 + 6n_4 + 2 \leqslant r^2$, since

$$f_1(\sqrt{(n_1^2+1)}) > f_4(\sqrt{(n_1^2+1)})$$

as we noted in §4. Thus we may assume that $r = \sqrt{(5n_4^2 + 6n_4 + 2)}$.

The class 1–class 4 tie implies that

$$n_1^2 - n_1 + 1 = 5n_4^2 + n_4 - 1 \,. \tag{13}$$

Moreover, since the relevant pair in class 2 does not determine $\theta(r)$, we have

$$2n_2^2 < 5n_4^2 + n_4 - 1 \,. \tag{14}$$

Conversely conditions (13) and (14) imply that there is a class 1–class 4 tie. (One must show that (13) implies that $\sqrt{((n_1+1)^2+1)} > r$.)

Letting $j = 2n_1 - 1$, $k = 10n_4 + 1$, we see that (13) is equivalent to

$$k^2 - 5j^2 = 36 \,. \tag{15}$$

All solutions of (15) with j odd and $k \equiv 1 \pmod{10}$ are given by expanding

$$(21 + 9\sqrt{5})(161 + 72\sqrt{5})^i \,, \qquad i = \ldots -1, 0, 1, 2, \ldots$$

as $a_i + b_i\sqrt{5}$ and then taking $k_i = |a_i|$, $j_i = |b_i|$. This can be seen by noting that $\frac{1}{2}(1+\sqrt{5})$ is a fundamental unit for $Q(\sqrt{5})$ and that all integers in $Q(\sqrt{5})$ with norm $+36$ have the form $\pm 6(\frac{1}{2}(1+\sqrt{5}))^{2l}$ for l any integer. All solutions of (15) with our side conditions and $k, j > 0$ will occur when $l \equiv 2 \pmod 6$.

It follows that

$$k_i = (21 + 9\sqrt{5})(161 + 72\sqrt{5})^i/2 + (21 - 9\sqrt{5})(161 - 72\sqrt{5})^i/2$$

or

$$k_i = \lceil (21 \pm 9\sqrt{5})(161 \pm 72\sqrt{5})^i/2 \rceil \,,$$

where $\lceil x \rceil = -[-x]$, the least integer not below x, and the $\pm$ signs are taken the same as the sign of i. For each such k_i we take $n_4 = (k_i - 1)/10$; with this choice (13) is satisfied for an appropriate n_1.

Since the n_4's which satisfy (13) lie in a near geometric progression, the number of ties achieved for all radii $r \leqslant x$ is $O(\log x)$. Also, each r interval for which $\theta(r)$ is achieved by a pair of class 4 vectors is of length at most 1.

The situation for class 1–class 3 ties is entirely analogous. (In fact we achieve (15) here too, but this time require $k \equiv -1 \pmod{10}$. The first occurrence of a class 1–class

3 tie is for the pairs (83, 1), (82, 1) and (73, 36), (75, 37).) Thus we have shown that $t(x) = O(\log x)$.

We have given all the solutions of (13). To show that ties occur for angles which are actually minimal, one must show that solutions of (13) also satisfy (14). The methods of §3 show that (14) holds, if $\sqrt{5} < 2\sqrt{2}t+\sqrt{2}+o(1)$, where $t = \{(r/\sqrt{2})-\frac{1}{2}\}$. This time, however, r is determined as a member of a near geometric progression, and we cannot assert that the t's are uniformly distributed as $r \to \infty$ through such a sequence. If the distribution is uniform in this case, then an arbitrary solution of (13) will yield a tie with probability $1-(\sqrt{5}-\sqrt{2})/(2\sqrt{2}) = 0{\cdot}709431$. In fact, assuming uniform distribution, we have $t(x) \sim c \log x$ for an appropriate positive constant c.

Table 3. *Data*

Maximal Norm	*Asymptotic Direction*	*Number of Occurrences**	*Total Length**	*Length Distribution**
58·69	(1, 0)	58	42·49	·7240
	(1, 1)	31	12·29	·2094
	(2, 1−)	6	1·19	·0203
	(2, 1+)	7	1·24	·0211
	other	4	1·08	·0184
114	(1, 0)	113	83·92	·7361
	(1, 1)	62	24·07	·2111
	(2, 1−)	14	2·43	·0213
	(2, 1+)	14	2·49	·0218
	other	4	1·08	·0095
224·94	(1, 0)	224	166·45	·7400
	(1, 1)	124	48·24	·2145
	(2, 1−)	28	4·53	·0201
	(2, 1+)	27	4·63	·0206
	other	4	1·08	·0048
447	(1, 0)	446	332·18	·7431
	(1, 1)	249	95·84	·2144
	(2, 1−)	55	9·16	·0205
	(2, 1+)	53	8·75	·0196
	other	4	1·08	·0024

* ties are counted with multiplicity

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