COPRIME PERMUTATIONS

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ABSTRACT. Let $C(n)$ denote the number of permutations σ of $[n] = \{1, 2, \ldots, n\}$ such that $\gcd(j, \sigma(j)) = 1$ for each $j \in [n]$. We prove that for n sufficiently large, $n!/3.73^n < C(n) < n!/2.5^n$.

1. INTRODUCTION

Several papers, some recent, have dealt with coprime matchings between two sets of n consecutive integers; that is a matching where corresponding pairs are coprime. For example in a paper [12] with Selfridge, we showed such a matching always exist if one of the intervals is $[n] = \{1, 2, \ldots, n\}$. In Bohman and Peng [2] it is shown that a matching always exists if n is even and the numbers involved are not too large as a function of n, with an interesting application to the lonely runner problem in Diophantine approximations. Their result was somewhat strengthened in [11].

The current paper considers the situation when both intervals are $[n]$. In this case it is trivial that a coprime matching exists, just take the cyclic permutation $(1, 2, \ldots, n)$. So instead we consider the enumeration problem. Let $C(n)$ denote the number of permutations σ of $[n]$ where $\gcd(j, \sigma(j)) = 1$ for each $j \in [n]$. This problem was considered in Jackson [5] where $C(n)$ was enumerated for $n \leq 24$. For example,

$$C(24) = 1{,}142{,}807{,}773{,}593{,}600.$$

After factoring his values, Jackson notes the appearance of sporadically large primes, which indicates there may not be a simple formula. The sequence also has an OEIS page, see [9], where the value of $C(25)$, due to A. P. Heinz, is presented (and the value for $C(16)$ is corrected). There are also links to further computations, especially those of Locke. In Section 6 we discuss how $C(n)$ can be computed and verify Locke's values.

Our principal result is the following.

Theorem 1. *For all large n, $n!/3.73^n < C(n) < n!/2.5^n$.*

Important in the proof of the lower bound is a numerically explicit estimation of the distribution function for $\varphi(n)/n$, where φ is Euler's function.

It would seem likely that there is a constant c with $2.5 \leq c \leq 3.73$ with $C(n) = n!/(c + o(1))^n$ as $n \to \infty$. In Section 5 we give some thoughts towards this possibility. (After a draft of this paper was posted on arXiv, Ashwin Sah and Mehtaab Sawhney, using different methods, have proved this, with the constant $c = c_0 = 2.65044\ldots$ that is identified in sub-section 5.1.)

In Section 6 we discuss the numerical calculation of $C(n)$. Finally, in Section 7 we discuss the number of permutations σ of $[n]$ where $\sigma(1) = 1$ and for $2 \leq j \leq n$, $\gcd(j, \sigma(j)) > 1$. It is possible here that the number of them is $O(n!/C^n)$ for every fixed C.

2. PRELIMINARIES

Regarding notation, we have

$$[n] = \{1, 2, \ldots, n\}, \quad [n]_o = \{1, 3, \ldots, 2n - 1\}.$$

Thus, $[n]_o$ is the set of the first n odd positive integers. Let $C_0(n)$ denote the number of one-to-one functions

$$f : [n]_o \longrightarrow [n]$$

such that each $\gcd(i, f(i)) = 1$. Similarly, let $C_1(n)$ denote the number of one-to-one functions

$$f : [n] \longrightarrow [n + 1]_o$$

such that each $\gcd(i, f(i)) = 1$.

Lemma 1. *We have $C(2n) = C_0(n)^2$ and for $n \geq 2$, $2C_0(n - 1)^2 \leq C(2n + 1) \leq C_1(n)^2$.*

Proof. Let σ be a coprime permutation of $[2n]$. Then σ maps evens to odds and odds to evens, so that σ corresponds to a pair of coprime matchings σ_0, σ_1 where σ_0 maps $\{2, 4, \ldots, 2n\}$ to $\{1, 3, \ldots, 2n-1\}$ and σ_1 maps $\{1, 3, \ldots, 2n-1\}$ to $\{2, 4, \ldots, 2n\}$. Then $f(2i-1) = \frac{1}{2}\sigma_1(2i-1)$ is one of the maps counted by $C_0(n)$ and so is $g(2i-1) = \frac{1}{2}\sigma_0^{-1}(2i-1)$. Conversely, each such pair of maps corresponds to a coprime permutation σ of $[2n]$. This proves that $C(2n) = C_0(n)^2$.

The upper bound for $C(2n + 1)$ follows in the same way. Let σ be a coprime permutation of $[2n + 1]$ and let σ_0 be σ restricted to even

numbers. Then define $f_\sigma(i) = \sigma_0(2i)$, so that f_σ is one of the functions counted by $C_1(n)$. Note that there is some $a \in \{1, 3, \ldots, 2n+1\}$ with $\sigma(a)$ odd, but all other members b of $\{1, 3, \ldots, 2n+1\}$ have $\sigma(b)$ even. Let σ_1 be σ restricted to $S_{a,n} := [n+1]_o \setminus \{a\}$ and let $g_\sigma(2i-1) = \frac{1}{2}\sigma_1(2i-1)$ for $i \in S_{a,n}$. Then g_σ^{-1} is one of the functions counted by $C_1(n)$. Note that if τ is a coprime permutation of $[2n+1]$ such that $f_\tau = f_\sigma$ and $g_\tau = g_\sigma$, then $\tau = \sigma$. This proves that $C(2n+1) \leq C_1(n)^2$. Note that the proof ignores the condition $\gcd(a, \sigma(a)) = 1$, so it only gives an upper bound.

For the lower bound, note that $C(2n+1) \geq 2C(2n-2)$. Indeed, corresponding to a coprime permutation of $[2n-2]$ we augment it with either the cycle $(2n-1, 2n, 2n+1)$ or its inverse, giving two coprime permutations of $[2n+1]$. The lower bound in the lemma for $C(2n+1)$ now follows from the first part of the lemma. □

We remark that the sequence $C(1), C(2), \ldots$ is not monotone, but it is monotone restricted to integers of the same parity. Indeed, augmenting a coprime permutation of $[n]$ with the cycle $(n+1, n+2)$ gives a coprime permutation of $[n+2]$, so that $C(n) \leq C(n+2)$.

Since $C_0(n) \leq n!$ and $C_1(n) \leq (n+1)!$, Lemma 1 immediately gives us that $C(2n) \leq (n!)^2$ and $C(2n+1) \leq (n+1)!^2$. With Stirling's formula this gives $C(n) \leq n!/(2 + o(1))^n$ as $n \to \infty$. Note that this argument considers only parity. By bringing in 3, 5, etc., we can improve this upper bound. In Section 5 we begin this process and show that $C(n) < n!/(5/2)^n$ for all large n.

It is much harder to get a comparable lower bound for $C(n)$, and this is our undertaking in the next two sections. From the thoughts above it suffices to get a lower bound for $C_0(n)$. The lower bound in Theorem 1 is a consequence of the following result.

Theorem 2. *For all large n, $C_0(n) \geq n!/1.864^n$.*

3. The distribution function

Let $\omega(n)$ denote the number of distinct prime factors of n.

Lemma 2. *For positive integers m, n, the number of $j \leq n$ with $\gcd(j, m) = 1$ is within $2^{\omega(m)-1}$ of $(\varphi(m)/m)n$.*

Proof. The result is clear if $m = 1$, so assume that $m > 1$. With μ the Möbius function, the exact number of j's is

$$\sum_{\substack{d|m}} \sum_{\substack{j \leq n \\ d|j}} \mu(d) = \sum_{d|m} \left(\frac{\mu(d)}{d} n + \mu(d)\theta_d \right),$$

where $0 \leq \theta_d < 1$. The sum of the main terms is $(\varphi(m)/m)n$. There are $2^{\omega(m)}$ error terms $\mu(d)\theta_d$ with $\mu(d) \neq 0$, and since $m > 1$, half of them are ≥ 0 and half are ≤ 0. So the sum of the error terms has absolute magnitude $< 2^{\omega(m)-1}$. $\qquad \square$

Corollary 1. *For $m \leq n$, the number of $j \leq n$ with $\gcd(j, m) = 1$ is greater than $(\varphi(m)/m)n - \sqrt{n}$.*

Proof. A short induction argument shows that $\sqrt{m} > 2^{\omega(m)-1}$, so the result follows directly from the lemma. $\qquad \square$

Lemma 3. *For all large n we have*

$$\sum_{\substack{m<2n \\ m\,\text{odd}}} \left(\frac{m}{\varphi(m)}\right)^2 < 1.78n.$$

Proof. Define a multiplicative function h with $h(p) = (2p-1)/(p-1)^2$ for each prime p and $h(p^a) = 0$ for $a \geq 2$. Then

$$\sum_{\substack{m<2n \\ m\,\text{odd}}} \left(\frac{m}{\varphi(m)}\right)^2 = \sum_{\substack{m<2n \\ m\,\text{odd}}} \sum_{d|m} h(d)$$

$$= \sum_{\substack{d<2n \\ d\,\text{odd}}} h(d) \sum_{\substack{j<2n/d \\ j\,\text{odd}}} 1 = \sum_{\substack{d<2n \\ d\,\text{odd}}} h(d) \left(\frac{n}{d} + O(1)\right).$$

The main term is

$$< n \prod_{p>2} \left(1 + \frac{2p-1}{(p-1)^2 p}\right),$$

and this infinite product converges to a constant smaller than 1.7725. For the error term a simple calculation shows that it is $O((\log n)^2)$, so our conclusion follows. $\qquad \square$

Let $\delta_\varphi(\alpha)$ be the distribution function for $\varphi(m)/m$; that is, for $0 \leq \alpha \leq 1$,

$$\delta_\varphi(\alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{m \leq n \\ \varphi(m)/m \leq \alpha}} 1.$$

It is known after various papers of Schoenberg, Behrend, Chowla, Erdős, and Erdős–Wintner that the limit exists, $\delta_\varphi(0) = 0$, $\delta_\varphi(1) = 1$, and δ_φ is strictly increasing and continuous. In addition, at a dense set of numbers in $[0,1]$, namely the values of $\varphi(m)/m$, the distribution function δ_φ has an infinite left derivative. This all can be generalized

to odd numbers. For $0 \leq \alpha \leq 1$, let $D(\alpha, n)$ denote the number of odd $m < 2n$ with $\varphi(m)/m \leq \alpha$. As with δ_φ,

$$\delta(\alpha) := \lim_{n \to \infty} D(\alpha, n)/n$$

exists, with δ continuous and strictly increasing on $[0, 1]$, with $\delta(0) = 0$ and $\delta(1) = 1$. By extending it to take the value 1 when $\alpha > 1$, we have

$$\delta_\varphi(\alpha) = \frac{1}{2}(\delta(\alpha) + \delta(2\alpha)),$$

as noted in [12]. In particular, for $\frac{1}{2} \leq \alpha \leq 1$,

(1) $$\delta(\alpha) = 2\delta_\varphi(\alpha) - 1.$$

A consequence of the argument in [12] is that $\delta(\alpha) \leq \alpha$ on $[0, 1]$. We shall need a somewhat stronger version of this inequality. In particular, note that Lemma 3 immediately gives

(2) $$\delta(\alpha) < 1.78\alpha^2,$$

which is stronger than $\delta(\alpha) \leq \alpha$ for $\alpha < 1/2$. It is certainly possible to get improvements on (2) by averaging higher moments of $m/\varphi(m)$, as was done in [6], which would lead to small improvements on our lower bound for $C(n)$.

We shall also need some estimates for $\delta(\alpha)$ when α is close to 1, and for this we use an argument of Erdős [3, Theorem 3]. There he shows, essentially, that $1 - \delta(1 - \epsilon) \sim 2/(e^\gamma |\log \epsilon|)$ as $\epsilon \to 0$, where γ is Euler's constant. We will need an estimate with somewhat more precision.

Let

$$\delta(\alpha, n) = \frac{1}{n} D(\alpha, n), \quad M(x) = \prod_{3 \leq p \leq x} \left(1 - \frac{1}{p}\right), \quad s_{j,x} = \sum_{4^{j-1}x < p \leq 4^j x} \frac{1}{p}.$$

Lemma 4. *Uniformly for $2 \leq x \leq \log n$ we have*

$$1 - \delta(1 - 1/x, n) \leq M(x) - 1/\sqrt{n}$$

and

$$1 - \delta(1 - 1/x, n) \geq M(2x) \left(1 - \sum_{j \geq 1} \frac{s_{j,2x}^{j+1}}{(j+1)!}\right) + O\left(\frac{1}{(\log n)^{\log \log \log n}}\right).$$

Proof. Note that $1 - \delta(1 - 1/x, n)$ denotes the fraction of numbers $m < 2n$ with $\varphi(m)/m > 1 - 1/x$ (all such m are odd). In fact, such numbers are not divisible by any prime $p \leq x$, which with Lemma 2 gives the upper bound.

For the lower bound we count the numbers $m < 2n$ that are not divisible by any prime $p \leq 2x$ and also divisible by at most j distinct

primes from each interval $I_j := (4^{j-1} \cdot 2x, \, 4^j \cdot 2x]$. Indeed, if m is such a number, then

$$\frac{\varphi(m)}{m} > \prod_{j \geq 1} \left(1 - \frac{1}{4^{j-1} \cdot 2x} \right)^j > 1 - \sum_{j \geq 1} \frac{j}{4^{j-1} \cdot 2x} = 1 - \frac{8}{9x}.$$

Let A_j be the set of products of $j+1$ distinct primes from I_j. For $a \in A_j$, the number of odd numbers $m < 2n$ with $a \mid m$ and m not divisible by any prime to $2x$, is by Lemma 2, within $2^{\pi(2x)-1}$ of $M(2x)n/a$. Note too that the sum of $1/a$ for $a \in A_j$ is at most $s_{j,2x}^{j+1}/(j+1)!$, by the multinomial theorem. Let $\pi(I_j)$ denote the number of primes in I_j. Thus, the number of odd $m < 2n$ not divisible by any prime to $2x$ and divisible by some $a \in A_j$ is uniformly

$$M(2x)ns_{j,2x}^{j+1}/(j+1)! + O\left(2^{\pi(2x)} \binom{\pi(I_j)}{j+1} \right).$$

The binomial coefficient here is bounded by $O(4^{j(j+1)}x^{j+1})$ using only that $\pi(I_j) < 4^j x$. Note that for $j \leq \log \log n$ this expression is $O_\epsilon(n^\epsilon)$ for any $\epsilon > 0$, as is $2^{\pi(2x)}$, so that the number of integers $m < 2n$ not divisible by any prime $p \leq 2x$ yet divisible by some $a \in A_j$ for $j \leq \log \log n$ is at least

$$M(2x)n \sum_{j \leq \log \log n} \frac{s_{j,2x}^{j+1}}{(j+1)!} + O(n^{1/2}).$$

Thus,

$$n - D(1 - 1/x, n) \geq M(2x)n \left(1 - \sum_{j \leq \log \log n} \frac{s_{j,2x}^{j+1}}{(j+1)!} \right)$$
$$- 2n \sum_{j > \log \log n} \frac{s_{j,2x}^{j+1}}{(j+1)!} + O(n^{1/2}).$$

Since $s_{j,2x} = O(1/j)$, we have

$$\sum_{j > \log \log n} \frac{s_{j,2x}^{j+1}}{(j+1)!} = O(1/(\log n)^{\log \log \log n}).$$

Thus, our count is

$$\geq M(2x)n \left(1 - \sum_{j \geq 1} \frac{s_{j,2x}^{j+1}}{(j+1)!} \right) + O(n/(\log n)^{\log \log \log n}),$$

which gives our lower bound. $\square$

Corollary 2. *Uniformly for $2 \leq x \leq \log n$, we have as $n \to \infty$,*

$$1 - \delta(1 - 1/x, n) \leq \frac{2}{e^\gamma \log x} \left(1 + \frac{1}{2(\log x)^2} + o(1) \right).$$

Further, for $150 \leq x \leq \log n$ and n sufficiently large,

$$1 - \delta(1 - 1/x, n) \geq \frac{2}{e^\gamma \log(2x)} \left(1 - \frac{7}{4(\log(2x))^2} \right).$$

Proof. By Rosser and Schoenfeld [13, (3.26)] we have

$$M(x) < \frac{2}{e^\gamma \log x} \left(1 + \frac{1}{2(\log x)^2} \right),$$

so our first assertion follows from the first part of Lemma 4. Further, using [13, (3.25)] we have

$$M(2x) > \frac{2}{e^\gamma \log(2x)} \left(1 - \frac{1}{2(\log(2x))^2} \right),$$

and so the second part our our assertion will follow from Lemma 4 if we show

$$\sum_{j \geq 1} \frac{s_{j,2x}^{j+1}}{(j+1)!} < \frac{1.4}{(\log(2x))^2}$$

for all sufficiently large x, noting that $1.4 < (2/e^\gamma)1.25$. Using [13, (3.17),(3.18)], we have

$$s_{j,2x} < \log\log(2 \cdot 4^j x) - \log\log(2 \cdot 4^{j-1}x) + \frac{1}{(\log(2 \cdot 4^{j-1}x))^2}$$

$$< \frac{\log 4}{\log(2 \cdot 4^{j-1}x)} + \frac{1}{(\log(2 \cdot 4^{j-1}x))^2} \leq \frac{\log 4}{\log(2x)} + \frac{1}{(\log(2x))^2} =: s.$$

Thus, using $x \geq 150$,

$$\sum_{j \geq 1} \frac{s_{j,2x}^{j+1}}{(j+1)!} < e^s - 1 - s < 0.55s^2$$

and $s^2 < 2.5/(\log(2x))^2$, so our claim follows. $\qquad \square$

In addition, we shall use the following numerical bounds. The first of these follows from Kobayashi [7], the last two from Lemma 4, and

the others from Wall [15].

$$0.02240 < \delta(0.5) < 0.02352,$$
$$0.1160 < \delta(0.6) < 0.1624,$$
$$0.3556 < \delta(0.7) < 0.3794,$$
(3) $$\qquad 0.4808 < \delta(0.8) < 0.5120,$$
$$0.5644 < \delta(0.9) < 0.6310$$
$$0.7593 < \delta(0.99) < 0.7949$$
$$0.8380 < \delta(0.999) < 0.8539.$$

4. THE LOWER BOUND

We partition $(0, 1]$ into consecutive intervals

$$(\alpha_0, \alpha_1], (\alpha_1, \alpha_2], \ldots, (\alpha_{k-1}, \alpha_k], \text{ where } 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1.$$

The parameter k will depend gently on n, namely $k = O(\log \log n)$. The partition of $(0, 1]$ will correspond to a partition of $[n]$ into subsets as follows. For $j = 0, 1, \ldots, k - 1$, let

$$S_j = \big\{m \in \{1, 3, \ldots, 2n - 1\} : \alpha_j < \varphi(m)/m \le \alpha_{j+1}\big\}.$$

In getting a lower bound for $C_0(n)$, we show that there are many ways to assign coprime companions for each member m of $\{1, 3, \ldots, 2n - 1\}$ that do not overlap with the choices for other values of m. In particular, we organize the odd numbers $m < 2n$ by increasing size of $\varphi(m)/m$, and so organize them into the sets $S_1, S_2, \ldots$. In particular, we will choose the parameters α_j in such a way that there are more ways to assign coprime companions for $m \in S_j$ than there are members in all of the sets S_i for $i \le j$ combined.

For an odd number $m < 2n$ let $F(m, n)$ denote the number of integers in $[n]$ coprime to m. Suppose $0 < \alpha < \beta < 1$ and we wish to find coprime assignments for members of

$$S = \{m \text{ odd} : m < 2n, \ \varphi(m)/m \in (\alpha, \beta]\} = \{m_1, m_2, \ldots, m_t\},$$

where $t = \#S = D(\beta, n) - D(\alpha, n)$. Let $M = \lceil \alpha n - \sqrt{n} \rceil$, so that for each $m \in S$ we have $F(m, n) \ge M$, via Corollary 1. Assume that those odd $m < 2n$ with $\phi(m)/m \le \alpha$ already have their coprime assignments. Then m_1 can be assigned to at least $M - D(\alpha, n)$ numbers in $[n]$, m_2 can be assigned to at least $M - 1 - D(\alpha, n)$ numbers in $[n]$, etc. In all, the numbers in S have at least

(4) $$\frac{(M - D(\alpha, n))!}{(M - D(\alpha, n) - \#S)!} = \frac{(M - D(\alpha, n))!}{(M - D(\beta, n))!}$$

coprime assignments that do not interfere with those for $\varphi(m)/m \leq \alpha$. If $0 < a < b < 1$ and an, bn are integers, then

$$(an - bn)! = \exp((a-b)n(\log n - 1) + (a-b)n\log(a-b) + O(\log n)).$$

Let $f(x) = x\log x$. Thus, the expression in (4) is equal to

$$\exp((\delta(\beta,n) - \delta(\alpha,n))n(\log n - 1) + E(\alpha,\beta,n)n + O(\log n)),$$

where

$$E(\alpha,\beta,n) = f(\alpha - \delta(\alpha,n)) - f(\alpha - \delta(\beta,n)).$$

We thus will have that $C_0(n) \geq n! \exp(nE + O(k\log n))$, where

$$(5) \qquad E = \sum_{1 \leq i \leq k-1} (f(\alpha_i - \delta(\alpha_i,n)) - f(\alpha_i - \delta(\alpha_{i+1},n))).$$

(We will choose $\alpha_1 = 1/\log\log n$ and for n sufficiently large, every odd $m < 2n$ will have $\varphi(m)/m > \alpha_1$, so the interval $(0, \alpha_1]$ does not contribute.)

The sum in (5) is almost telescoping. In particular the density $\delta(\alpha_{i+1}, n)$ when $1 \leq i \leq k-2$ appears twice, the two f-values being

$$-f(\alpha_i - \delta(\alpha_{i+1},n)) + f(\alpha_{i+1} - \delta(\alpha_{i+1},n)).$$

We do not have a completely accurate evaluation for $\delta(\alpha_{i+1}, n)$ nor for the limiting value of $\delta(\alpha_{i+1})$, but we do have a fairly narrow interval where this limit lives. Note that the expression

$$-f(\alpha_i - x) + f(\alpha_{i+1} - x)$$

is decreasing in x when $0 < x < \alpha_i$, so if we use an upper bound for $\delta(\alpha_{i+1}, n)$ in (5), we will get a lower bound for the sum.

4.1. **The interval** $(0, 1/2]$. Let j_0 be the least integer with $2^{j_0} > \log\log n$ and let $\alpha_1 = 1/2^{j_0}$. Further, let $\alpha_j = 2^{j-1}\alpha_1 = 1/2^{j_0-j+1}$, for $j \leq j_0$. This gives the first part of our partition of $(0, 1]$, namely the sets $(\alpha_j, \alpha_{j+1}]$ for $j \leq j_0$ give a partition of $(0, 1/2]$.

Using (2) and the upper bound for $\delta(1/2)$ in (3), we have

$$\delta(1/2^i, n) \leq \min\{1.78/4^i, \, 0.02352\}$$

for all i and all large n. We find the E-sum from (5) for the portion for $(0, 1/2]$ is > -0.0538. So the contribution for this part of the count is greater than

$$(6) \qquad \exp\big(D(1/4, n)(\log n - 1) - 0.0538n\big)$$

for all large n.

4.2. **The interval** $(1/2, 0.999]$. We split the interval $(1/2, 0.999]$ at

$$0.6, \ 0.7, \ 0.8, \ 0.9, \ 0.99.$$

Using the upper bounds for our various densities from (3), we have the E-sum from (5) is

$$
\begin{aligned}
&f(.5 - .02352) - f(0.5 - .1624) + f(.6 - .1624) - f(.6 - .3794) \\
&+ f(.7 - .3794) - f(.7 - .5120) + f(.8 - .5120) - f(.8 - .6310) \\
&+ f(.9 - .6310) - f(.9 - .7949) + f(.99 - .7949) - f(.99 - .8539) \\
&> -0.2873.
\end{aligned}
$$

Thus, the contribution from $(1/2, 0.999]$ is greater than

$$(7) \qquad \exp\big((D(0.999, n) - D(1/4, n)(\log n - 1) - 0.2873n\big)$$

for all large n.

4.3. **The interval** $(0.999, 1 - 1/\log n]$. Let j_1 be the least integer with $10^{j_1} > \log n$. Let $\epsilon_i = 10^{-i}$. We deal with the intervals

$$(1 - \epsilon_{i-1}, 1 - \epsilon_i] \quad \text{for} \quad 4 \le i \le j_1 - 1.$$

For our argument to work we will need to show that $D(1 - \epsilon_i, n) < (1 - \epsilon_{i-1})n - \sqrt{n}$, that is,

$$(8) \qquad\qquad \epsilon_{i-1} n - \sqrt{n} < n - D(1 - \epsilon_i, n) \text{ for } i \ge 4.$$

From Corollary 2 we have

$$n - D(1 - \epsilon_i, n) > \frac{2n}{e^\gamma \log(2/\epsilon_i)} \left(1 - \frac{7}{4(\log(2/\epsilon_i))^2}\right).$$

Note that $\log(2/\epsilon_i) = \log 2 + i \log 10$, so that an expression of magnitude $1/\log(2/\epsilon_i)$ is much larger than ϵ_{i-1} when $i \ge 4$, so we have (8).

We now compute the contribution from the intervals $(1 - \epsilon_{i-1}, 1 - \epsilon_i]$ for $i = 4, 5, \ldots, j_1 - 1$. This is at least

$$\exp(D(1 - \epsilon_{j_1-1}, n) - D(0.999, n)(\log n - 1) + En),$$

where

$$E = \sum_{4 \le i \le j_1 - 1} f(1 - \epsilon_{i-1} - \delta(1 - \epsilon_{i-1}, n) - f(1 - \epsilon_{i-1} - \delta(1 - \epsilon_i, n)).$$

Using our bound 0.8539 for $\delta(0.999)$ from (3) and Corollary 2 for $\delta(1 - \epsilon_i)$ for $i \ge 4$, we have $E > -0.2814$, so the contribution for all large n is at least

$$\exp((D(1 - \epsilon_{j_1-1}, n) - D(0.999, n))(\log n - 1) - 0.2814n).$$

The final interval $(1 - \epsilon_{j_1-1}, 1 - 1/\log n]$ contributes

$$\exp(D(1 - 1/\log n, n) - D(1 - \epsilon_{j_1-1}, n)(\log n - 1) + O(n/\log \log n)),$$

so our total contribution from $(.999, 1 - 1/\log n]$ is at least

(9) $\exp((D(1 - 1/\log n, n) - D(0.999, n)(\log n - 1) - 0.2815n)$

for all large n.

4.4. **The interval** $(1 - 1/\log n, 1]$. We break this interval at $1 - 1/\sqrt{2n}$. It is evident that if $m < 2n$ is odd and $\varphi(m)/m > 1 - 1/\sqrt{2n}$, then $m = 1$ or m is a prime in the interval $(\sqrt{2n}, 2n)$. Thus,

(10) $D(1 - 1/\sqrt{2n}, n) = n - 2n/\log n + O(n/(\log n)^2)$

by the prime number theorem. Thus, $\delta(1 - 1/\sqrt{2n}, n) < 1 - 1/\log n$ for all large n. A calculation shows that the contribution is at least

$$\exp\left(\left(D\left(1 - \frac{1}{\sqrt{2n}}, n\right) - D\left(1 - \frac{1}{\log n}, n\right)\right)(\log n - 1) + E\right),$$

where $E = O(n \log \log \log n / \log \log n)$, this term coming from $f(1 - 1/\log n - \delta(1 - 1/\log n, n))$.

For the final interval, we have already noted that the numbers in $[n]_{\mathrm{o}}$ remaining are 1 and the primes in $(\sqrt{2n}, 2n)$. We follow the argument in [12, Proposition 1]. Label the primes in $(\sqrt{2n}, 2n)$ in decreasing order $p_1, p_2, \ldots, p_t$, so that, by (10), $t = 2n/\log n + O(n/(\log n)^2)$. Each p_i has $< 2n/p_i$ multiples to $2n$, of which $< n/p_i + 1/2$ are odd. Let $u = \lfloor t/2 \rfloor = n/\log n + O(n/(\log n)^2)$, so that $p_u \sim n$. We count assignments for p_i for $i = t, t - 1, \ldots, u$ in order. At each i there are $i + 1$ numbers remaining to be associated with p_i of which at most $n/p_i + 1/2$ are multiples of p_i. So, there are at least $i - \sqrt{n}$ coprime choices for p_i's assignment. Multiplying these counts, we have at least

$$(u - \sqrt{n})^u = \exp(n + O(n/\log n))$$

choices. For each of the remaining primes p_i there are $i + 1$ numbers left as possible assignments, with at most one of these divisible by (actually, equal to) p_i. So the contribution of these primes is $(u - 1)! = \exp(n + O(n/\log n))$. The final number to assign is 1, and it goes freely to the remaining number left. So for this interval we have at least

$$\exp(2n + O(n/\log n))$$

possibilities. By (10) the count can be rewritten as

$$\exp\left(\left(n - D\left(1 - \frac{1}{\sqrt{2n}}, n\right)\right)(\log n - 1) + O\left(\frac{n}{\log n}\right)\right)$$

With the prior calculation, we have at least

(11) $\exp((n - D(1 - 1/\log n, n)(\log n - 1) + O(n \log \log \log n/ \log n))$

assignments.

To conclude the proof we multiply the expressions in (6), (7), (9), and (11), getting at least

$$\exp(n(\log n - 1) - 0.6226n)$$

coprime matchings from $[n]_\mathrm{o}$ to $[n]$ for all large n. Since $e^{0.6226} > 1.8637$, this completes the proof of the lower bound in Theorem 2.

5. The upper bound and a conjecture

For each integer $k \geq 2$, let $C_k(n)$ denote the number of permutations σ of $[n]$ where $\gcd(j, \sigma(j), k!) = 1$ for each $j \in [n]$. Thus, $C(n) \leq C_k(n)$ for every k. In fact, $C(n) = C_k(n)$ when $k \geq n$, but we are interested here in the situation when k is fixed and n is large. We claim that for each fixed $k \geq 2$ there is a positive constant c_k such that $C_k(n) = n!/(c_k + o(1))^n$ as $n \to \infty$.

Here is a possible plan for the proof of this claim. Let K be the product of the primes to k. If $dd' \mid K$, then one can count the number of $m \in [n]$ with $\gcd(m, K) = d$ that get mapped to an m' with $\gcd(m', K) = d'$. The product of all of the positive counts is $n^{O(1)}$, so basically, up to a factor of this shape, the number of permutations is given by those with one optimal suite of counts.

Let I_d be the set of m with $\gcd(m, K) = d$ and let $\beta(d, d')$ be the proportion of members of I_d that get sent to $I_{d'}$ by a given permutation. Then for a fixed d, the numbers $\beta(d, d')$ have sum 1 for $d' \mid K/d$, and sum 1 for a fixed d' and $d \mid K/d'$. One can start with some suite of proportions $\beta(d, d')$ that are "legal" and consider permutations which approximate these proportions, and see the count as some complicated, but continuous function of the variables $\beta(d, d')$. So, there is an optimal suite of proportions, via calculus, and this gives rise to c_k.

Assume that c_k exists. Note that the sequence (c_k) is monotone nondecreasing and that if $p < p'$ are consecutive primes, then $c_k = c_p$ for $p \leq k < p'$. It follows from our lower bound for $C(n)$ that the numbers c_k are bounded above. Let $c_0 = \lim_{k \to \infty} c_k$.

Conjecture 1. We have $C(n) = n!/(c_0 + o(1))^n$ as $n \to \infty$.

We now prove for $k = 2, 3, 5$ that c_k exists and we compute it. Our value for c_5 gives our upper bound theorem for $C(n)$.

The results in Section 2 largely carry over in the case $k = 2$. Indeed, note that $C_0(n) \leq n!$ and $C_1(n) \leq (n + 1)!$, so that $C(2n) \leq n!^2$ and

$C(2n + 1) \leq (n + 1)!^2$. From this we immediately get that $C_2(n) \leq n!/(2 + o(1))^n$ as $n \to \infty$. In fact, from the proof of Lemma 1 we have $C_2(2n) = n!^2$ and $C_2(2n + 1) = (n + 1)!^2$, so that $c_2 = 2$.

For $k = 3$, we first deal with $6n$ and count one-to-one functions σ from $\{1, 2, \ldots, 3n\}$ to $\{1, 3, \ldots, 6n - 1\}$ that map multiples of 3 to non-multiples of 3. There are precisely $(2n)!^2/n!$ of them, so $C_3(6n) = ((2n)!^2/n!)^2$. Similarly we get $C_3(6n + 3) = ((2n + 1)!^2/(n + 1)!)^2$, so these two formulas lead to $C_3(n) = n!/(3/2^{1/3} + o(1))$ as $n \to \infty$ with $3 \mid n$. To get to other cases, note that $C_3(n) \leq C_3(n + 2)$ for all n, so we can sandwich n between 2 consecutive multiples of 3 and absorb the error in the "$o(1)$". We thus have

$$c_3 = 2^{-1/3}3 = 2.381101\ldots.$$

The case $k = 5$ is considerably harder. We only treat multiples of 30, the case 15 (mod 30) is similar, and since $C_k(n) \leq C_k(n + 2)$, we can extend to all n readily. The problem is reduced to counting matchings from $[15n]$ to $[15n]_{\mathrm{o}}$ where corresponding terms have gcd coprime to 15. We split $[15n]$ into the n multiples of 15, the $2n$ numbers that are divisible by 5 but not 3, the $4n$ numbers divisible by 3 but not 5, and the $8n$ numbers coprime to 15. We have the corresponding decomposition for $\{1, 3, \ldots, 30n - 1\}$. The first group consisting of the multiples of 15 must be mapped to the numbers coprime to 15, and this can be done in

$$\frac{(8n)!}{(7n)!}$$

ways. The next case we consider is the $2n$ multiples of 5 but not 3. They must be mapped to the numbers coprime to 5, where some of them are mapped to numbers coprime to 15 and the rest of them are mapped to numbers divisible by 3 but not 5. A calculation shows that the most numerous case is when it is half and half, also considering the next step which is to place the multiples of 3 but not 5. So the total will be within a factor $2n$ of this most numerous case, which has

$$\binom{2n}{n} \frac{(7n)!}{(6n)!} \frac{(4n)!}{(3n)!}$$

matchings. For the multiples of 3 but not 5, these are mapped into the union of the remaining $6n$ numbers coprime to 15 and the $2n$ numbers divisible by 5 but not 3, for a total of

$$\frac{(8n)!}{(4n)!}$$

matchings. The remaining $8n$ numbers are all coprime to 15 and can be mapped to the remaining $8n$ numbers in every possible way, giving $(8n)!$ matchings. In all we thus have

$$\frac{(8n)!}{(7n)!}\binom{2n}{n}\frac{(7n)!}{(6n)!}\frac{(4n)!}{(3n)!}\frac{(8n)!}{(4n)!}(8n)!n^{O(1)} = \frac{(8n)!^3(2n)!}{(6n)!(3n)!n!^2}n^{O(1)}$$

matchings. The log of this expression is within $O(\log n)$ of

$$= \exp(15n(\log n - 1) + (24\log 8 + 2\log 2 - 6\log 6 - 3\log 3)n).$$

Our count is then squared and $(30n)!$ is factored out, giving

$$(30n)!\exp((136\log 2 - 18\log 3 - 30\log 30)n + O(\log n)).$$

This then gives that

$$C_5(n) = (n!/c_5^n)n^{O(1)},$$

where

$$c_5 = \exp\Big(-\frac{53}{15}\log 2 + \frac{8}{5}\log 3 + \log 5\Big) = 2^{-53/15}3^{8/5}5 = 2.504521\ldots.$$

5.1. **A possible value for c_0.** Nathan McNew has suggested the following argument. First, for a prime p, let $N_p(n)$ be the number of permutations σ of $[n]$ with each $\gcd(j,\sigma(j),p) = 1$. So the constraint is that the multiples of p get mapped to the non-multiples of p, and so we have

$$N_p(n) = \frac{(\lfloor(1-1/p)n\rfloor!)^2}{\lfloor(1-2/p)n\rfloor!}n^{O(1)}.$$

Then, up to a factor $n^{O(1)}$, we have

$$\frac{n!}{N_p(n)} = \left(\frac{p(p-2)^{1-2/p}}{(p-1)^{2(1-1/p)}}\right)^n,$$

which suggests by independence that

$$c_k = \prod_{p\le k}\frac{p(p-2)^{1-2/p}}{(p-1)^{2(1-1/p)}}.$$

(We the factor at $p=2$ as 2.) This expression agrees with our computation of c_k for k up to 5. And it suggests that c_0 is the infinite product over all primes p, so that $c_0 = 2.65044\ldots$.

TABLE 1. Values of $C_0(n) = \sqrt{C(2n)}$ and r_{2n}.

n	$C_0(n)$	r_{2n}
1	1	1.4142
2	2	1.5651
3	4	1.8860
4	18	1.8276
5	60	1.9969
6	252	2.1044
7	1,860	2.0625
8	9,552	2.1629
9	59,616	2.2260
10	565,920	2.2082
11	4,051,872	2.2707
12	33,805,440	2.3118
13	465,239,808	2.2727
14	4,294,865,664	2.3171
15	35,413,136,640	2.3850
16	768,372,168,960	2.3122
17	8,757,710,173,440	2.3451
18	79,772,814,777,600	2.4122
19	1,986,906,367,584,000	2.3531
20	22,082,635,812,268,800	2.4029
21	280,886,415,019,776,000	2.4374
22	7,683,780,010,315,046,400	2.3905
23	102,400,084,005,498,547,200	2.4278
24	1,774,705,488,555,494,476,800	2.4401
25	40,301,474,964,335,327,232,000	2.4291

6. COMPUTING $C(n)$

In this section we discuss the numerical computation of $C(n)$ for modest values of n. In [9] it is remarked that $C(n)$ has been computed to $n = 30$ by Seiichi Manyama, and extended to $n = 50$ by Stephen Locke, see https://oeis.org/A005326/b005326.txt. We have verified these values using the methods of this section and Mathematica.

As is easy to see, the permanent of the incidence matrix of a bipartite graph of two n-sets gives the number of perfect matchings contained in the graph. Let $\mathbf{B}(n)$ be the $n \times n$ "coprime matrix", where $\mathbf{B}(n)_{i,j} = 1$ when i, j are coprime and 0 otherwise. So, in particular, and as noted by Jackson [5],

(12) $$C(n) = \operatorname{perm}(\mathbf{B}(n)).$$

However, it is not so simple to compute a large permanent, though we do have some algorithms that are better than brute force, for example [14] and [1].

Recall from Lemma 1 that $C(2n) = C_0(n)^2$, where $C_0(n)$ is the number of coprime matchings between $[n]$ and $[n]_{\mathrm{o}}$. Thus, $C(2n)$ can be obtained from an $n \times n$ permanent, which is considerably easier than the more naive $2n \times 2n$ permanent required when applying (12) to $C(2n)$.

There is a similar reduction for computing $C(2n + 1)$. For each $a \in [n+1]_{\mathrm{o}}$, let $C_{(a)}(n)$ denote the number of coprime matchings between $[n]$ and $[n+1]_{\mathrm{o}} \setminus \{a\}$. Then $C_1(n) = \sum_{a \in [n+1]_{\mathrm{o}}} C_{(a)}(n)$ and

$$C(2n + 1) = \sum_{a \in [n+1]_{\mathrm{o}}} \sum_{\substack{b \in [n+1]_{\mathrm{o}} \\ \gcd(a,b)=1}} C_{(a)}(n) C_{(b)}(n).$$

Thus, $C(2n+1)$ can be easily computed from $n+1$ permanents of size $n \times n$.

Let $r_n = (n!/C(n))^{1/n}$, so that $C(n) = n!/r_n^n$. We have shown that for all large n we have $2.5 < r_n < 3.73$. In the following tables we have computed the actual values of r_n for $n \leq 50$ rounded to 4 decimal places.

It is easy to see that $C_0(n)$ is the number of partitions of $[2n]$ into coprime unordered pairs. This has its own OEIS page: A009679, and has been enumerated there up to $n = 30$.

7. Anti-coprime permutations

One might also wish to consider permutations σ of $[n]$ where each $\gcd(j, \sigma(j)) > 1$. Of course, none exist, since $1 \in [n]$. Instead we can count the number $A(n)$ where $\gcd(j, \sigma(j)) > 1$ for $2 \leq j \leq n$. This seems like an interesting problem. We can prove the following lower bound.

Proposition 1. *As $n \to \infty$, we have*

(13) $$A(n) \geq n!/\exp((e^{-\gamma} + o(1))n \log \log n).$$

We sketch the proof. Let $\epsilon_n = 1/\sqrt{\log \log n}$ and let $g(x) = \prod_{p<x}(1 - 1/p)$, so that $g(x)$ is similar to the function $M(x)$ we considered earlier. For each prime $p < n^{\epsilon_n}$ consider the set $L_n(p)$ of integers $m \leq n$ with least prime factor p, and let

$$\lambda(p, n) = \frac{1}{n} \# L_n(p).$$

TABLE 2. Values of $C(n)$ for n odd and r_n.

n	$C(n)$	r_n
1	1	1
3	3	1.2599
5	28	1.3378
7	256	1.5307
9	3,600	1.6696
11	129,774	1.6834
13	3,521,232	1.7776
15	60,891,840	1.9444
17	8,048,712,960	1.8761
19	425,476,094,976	1.9372
21	12,474,417,291,264	2.0648
23	2,778,580,249,611,264	2.0090
25	172,593,628,397,420,544	2.0804
27	17,730,530,614,153,986,048	2.1159
29	4,988,322,633,552,214,818,816	2.0841
31	427,259,978,841,815,654,400,000	2.1466
33	57,266,563,000,754,880,493,977,600	2.1818
35	14,786,097,120,330,296,843,693,260,800	2.1798
37	3,004,050,753,199,657,126,879,764,480,000	2.1988
39	536,232,134,065,318,935,894,365,552,640,000	2.2295
41	274,431,790,155,416,580,402,144,584,785,920,000	2.2058
43	51,681,608,012,142,138,983,265,921,023,262,720,000	2.2409
45	7,417,723,304,411,612,192,092,096,851,178,291,200,000	2.2918
47	7,896,338,788,322,918,879,731,318,625,512,774,041,600,000	2.2459
49	1,989,208,671,980,285,257,956,064,090,726,080,876,380,160,000	2.2743

Note that

$$\bigcup_{p<n^{\epsilon_n}} L_n(p)$$

is the set of integers with least prime factor $< n^{\epsilon_n}$, so the number of integers $m \leq n$ not in this union is $O(n/(\epsilon_n \log n))$. In particular,

(14)
$$\sum_{p<n^{\epsilon_n}} \lambda(p,n) = 1 + O(1/(\epsilon_n \log n)).$$

Each of the $(\#L_n(p))!$ permutations of $L_n(p)$ is anti-coprime, and gluing these together for $p < n^{\epsilon_n}$ and having the remaining elements of $[n]$ as fixed points, gives an anti-coprime permutation of $[n]$. So we have

$$A(n) \geq \prod_{p<n^{\epsilon_n}} (\#L_n(p))!.$$

Thus, by the inequality $k! > (k/e)^k$,

$$A(n) \geq \exp\left(\sum_{p < n^{\epsilon_n}} \lambda(p,n)n(\log n + \log(\lambda(p,n)) - 1)\right)$$

$$= \exp\left(\sum_{p < n^{\epsilon_n}} \lambda(p,n)n(\log n - 1) + nE\right)$$

where

$$E = \sum_{p < n^{\epsilon_n}} \lambda(p,n)\log(\lambda(p,n)).$$

Note that by (14)

$$\sum_{p < n^{\epsilon_n}} \lambda(p,n)n(\log n - 1) = n\log n + O(n/\epsilon_n).$$

To deal with E, we have that for $p < n^{1/\epsilon_n}$,

$$\lambda(p,n) = \frac{g(p)}{p}(1 + O(e^{-1/\epsilon_n}))$$

uniformly for large n. Indeed each $m \in L_n(p)$ is of the form pk where $k \leq n/p$ is an integer not divisible by any prime $q < p$. Such integers k are easily counted by the fundamental lemma of either Brun's or Selberg's sieve, which gives the above estimate.

We have $g(p)$ of magnitude $1/\log p$, in fact $g(p) = 1/(e^\gamma \log p)(1 + O(1/\log p))$. Thus, we have

$$E = \sum_{p < n^{\epsilon_n}} \frac{g(p)}{p}(\log(g(p)/p)(1 + O(e^{-1/\epsilon_n}))$$

$$= \sum_{p < n^{\epsilon_n}} \frac{1}{e^\gamma p \log p}(-\log p - \log\log p - \gamma)(1 + O(1/\log p) + O(e^{-1/\epsilon_n}))$$

$$= -\sum_{p < n^{\epsilon_n}} \frac{1}{e^\gamma p}(1 + O(e^{-1/\epsilon_n}) + O(1).$$

It remains to note that

$$\sum_{p < n^{\epsilon_n}} \frac{1}{p} = \log\log n - \log\epsilon_n + O(1).$$

Thus, we have Proposition 1.

We conjecture that $A(n) = n!/\exp((e^{-\gamma} + o(1))n\log\log n)$ as $n \to \infty$, that is, Proposition 1 is best possible. Though it is difficult to "see" $\log\log n$ tending to infinity, we have some scant evidence in Table 3. Let $u_n = (n!/A(n))^{1/n}$, so the conjecture is that $u_n \sim e^{-\gamma}\log\log n$.

TABLE 3. Values of $A(n)$ for n composite and u_n.

n	$A(n)$	u_n
4	2	1.8612
6	8	2.1170
8	30	2.4607
9	72	2.5786
10	408	2.4826
12	4,104	2.6440
14	29,640	2.8976
15	208,704	2.8388
16	1,437,312	2.8034
18	22,653,504	2.9479
20	318,695,040	3.1199
21	2,686,493,376	3.0866
22	27,628,410,816	3.0356
24	575,372,874,240	3.1722
25	1,775,480,841,216	3.2935
26	21,115,550,048,256	3.2420
27	132,879,856,582,656	3.2758
28	2,321,256,928,702,464	3.1932
30	83,095,013,944,442,880	3.2870

The computation of $A(n)$ is helped by the realization that all of the permutations counted have 1 and the primes in $(n/2, n]$ as fixed points, so one can deal with a somewhat smaller adjacency matrix than $n \times n$. In particular, if n is prime, then $A(n) = A(n-1)$, so in Table 3 we only consider n composite (the cases $n = 1, 2$ being trivial). In addition, for a prime $p \in (n/3, n/2]$ either p is a fixed point or $(p, 2p)$ is a 2-cycle, which gives another reduction.

7.1. **Other types of permutations.** One might consider other arithmetic constraints on permutations. For example, what can be said about the number of permutations σ of $[n]$ where for each $j \in [n]$, either $j \mid \sigma(j)$ or $\sigma(j) \mid j$? Or, the number where each $\text{lcm}[j, \sigma(j)] \leq n$? Problems such as the longest possible cycle in such permutations, the minimum number of disjoint cycles, etc. were studied in [10], [4], [8] and elsewhere. The enumeration problems have not been well-studied, though the first one has an OEIS page: A320843.

ACKNOWLEDGMENTS

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