Erdős Lecture Series

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Erdős and primitive sets

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A subset $\mathcal{A}$ of the positive integers is said to be *primitive* if no member of $\mathcal{A}$ divides another.

Some examples:

1. The set $\mathcal{P}$ of prime numbers.

2. More generally, the set $\mathbb{N}_k = \{n : \Omega(n) = k\}$, where $\Omega(n)$ is the number of prime factors of n counted with repetition.

3. The set $(x, 2x] \cap \mathbb{N}$.

4. With σ the sum-of-divisors function, the set

$$\mathcal{A} = \{n \in \mathbb{N} : \sigma(n)/n \geq 2,\ \ \sigma(d)/d < 2 \text{ for all } d \mid n,\, d < n\}.$$

This last example goes back to **Pythagoras**.

He was quite interested in the sum-of-divisors function σ, and he and his followers classified the natural numbers into 3 categories:

- *deficient*: $\sigma(n)/n < 2$. E.g., $n = 1, 2, 3, 4, 5, 7, 8, 9, 10, \dots$.

- *perfect*: $\sigma(n)/n = 2$. E.g., $n = 6, 28, 496, 8128, \dots$.

- *abundant*: $\sigma(n)/n > 2$. E.g., $n = 12, 18, 20, 24, \dots$.

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What's with "$\mathbf{2}$"? Well, it was actually $s(n) = \sigma(n) - n$ that was considered, and then a number is deficient, perfect, or abundant if $s(n)/n < 1, = 1, > 1$, respectively.

Do these 3 sets have asymptotic densities?

It's easy to see that the perfect numbers have density 0, since already from **Euler** we know that a perfect number must be of the form pm^2 where p is a prime that divides $\sigma(m^2)$.

An easy observation: If n is not deficient, i.e., $\sigma(n)/n \geq 2$, then every multiple of n is not deficient.
Proof. $\sigma(n)/n = \sum_{d|n} 1/d$, and this sum is only larger when applied to a multiple of n. $\square$

An easy consequence: The nondeficient numbers are completely determined by the *minimal* ones in the divisibility relation, that is, those nondeficient numbers that are *not* a multiple of a smaller nondeficient number: 6, 20, 28, 70, 88 . . . .

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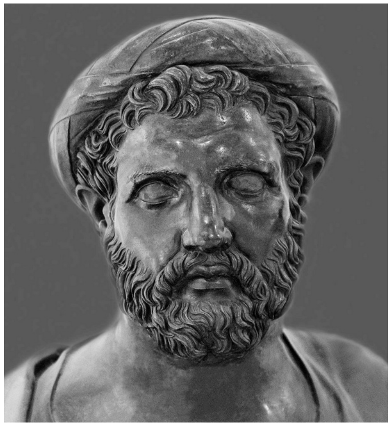
They form a primitive set and it is where **Erdős** entered the picture.

Erdős (1934). *The reciprocal sum of the primitive nondeficient numbers is finite.*

Corollary. *The set of nondeficient numbers have a positive density.*

(Since $(1/x)\sum_{n\le x}\sigma(n)/n \sim \pi^2/6 < 2$ as $x \to \infty$, the density of the nondeficient numbers is < 1, so the set of deficient numbers also has a positive density.)



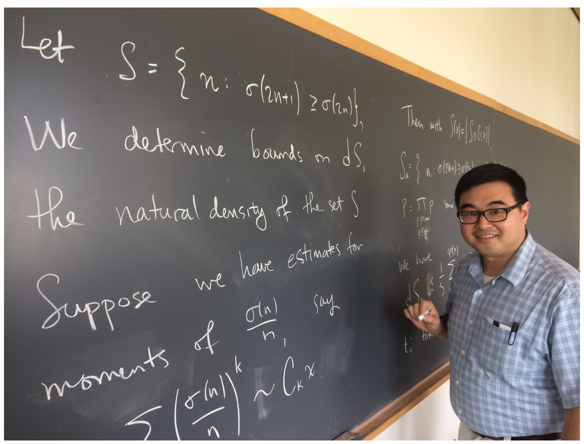
Pythagoras of Samos

Leonhard Euler

Paul Erdős

We now know (**Kobayashi**, 2010) that the density of the abundant numbers (= the density of the nondeficient numbers) lies between 0.2476171 and 0.2476475.

And we know (**Lichtman**, 2018) that the sum of reciprocals of the primitive nondeficient numbers is between 0.34842 and 0.37937.



Mitsuo Kobayashi

Jared Lichtman

Actually, it was known before **Erdős** that the density of the nondeficient numbers exists:

Davenport (1933) showed the density $D(u)$ of $\{n : \sigma(n)/n \le u\}$ exists, and that $D(u)$ is continuous.

Davenport strongly used a technical criterion of **Schoenberg**, who in 1928 proved an analogous result for the density of numbers n with $n/\varphi(n) \le u$. Here φ is Euler's function.

With his paper on primitive nondeficient numbers in1934, **Erdős** began studying this subject, looking for the great theorem that would unite these threads. His elementary approach through primitive sets led him to believe that non-technical methods could be used.

Harold Davenport

Isaac J. Schoenberg

This culminated in the **Erdős–Wintner** theorem in 1939 and the **Erdős–Kac** theorem the same year. And so was born the subject of probabilistic number theory.

But as usual, **Erdős** was interested in primitive sets for their own sake, and this led in interesting directions as well.

Aurel Wintner

Mark Kac

One might guess that a primitive set always has asymptotic density 0. It's true for our 4 examples, and more generally, it's true that the lower asymptotic density of a primitive set must be 0. Somewhat counter-intuitively, the upper asymptotic density need not be zero!

Here's a construction. The set of integers in the interval $(x, 2x]$ is primitive; let $D(x)$ be the asymptotic density of the set of multiples. We know after **Besicovitch** (1934) that $\liminf D(x) = 0$. In fact, after work of **Erdős**, **Tenenbaum**, and **Ford**, we now know that

$$D(x) \asymp \frac{1}{(\log x)^c (\log\log x)^{3/2}}, \quad c = 1 - \frac{1 + \log\log 2}{\log 2} = 0.08607\ldots.$$

Abram Besicovitch

Kevin Ford

Gérald Tenenbaum

But already, just from Besicovitch's result, we can construct primitive sets with upper asymptotic density arbitrarily close to 1/2. Namely, choose a very briskly increasing sequence $x_1, x_2, \ldots$ with $D(x_1)$ very small and $D(x_j) \downarrow 0$. Take all numbers in $(x_1, 2x_1]$, all numbers in $(x_2, 2x_2]$ not divisible by any number in $(x_1, 2x_1]$, all numbers in $(x_3, 2x_3]$ not divisible by any number previously chosen, etc.

This result is best possible: The upper asymptotic density of a primitive set is always $< 1/2$.

Primitive sets continued to be of interest to **Erdős** throughout his life.

In 1988, he and **Cameron** considered counting the subsets of $[1, n]$ that are primitive, conjecturing the number is $\alpha^{(1+o(1))n}$ for some α between 1 and 2. They proved that if α exists, it is between 1.55967 and 1.6.

Very recently, **Angelo** showed that α exists, and **McNew** was able to get a decent error estimate for the "$o(1)$". We also have

$$1.572939 < \alpha < 1.574445,$$

(**McNew** for the lower bound, **Liu, Pach, & Palincza** for the upper bound).

Peter Cameron

Rodrigo Angelo

Nathan McNew

Hong Liu

Péter Pál Pach

Richárd Palincza

And there is the famous **Erdős Conjecture** on primitive sets, based on the following old theorem.

Theorem (Erdős, 1935). *If $\mathcal{A}$ is a primitive set, then*

$$f(\mathcal{A}) := \sum_{\substack{a \in \mathcal{A} \\ a > 1}} \frac{1}{a \log a} < \infty.$$

In fact, $f(\mathcal{A})$ is uniformly bounded as $\mathcal{A}$ varies over primitive sets.

With $\mathcal{P}$ the set of primes, let $C = f(\mathcal{P}) = 1.63661632336\ldots$, the calculation done by **Cohen**.

Conjecture (Erdős, 1988). *For $\mathcal{A}$ primitive, $f(\mathcal{A}) \le C$.*

Conjecture (Erdős, 1988). *For $\mathcal{A}$ primitive,*
$f(\mathcal{A}) \le C = 1.63661632336\ldots$, *where* $f(\mathcal{A}) = \sum_{a\in\mathcal{A},\, a>1} 1/(a \log a)$.

What do we know about $f(\mathcal{A})$?

Erdős, Zhang (unpublished): $f(\mathcal{A}) < 2.886$.

Robin (unpublished): $f(\mathcal{A}) < 2.77$.

Erdős, Zhang (1993): $f(\mathcal{A}) < 1.84$.

The first two results used the original Erdős argument, but the 1993 paper used a new argument.

Zhenxiang Zhang

Let $\mathcal{P}(\mathcal{A})$ denote the set of primes which divide some member of $\mathcal{A}$. Note that the Erdős conjecture is equivalent to the assertion that

$$f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$$

for all primitive sets $\mathcal{A}$.

Proof. Say $f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$ for all primitive sets $\mathcal{A}$. If $\mathcal{Q}$ is the set of primes not in $\mathcal{P}(\mathcal{A})$, then add $1/(q \log q)$ for $q \in \mathcal{Q}$ to both sides. Conversely, say $f(\mathcal{A}) \le C = f(\mathcal{P})$ for all primitive sets $\mathcal{A}$. Then subtract $1/(q \log q)$ for $q \in \mathcal{A}$ from both sides. $\square$

Recall that $\mathbb{N}_k = \{n : \Omega(n) = k\}$,
$\mathcal{P}(\mathcal{A}) = \{p \text{ prime} : p \text{ divides some member of } \mathcal{A}\}$.

Zhang (1991): *If* $f(\mathcal{A}) \le C$ *if each* $a \in \mathcal{A}$ *has* $\Omega(a) \le 4$.

Zhang (1993): *For each* $k \ge 2$, $f(\mathbb{N}_k) < f(\mathbb{N}_1) = C$.

Banks, Martin (2013): *If* $\sum_{p \in \mathcal{P}(\mathcal{A})} 1/p < 1.7401\ldots$, *then* $f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$.

Banks, Martin (2013): **Conjecture**: $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \ldots$.

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Banks, Martin (2013): **Conjecture**: $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \ldots$.

Lichtman (2019): *The Banks–Martin Conjecture is false.*

Bill Banks Greg Martin

Lichtman, P (2019). *For $\mathcal{A}$ primitive,*

- $f(\mathcal{A}) < e^{\gamma} = 1.78109\ldots$.

- *If no member of $\mathcal{A}$ is divisible by 8, then* $f(\mathcal{A}) < f(\mathcal{P}(\mathcal{A})) + 2.37 \times 10^{-7}$.

- *Assuming RH and LI, there is a set of primes $\mathcal{P}_0$ of relative lower logarithmic density ≥ 0.995 such that $f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$ when $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}_0$. Unconditionally, $\mathcal{P}_0$ contains all of the odd primes up to* $\exp(10^6)$.

Note: The relative lower logarithmic density of a set of primes $\mathcal{P}_0$ is

$$\liminf_{x\to\infty} \frac{1}{\log\log x} \sum_{\substack{p \in \mathcal{P}_0 \\ p \le x}} \frac{1}{p}.$$

In a new paper **Lichtman, Martin, & P** show that $\mathcal{P}_0$ has relative lower asymptotic density at least $0.99999973\ldots$.

Notation: For an integer $a \ge 2$, let

$$p(a) := \min\{p \text{ prime} : p \mid a\},$$
$$P(a) := \max\{p \text{ prime} : p \mid a\}.$$

A version of the 1935 Erdős argument:

Let $S_a = \{ba : p(b) \geq P(a)\}$. The asymptotic density of S_a is

$$\delta(S_a) = \frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right).$$

Moreover the sets S_a, as a varies over a primitive set $\mathcal{A}$, are pairwise disjoint. So

$$\sum_{a\in\mathcal{A}} \frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right) = \sum_{a\in\mathcal{A}} \delta(S_a) = \delta\left(\bigcup_{a\in\mathcal{A}} S_a\right) \leq 1.$$

But

$$\frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right) \gg \frac{1}{a \log P(a)} \geq \frac{1}{a \log a},$$

so that $f(\mathcal{A}) \ll 1$.

To do a little better, we should be more careful with the step where we say $\prod_{p<P(a)}(1-1/p) \gg 1/\log(P(a)) \geq 1/\log a$.

Note that as $x \to \infty$, we have $\prod_{p<x}(1-1/p) \sim 1/(e^\gamma \log x)$. Also, if a is composite, then $a \geq 2P(a)$.

Lemma. *For $x \geq 2$, we have $1/\log(2x) < e^\gamma \prod_{p<x}(1-1/p)$.*

Conclude: If a is composite, then $1/\log a < e^\gamma \prod_{p<P(a)}(1-1/p)$. So, if every member of $\mathcal{A}$ is composite, then $f(\mathcal{A}) < e^\gamma$, since

$$f(\mathcal{A}) = \sum_{a \in \mathcal{A}} \frac{1}{a \log a} < e^\gamma \sum_{a \in \mathcal{A}} \frac{1}{a} \prod_{p<P(a)} \left(1 - \frac{1}{p}\right) = e^\gamma \sum_{a \in \mathcal{A}} \delta(S_a) \leq e^\gamma.$$

With a non-strict inequality this much was proved by **Clark** (1995). But we show $f(\mathcal{A}) < e^\gamma$ for *all* primitive sets $\mathcal{A}$.

One path we followed was suggested by the work of **Erdős** and **Zhang**: partition $\mathcal{A}$ by the least prime factor of the elements: Let $\mathcal{A}(q) = \{a \in \mathcal{A} : p(a) = q\}$. We'd love to show that $f(\mathcal{A}(q)) \le 1/(q \log q)$, and this is clear if $q \in \mathcal{A}(q)$. Now

$$f(\mathcal{A}(q)) < \frac{e^\gamma}{q} \prod_{p<q} \left(1 - \frac{1}{p}\right) \text{ if } q \notin \mathcal{A}(q).$$

Indeed, note that

$$\bigcup_{a \in \mathcal{A}(q)} S_a \subset \{ba : a \in \mathcal{A}(q), p(b) \ge q\} \subset \{n : p(n) = q\},$$

so that

$$\sum_{a \in \mathcal{A}(q)} \delta(S_a) \le \frac{1}{q} \prod_{p<q} \left(1 - \frac{1}{p}\right).$$

But in this case every $a \in \mathcal{A}(q)$ is composite, and we've seen then that

$$\frac{1}{a \log a} < e^{\gamma} \delta(S_a),$$

so the claim above holds, i.e., if $q \notin \mathcal{A}(q)$,

$$f(\mathcal{A}(q)) = \sum_{a \in \mathcal{A}(q)} \frac{1}{a \log a} < e^{\gamma} \sum_{a \in \mathcal{A}(q)} \delta(S_a) \le \frac{e^{\gamma}}{q} \prod_{p<q} \left(1 - \frac{1}{p}\right).$$

We'd be laughing if

$$e^{\gamma} \prod_{p<q} \left(1 - \frac{1}{p}\right) \le \frac{1}{\log q}.$$

In fact, the famous theorem of **Mertens** says that the left side is asymptotically equal to the right side as $q \to \infty$.

Say a prime q is **Mertens** if

$$e^{\gamma}\prod_{p<q}\left(1-\frac{1}{p}\right)\le\frac{1}{\log q}.$$

So, if q is Mertens, then $f(\mathcal{A}(q))\le 1/(q\log q)$ regardless if $q\in\mathcal{A}(q)$ or $q\notin\mathcal{A}(q)$. Thus, if every prime in $\mathcal{P}(\mathcal{A})$ is Mertens, then

$$f(\mathcal{A})=\sum_{q\in\mathcal{P}(\mathcal{A})}f(\mathcal{A}(q))\le\sum_{q\in\mathcal{P}(\mathcal{A})}\frac{1}{q\log q}=f(\mathcal{P}(\mathcal{A})).$$

Franz Mertens

A prime q is **Mertens** if $e^\gamma \prod_{p<q}(1-1/p) \le 1/\log q$. And, if every prime in $\mathcal{P}(\mathcal{A})$ is Mertens, then $f(\mathcal{A}) \le f(\mathcal{P}(\mathcal{A}))$. That is, the Erdős conjecture is true for sets supported on the Mertens primes.

Let's try it out. Is 2 Mertens?

$$\prod_{p<2}\left(1-\frac{1}{p}\right) = 1, \qquad \frac{1}{e^\gamma \log 2} = 0.81001\ldots.$$

So, 2 is not Mertens. ☹

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Theorem (Lamzouri, 2016). *Assuming RH and LI, the set of real numbers x with $e^\gamma \prod_{p\le x}(1-1/p) < 1/\log x$ has logarithmic density* $0.99999973\ldots$.

Corollary (Lichtman, Martin, P, 2019). *Assuming RH and LI, the set of Mertens primes has relative logarithmic density* 0.99999973... .

Note that 0.99999973... is the exact same log density that **Rubinstein & Sarnak** found in their famous 1994 paper for the set of x where $\mathrm{li}(x) > \pi(x)$, on assumption of RH and LI, though the two sets are not the same. (For technical reasons, the density of the Mertens race and the density of the π/li race turn out to be the same number.)

When we started investigating primitive sets we had no idea that we would find a connection to "Chebyshev's bias" and "prime number races".

Youness Lamzouri

Michael Rubinstein

Peter Sarnak

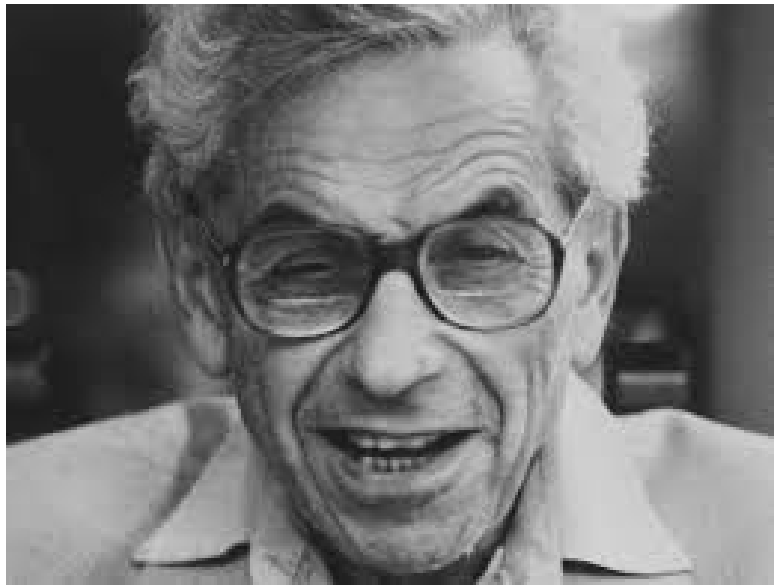
We could push the verification of Mertens primes beyond the 10^8th prime, but instead, we found an alternative that allows us to push things *much* higher. As mentioned earlier:

Theorem (Lichtman, P, 2019). *The Erdős conjecture holds for primitive sets $\mathcal{A}$ supported on the odd primes up to e^{10^6}.*

For details and our other results, see our papers:

J. D. Lichtman and C. Pomerance, *The Erdős conjecture for primitive sets*, Proc. Amer. Math. Soc. Ser. B **6** (2019), 1–14.

J. D. Lichtman, G. Martin, and C. Pomerance, *Primes in prime number races*, Proc. Amer. Math. Soc. **147** (2019), 3743–3757,



Thank you