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The Erdős conjecture on primitive sets

Carl Pomerance

Dartmouth College (emeritus)

University of Georgia (emeritus)

Joint work with **Jared Duker Lichtman**

A subset \mathcal{A} of the positive integers is said to be *primitive* if no member of \mathcal{A} divides another.

Some examples:

1. The set \mathcal{P} of prime numbers.
2. The set $\mathbb{N}_k = \{n : \Omega(n) = k\}$, where $\Omega(n)$ is the number of prime factors of n counted with repetition.
3. The squarefree members of \mathbb{N}_k .
4. The set $(x, 2x] \cap \mathbb{N}$.

5. With σ the sum-of-divisors function, the set

$$\mathcal{A} = \{n \in \mathbb{N} : \sigma(n)/n \geq 2, \sigma(d)/d < 2 \text{ for all } d \mid n, d < n\}.$$

The last example is the set of *primitive nondeficient numbers*. They have a finite reciprocal sum ([Erdős, 1934](#)), so the nondeficient numbers (i.e., $\sigma(n)/n \geq 2$) have an asymptotic density. We now know ([Kobayashi, 2010](#)) that this density is between 0.2476171 and 0.2476475. And we know ([Lichtman, 2018](#)) that the sum of reciprocals of primitive nondeficient numbers is between 0.34842 and 0.37937.

One might guess that a primitive set always has asymptotic density 0. It's true for our 5 examples, and more generally, it's true that the lower asymptotic density of a primitive set must be 0. Somewhat counter-intuitively, the upper asymptotic density need not be zero!

Here's a construction. The set of integers in the interval $(x, 2x]$ is primitive; let $D(x)$ be the asymptotic density of the set of multiples. We know after [Besicovitch](#) (1934) that $\liminf D(x) = 0$. In fact, after work of [Erdős](#), [Tenenbaum](#), and [Ford](#), we now know that

$$D(x) \asymp \frac{1}{(\log x)^c (\log \log x)^{3/2}}, \quad c = 1 - \frac{1 + \log \log 2}{\log 2} = 0.08607 \dots$$

But already, just from Besicovitch's result, we can construct primitive sets with upper asymptotic density arbitrarily close to $1/2$. Namely, choose a very briskly increasing sequence x_1, x_2, \dots with $D(x_1)$ very small and $D(x_j) \downarrow 0$. Take all numbers in $(x_1, 2x_1]$, all numbers in $(x_2, 2x_2]$ not divisible by any number in $(x_1, 2x_1]$, all numbers in $(x_3, 2x_3]$ not divisible by any number previously chosen, etc.

This result is best possible: The upper asymptotic density of a primitive set is always $< 1/2$.

From now on we only consider primitive sets $\mathcal{A} \neq \{1\}$.

Theorem (Erdős, 1935). *If \mathcal{A} is a primitive set, then*

$$f(\mathcal{A}) := \sum_{a \in \mathcal{A}} \frac{1}{a \log a} < \infty.$$

In fact, $f(\mathcal{A})$ is uniformly bounded as \mathcal{A} varies over primitive sets.

With \mathcal{P} the set of primes, let $C = f(\mathcal{P}) = 1.63661632336\dots$, the calculation done by [Cohen](#).

Conjecture (Erdős, 1988). *For \mathcal{A} primitive, $f(\mathcal{A}) \leq C$.*

Conjecture (Erdős, 1988). For \mathcal{A} primitive,
 $f(\mathcal{A}) \leq C = 1.63661632336\dots$, where $f(\mathcal{A}) = \sum_{a \in \mathcal{A}} 1/(a \log a)$.

What do we know about $f(\mathcal{A})$?

Erdős, Zhang (unpublished): $f(\mathcal{A}) < 2.886$.

Robin (unpublished): $f(\mathcal{A}) < 2.77$.

Erdős, Zhang (1993): $f(\mathcal{A}) < 1.84$.

The first two results used the original Erdős argument, but the 1993 paper used a new argument.

Let $\mathcal{P}(\mathcal{A})$ denote the set of primes which divide some member of \mathcal{A} . Note that the Erdős conjecture is equivalent to the assertion that

$$f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$$

for all primitive sets \mathcal{A} . Indeed, if $f(\mathcal{A}) > f(\mathcal{P}(\mathcal{A}))$ for some primitive set \mathcal{A} , let \mathcal{A}' be \mathcal{A} together with every prime not in $\mathcal{P}(\mathcal{A})$, so that \mathcal{A}' is primitive, $\mathcal{P}(\mathcal{A}') = \mathcal{P}$, and $f(\mathcal{A}') > f(\mathcal{P})$.

Recall that $\mathbb{N}_k = \{n : \Omega(n) = k\}$.

Zhang, (1991): $f(\mathcal{A}) \leq C$ if each $a \in \mathcal{A}$ has $\Omega(a) \leq 4$.

Zhang, (1993): For each $k \geq 2$, $f(\mathbb{N}_k) < f(\mathbb{N}_1) = C$.

Banks, Martin, (2013): If $\sum_{p \in \mathcal{P}(\mathcal{A})} 1/p < 1.7401\dots$, then $f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$.

Banks, Martin, (2013): **Conjecture:** $f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) \dots$

Bayless, Kinlaw, Klyve, (2018): $f(\mathbb{N}_2) > f(\mathbb{N}_3)$.

Lichtman, P, (2018). *For \mathcal{A} primitive,*

- $f(\mathcal{A}) < e^\gamma$.
- *If no member of \mathcal{A} is divisible by 8, then $f(\mathcal{A}) < f(\mathcal{P}(\mathcal{A})) + 2.37 \times 10^{-7}$.*
- *There is an absolute constant $c > 0$ such that $f(\mathbb{N}_k) > c$ for all k .*
- *Assuming RH and LI, there is a set of primes \mathcal{P}_0 of relative lower logarithmic density ≥ 0.995 such that $f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$ when $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}_0$. Unconditionally, \mathcal{P}_0 contains all of the odd primes up to $\exp(10^6)$.*

Note: The relative lower logarithmic density of a set of primes \mathcal{P}_0 is

$$\liminf_{x \rightarrow \infty} \frac{1}{\log \log x} \sum_{\substack{p \in \mathcal{P}_0 \\ p \leq x}} \frac{1}{p}.$$

In a new paper **Lichtman, Martin, & P** show that \mathcal{P}_0 has relative lower asymptotic density at least 0.99999973....

Recall the theorem of **Mertens**:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{1}{e^\gamma \log x}, \quad \text{as } x \rightarrow \infty.$$

Say a prime q is **Mertens** if

$$e^\gamma \prod_{p < q} \left(1 - \frac{1}{p}\right) \leq \frac{1}{\log q}.$$

Lichtman & P: *If every prime in $\mathcal{P}(\mathcal{A})$ is **Mertens**, then $f(\mathcal{A}) \leq f(\mathcal{P}(\mathcal{A}))$. That is, the Erdős conjecture is true for sets supported on the **Mertens** primes.*

A prime q is **Mertens** if $e^\gamma \prod_{p < q} (1 - 1/p) \leq 1/\log q$.

Let's try it out. Is 2 **Mertens**?

$$e^\gamma \prod_{p < 2} \left(1 - \frac{1}{p}\right) = e^\gamma = 1.78107\dots, \quad \frac{1}{\log 2} = 1.44269\dots$$

So, 2 is not **Mertens**. 😞

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But every odd prime up to p_{10^8} is indeed **Mertens**. 😊

Theorem (Lamzouri, 2016). *Assuming RH and LI, the set of real numbers x with $e^\gamma \prod_{p \leq x} (1 - 1/p) < 1/\log x$ has logarithmic density $0.99999973\dots$.*

Corollary (Lichtman, Martin, P, 2018). *Assuming RH and LI, the set of **Mertens** primes has relative logarithmic density 0.99999973... .*

Note that 0.99999973... is the exact same log density that **Rubinstein & Sarnak** found in their famous 1994 paper for the set of x where $\text{li}(x) > \pi(x)$, on assumption of RH and LI, though the two sets are not the same. (For technical reasons, the density of the Mertens race and the density of the π/li race turn out to be the same number.)

When we started investigating primitive sets we had no idea that we would find a connection to “Chebyshev’s bias” and “prime number races” .

We could push the verification of **Mertens** primes beyond the 10^8 th prime, but instead, we found an alternative that allows us to push things *much* higher. As mentioned earlier:

Theorem (Lichtman, P, 2018). *The Erdős conjecture holds for primitive sets A supported on the odd primes up to e^{10^6} .*

For details and our other results, see our papers posted on arXiv or on my home page:

J. D. Lichtman and C. Pomerance, *The Erdős conjecture for primitive sets*, 2018.

J. D. Lichtman, G. Martin, and C. Pomerance, *Primes in prime number races*, 2018.

Thank you