Counting in number theory

# Fibonacci integers

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Leonardo of Pisa (Fibonacci) (c. 1170 – c. 1250)

We all know the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 510, \ldots .$$

The $n$-th one is given by Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ are the roots of $t^2 - t - 1$.

Thus, $F_n \sim \alpha^n/\sqrt{5}$ and the number of Fibonacci numbers in $[1, x]$ is $\log x/\log \alpha + O(1)$.

Thus, it is quite special for a natural number to be in the Fibonacci sequence; it is a rare event.

Say we try to “spread the wealth” by also including integers we can build up from the Fibonacci numbers using multiplication and division. Some examples that are not themselves Fibonacci numbers:

$$4 = 2^2,\ 6 = 2 \cdot 3,\ 7 = \frac{21}{3},\ 8 = 2^3,\ 9 = 3^2, \dots .$$

Call such numbers Fibonacci integers. These are the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \dots .$$

Perhaps we have spread the wealth too far?

Well, not every natural number is a Fibonacci integer: the first one missing is 37. To see this, first note that

$$F_{19} = 4181 = 37 \cdot 113,$$

so that the *rank of appearance* of 37 (and 113) is 19.

Some explanation: The Fibonacci numbers have the property that for any integer $m$ there is some $n$ where $m \mid F_n$. In fact, the set of such numbers $n$ is the set of multiples of the least positive $n$, denote it $z(m)$. This is the rank of appearance of $m$. A corollary is that if $m \mid F_p$ for some prime $p$ and $m > 1$, then $z(m) = p$.

So from $F_{19} = 37 \cdot 113$ we deduce that 19 is the rank of appearance of 37. Suppose that 37 is a Fibonacci integer, so that

$$37 = \frac{F_{n_1} \dots F_{n_k}}{F_{m_1} \dots F_{m_l}},$$

where $n_1 \le \cdots \le n_k$ and $m_1 \le \cdots \le m_l$, $n_k \ne m_l$. Then $n_k \ge 19$. Carmichael showed that each $F_n$ has a *primitive* prime factor (i.e., not dividing a smaller $F_k$) when $n \ne 1, 2, 6, 12$. Thus, $F_{n_k}$ has a primitive prime factor $p \ne 37$ (if $n_k = 19$, then $p = 113$). So $p$ must appear in the denominator, so that $m_l \ge n_k$ and indeed $m_l > n_k$. Then repeat with a primitive prime factor $q$ of $m_l$, getting $n_k > m_l$.

Let $N(x)$ denote the number of Fibonacci integers in $[1, x]$. We have

$N(10) = 10$, $N(100) = 88$, $N(1000) = 534$, $N(10{,}000) = 2{,}681$.

So, what do you think?

$$N(x) \approx x/(\log x)^c$$
$$N(x) \approx x/\exp((\log x)^c)$$
$$N(x) \approx x^c$$
$$N(x) \approx \exp((\log x)^c)$$
$$N(x) \approx (\log x)^c$$

**???**

Luca, Porubský (2003). *With* $N(x)$ *the number of Fibonacci integers in* $[1, x]$*, we have*

$$N(x) = O_c\left(x/(\log x)^c\right)$$

*for every positive number* $c$*.*

Florian Luca

Štefan Porubský

Luca, P, Wagner (2010). *With* $N(x)$ *the number of Fibonacci integers in* $[1, x]$*, for each* $\epsilon > 0$,

$$\exp\left(C(\log x)^{1/2} - (\log x)^{\epsilon}\right) < N(x) < \exp\left(C(\log x)^{1/2} + (\log x)^{1/6+\epsilon}\right)$$

*for* $x$ *sufficiently large, where* $C = 2\zeta(2)\sqrt{\zeta(3)/(\zeta(6)\log\alpha)}$.

Stephan Wagner

The problem of counting Fibonacci integers is made more difficult because of allowing denominators. That is, if we just looked at the *semigroup* generated by the Fibonacci numbers, rather than integers in the group that they generate, life would be simpler.

In fact, because of Carmichael's primitive prime factors, if we throw out $F_n$ for $n = 1, 2, 6, 12$, then an integer represented as a product of Fibonacci numbers is done so uniquely up to order; the semigroup is freely generated by $F_n$ for $n \neq 1, 2, 6, 12$.

And we can now begin to see the shape of the counting function (for this restricted problem). It is essentially a partition problem.

Let $p(n)$ denote the number of partitions of $n$; that is, the number of ordered tuples $a_1 \le a_2 \le \cdots \le a_k$ of positive integers with

$$a_1 + a_2 + \cdots + a_k = n.$$

If we were just considering the multiplicative semigroup generated by the Fibonacci numbers, then we would be considering ordered tuples $a_1 \le a_2 \le \cdots \le a_k$ with

$$F_{a_1} F_{a_2} \ldots F_{a_k} \le x.$$

Using the Binet formula and taking logs, this becomes

$$a_1 + a_2 + \cdots + a_k \le \frac{\log x}{\log \alpha} + O(k).$$

Well, what do we know about $p(n)$, and more specifically, for

$$\sum_{n \le x} p(n)?$$

Hardy & Ramanujan (1918): *Let* $c = \pi\sqrt{2/3}$*. As* $n \to \infty$,

$$p(n) = (1 + o(1))\frac{\exp(c\sqrt{n})}{4n\sqrt{3}}.$$

In fact, they gave a formula for $p(n)$ involving a divergent series. Stopping the series near $\sqrt{n}$ gives a close approximation to $p(n)$ with error $O(n^{-1/4})$. In 1937, Rademacher found a series that actually converges to $p(n)$, which if stopped at the $k$th term produces an error that is $O(k^{-1/2})$.

G. H. Hardy

S. Ramanujan

H. Rademacher

Using these formulas, we have

$$\sum_{n\le x} p(n) = (1+o(1))\frac{\text{exp}(c\sqrt{x})}{2\pi\sqrt{2x}},$$

as $x \to \infty$.

And taking this over to the multiplicative semigroup generated by the Fibonacci numbers, we would have that the number of such integers in $[1, x]$ would be about $\text{exp}(c'\sqrt{\log x})$, where $c' = c/\sqrt{\log \alpha}$. With more care an asymptotic formula could be worked out.

However, the problem of Fibonacci integers allows more numbers since we can also divide, not just multiply. Is there some way to allow for this and keep our insight into partitions?

**Ban denominators? The cyclotomic factorization:**

Let $\Phi_n(x)$ denote the $n$-th cyclotomic polynomial, so that

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \qquad \Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

Let $\Phi_n(x, y) = y^{\varphi(n)} \Phi_n(x/y)$ be the homogenization of $\Phi_n(x)$. Then for $n > 1$,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \prod_{d|n,\ d>1} \Phi_d(\alpha, \beta), \qquad \Phi_n(\alpha, \beta) = \prod_{d|n} F_d^{\mu(n/d)}.$$

Abbreviate $\Phi_n(\alpha, \beta)$ as $\Phi_n$. For $n > 1$, $\Phi_n$ is a natural number, and in fact it is a Fibonacci integer.

Thus, the Fibonacci integers are also generated by the cyclotomic numbers $\Phi_n = \Phi_n(\alpha, \beta)$ for $n > 1$. The number $\Phi_n = \Phi_n(\alpha, \beta)$ divides $F_n$, and it has all of the primitive prime factors of $F_n$ (with the same exponents as in $F_n$). So they too (for $n \neq 1, 2, 6, 12$) freely generate a semigroup that now contains many more Fibonacci integers than the semigroup generated by just the Fibonacci numbers themeselves.

Is this the whole story? Assume so. We have $\Phi_n \approx \alpha^{\varphi(n)}$, so now our partition problem would be of the shape

$$\varphi(a_1) + \varphi(a_2) + \cdots + \varphi(a_k) \lessapprox \frac{\log x}{\log \alpha}.$$

We would have to make sense of these approximations, but perhaps things would be still tractable.

But consider the Fibonacci integer 23. We can see that it is one, since

$$F_{24} = 2^5 \cdot 3^2 \cdot 7 \cdot 23, \text{ so that } 23 = \frac{F_{24}}{F_3^5 F_4 F_8}.$$

The primitive part of $F_{24}$ is 23, so this will appear in $\Phi_{24}$. However $\Phi_{24} = 46$, and we have

$$23 = \frac{\Phi_{24}}{\Phi_3}.$$

Thus, even with the cyclotomics, denominators are still necessary.

Recall that $z(p)$ is the rank of appearance of the prime $p$; that is, the least positive $n$ with $p \mid F_n$. Then for any positive integer $k$,

$$\Phi_{p^k z(p)} = p \times (\text{the primitive part of } F_{p^k z(p)}).$$

And if $n$ is not in the form $p^k z(p)$, then $\Phi_n$ is exactly the primitive part of $F_n$.

For example, $\Phi_{19} = F_{19} = 37 \cdot 113$, as we've seen. Thus,

$$\frac{\Phi_{37^k \cdot 19}\Phi_{113^l \cdot 19}}{\Phi_{19}}$$

is a Fibonacci integer for any choice of positive integers $k, l$.

It is possible to figure out the *atoms* for the Fibonacci integers, namely those Fibonacci integers exceeding 1 that cannot be factored into smaller Fibonacci integers. And with these atoms, we would not need denominators; that is, the Fibonacci integers would indeed be the semigroup generated by the atoms.

However, we do not have unique factorization into atoms. Call the above example $n(k, l)$. It is easy to see that they are atoms, but $n(1, 1)n(2, 2) = n(1, 2)n(2, 1)$.

Our strategy: ignore the difficulties and plow forward.

First, let $N_\Phi(x)$ be the number of integers in $[1, x]$ representable as a product of $\Phi_n$'s (for $n \neq 1, 2, 6, 12$). Clearly $N(x) \geq N_\Phi(x)$.

Since different words in these factors (order of factors not counting) give different Fibonacci integers and

$$\Phi_n \approx \alpha^{\varphi(n)} \quad \text{(in fact, } \alpha^{\varphi(n)-1} \leq \Phi_n \leq \alpha^{\varphi(n)+1}\text{)},$$

we can tap into the partition philosophy mentioned earlier. Following the analytic methods originally laid out by Hardy, Ramanujan, and Rademacher we can show that

$$\exp\left(C(\log x)^{1/2} - (\log x)^{\epsilon}\right) \leq N_\Phi(x) \leq \exp\left(C(\log x)^{1/2} + (\log x)^{\epsilon}\right).$$

The basic plan is to consider the generating function

$$D(z) = \sum_{n \in \Phi} n^{-z} = \prod_{n \neq 1,2,6,12} (1 - \Phi_n^{-z})^{-1},$$

where $\Phi$ is the multiplicative semigroup generated by the $\Phi_n$'s. By a standard argument, the Mellin transform of $\log D(z)$ is $\Gamma(s)\zeta(s+1)C(s)$, where

$$C(s) = \sum_{n \neq 1,2,6,12} (\log \Phi_n)^{-s}.$$

Then $C(s)$ differs from $(\log \alpha)^{-s} \sum_n \varphi(n)^{-s}$ by a function analytic in $\Re(s) > 0$ with nice growth behavior in the vertical aspect.

Then comes the saddle point method, and so on.

And with more work we believe we can attain an asymptotic formula for $N_\Phi(x)$.

Could there be an elementary approach for this part of the proof? There very well might be, since Erdős showed in 1942 that by elementary methods one can get an asymptotic formula for $p(n)$ (but not a determination of the constant outside of the exponential).

Our next step in the plowing-ahead program is to estimate the number of extra Fibonacci integers that are not words in the $\Phi_n$'s. We show, via a fairly delicate combinatorial argument, that these extra Fibonacci integers introduce a factor of at most

$$\exp\left((\log x)^{1/6+\epsilon}\right).$$

Further, we can show that if $\Phi_n$ has a prime factor larger than $n^K$ for each fixed $K$ and all sufficiently large $n$, depending on $K$, then "1/6" may be replaced with 0.

To understand this problem, let us look at an easier form of it.

What can one say about $P(2^n - 1)$? Here $P(m)$ is the largest prime factor of $m$ (and $P(1) = 0$).

Any right-thinking person can see quite clearly that $P(2^n - 1)$ goes to infinity much faster than $n$ does.

However, the strongest theorem known is that $P(2^n - 1) \geq 2n + 1$ for all $n > 12$. This is due to Schinzel in 1962.

However, since we are thinking statistically, we can also ask about “almost all” results. That is, ignoring a thin set of exceptional integers $n$, what can we say about $P(n)$?

Or we can think conditionally; that is, under suitable hypotheses, what is the case?

Or we can think statistically and conditionally . . . .

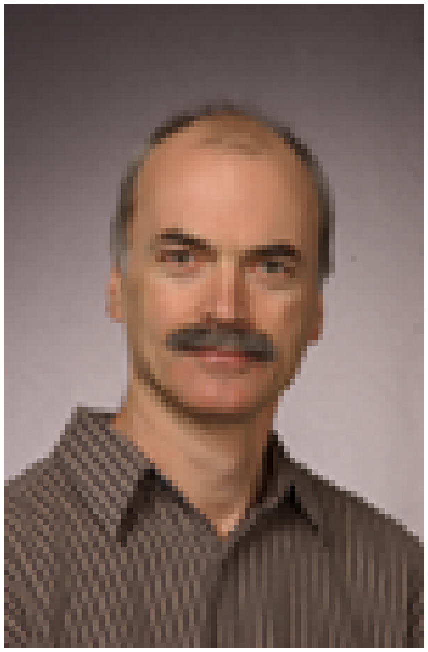
Some results:

Schinzel (1962): *For all* $n > 12$, $P(2^n - 1) \geq 2n + 1$.

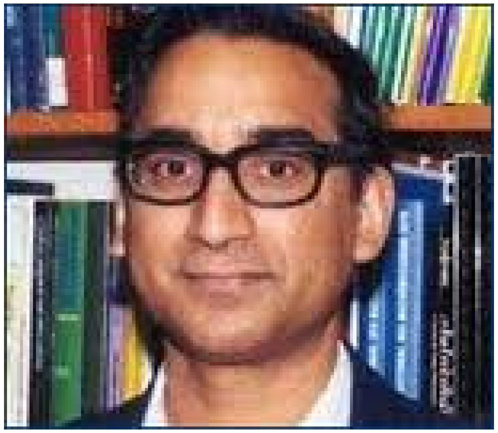
Stewart (2010): *For all large* $n$, $P(2^n - 1) > n^{1+c/\log\log n}$.

P & Murata (2004): *Assuming the GRH, almost all integers* $n$ *have* $P(2^n - 1) > n^{4/3}/\log\log n$.

Murty & Wong (2002): *Assuming the ABC conjecture, for all large* $n$, $P(2^n - 1) > n^{2-\epsilon}$.



C. L. Stewart



M. Ram Murty

Andrzej Schinzel

Essentially the same results hold for $P(\Phi_n)$, but to get the ABC-conditional result we would need to assume a stronger form of this conjecture in a preprint of Stewart and Tenenbaum.

So the (Luca, P, Wagner) theorem is again:
*The number of Fibonacci integers in* $[1, x]$ *is between* $\exp(C\sqrt{\log x} - (\log x)^{\epsilon})$ *and* $\exp(C\sqrt{\log x} + (\log x)^{1/6+\epsilon})$ *for all large* $x$*, where* $C = 2\zeta(2)\sqrt{\zeta(3)/(\zeta(6)\log\alpha)}$.

Assuming the ABC conjecture, 1/6 may be replaced with 1/8.

Finally: Our result for Fibonacci integers carries over to "Mersenne integers" (integers in the multiplicative group generated by the Mersenne numbers $2^n - 1$) and other similar constructs created from binary recurrent sequences. Only the coefficient of $(\log x)^{1/2}$ in the exponent changes.

**THANK YOU!**