The ranges of various familiar functions

Carl Pomerance, Dartmouth College

based on joint work with
K. Ford, F. Luca, and P. Pollack
Let us introduce our cast of characters:

- **Euler**’s function: \( \varphi(n) \) is the cardinality of \( (\mathbb{Z}/n\mathbb{Z})^\times \).

- **Carmichael**’s function: \( \lambda(n) \) is the exponent of \( (\mathbb{Z}/n\mathbb{Z})^\times \).

- \( \sigma \): the sum-of-divisors function.

- \( s(n) = \sigma(n) - n \): the sum-of-proper-divisors function.
The oldest of these functions is $s(n) = \sigma(n) - n$, going back to Pythagoras. He was interested in fixed points ($s(n) = n$) and 2-cycles ($s(n) = m$, $s(m) = n$) in the dynamical system given by iterating $s$.

Very little is known after millennia of study, but we do know that the number of $n$ to $x$ with $s(n) = n$ is at most $x^\epsilon$ (Hornfeck & Wirsing, 1957) and that the number of $n$ to $x$ with $n$ in a 2-cycle is at most $x/\exp((\log x)^{1/2})$ for $x$ large (P, 2014).

The study of the comparison of $s(n)$ to $n$ led to the theorems of Schoenberg, Davenport, and Erdős & Wintner.
Erdős was the first to consider the set of values of \( s(n) \). Note that if \( p \neq q \) are primes, then \( s(pq) = p + q + 1 \), so that:

\[
\text{All even integers at least 8 are the sum of 2 unequal primes,}
\]

\[
\implies
\]

\[
\text{All odd numbers at least 9 are values of } s.
\]

Also, \( s(2) = 1 \), \( s(4) = 3 \), and \( s(8) = 7 \), so presumably the only odd number that’s not an \( s \)-value is 5. It’s known that this slightly stronger form of Goldbach is almost true in that the set of evens not so representable as \( p + q \) has density 0.

Thus: \textit{the image of } s \textit{ contains almost all odd numbers.}
But what of even numbers? Erdős (1973): \textit{There is a positive proportion of even numbers missing from the image of }s.
But what of even numbers? Erdős (1973): *There is a positive proportion of even numbers missing from the image of $s$.*

Y.-G. Chen & Q.-Q. Zhao (2011): *At least $(0.06 + o(1))x$ even numbers in $[1, x]$ are not of the form $s(n)$.*

P & H.-S. Yang (2014): Computationally it is appearing that about $\frac{1}{6}x$ even numbers to $x$ are not of the form $s(n)$. 
But what of even numbers? Erdős (1973): There is a positive proportion of even numbers missing from the image of $s$.

Y.-G. Chen & Q.-Q. Zhao (2011): At least $(0.06 + o(1))x$ even numbers in $[1, x]$ are not of the form $s(n)$.

P & H.-S. Yang (2014): Computationally it is appearing that about $\frac{1}{6}x$ even numbers to $x$ are not of the form $s(n)$.

Unsolved: Does the image of $s$ have an asymptotic density? Does the image of $s$ contain a positive proportion of even numbers?
But what of even numbers? Erdős (1973): *There is a positive proportion of even numbers missing from the image of* $s$.

Y.-G. Chen & Q.-Q. Zhao (2011): *At least* $(0.06 + o(1))x$ *even numbers in* $[1, x]$ *are not of the form* $s(n)$.

P & H.-S. Yang (2014): Computationally it is appearing that about $\frac{1}{6}x$ even numbers to $x$ are not of the form $s(n)$.

Unsolved: Does the image of $s$ have an asymptotic density? Does the image of $s$ contain a positive proportion of even numbers?

Late breaking news: Yes to the second question. (Luca & P, 2014)
The set of values of $\varphi$ was first considered by Pillai (1929): 

*The number $V_{\varphi}(x)$ of $\varphi$-values in $[1, x]$ is $O(x/(\log x)^c)$, where $c = \frac{1}{e} \log 2 = 0.254 \ldots$.*

Pillai’s idea: There are not many values $\varphi(n)$ when $n$ has few prime factors, and if $n$ has more than a few prime factors, then $\varphi(n)$ is divisible by a high power of 2.
The set of values of $\varphi$ was first considered by Pillai (1929): The number $V_\varphi(x)$ of $\varphi$-values in $[1, x]$ is $O(x/(\log x)^c)$, where $c = \frac{1}{e} \log 2 = 0.254 \ldots$.

Pillai’s idea: There are not many values $\varphi(n)$ when $n$ has few prime factors, and if $n$ has more than a few prime factors, then $\varphi(n)$ is divisible by a high power of 2.

Erdős (1935): $V_\varphi(x) = x/(\log x)^{1+o(1)}$.

Erdős’s idea: Deal with $\Omega(\varphi(n))$ (the total number of prime factors of $\varphi(n)$, with multiplicity). This paper was seminal for the various ideas introduced. For example, the proof of the infinitude of Carmichael numbers owes much to this paper.
Again: $V_{\varphi}(x) = x/(\log x)^{1+o(1)}$.

But: A great deal of info may be lurking in that “$o(1)$”.

After work of Erdős & Hall, Maier & P, and Ford, we now know that $V_{\varphi}(x)$ is of magnitude

$$\frac{x}{\log x} \exp \left( A(\log_3 x - \log_4 x)^2 + B \log_3 x + C \log_4 x \right),$$

where $\log_k$ is the $k$-fold iterated log, and $A, B, C$ are explicit constants.

Unsolved: Is there an asymptotic formula for $V_{\varphi}(x)$?
Do we have $V_{\varphi}(2x) \sim 2V_{\varphi}(x)$?
The same results and unsolved problems pertain as well for the image of $\sigma$.

In 1959, Erdős conjectured that the image of $\sigma$ and the image of $\varphi$ has an infinite intersection; that is, there are infinitely many pairs $m, n$ with

$$\sigma(m) = \varphi(n).$$

It is amazing how many famous conjectures imply that the answer is yes!
Yes, if there are infinitely many twin primes:

If $p, p + 2$ are both prime, then

$$\varphi(p + 2) = p + 1 = \sigma(p).$$
Yes, if there are infinitely many twin primes:

If \( p, p + 2 \) are both prime, then
\[
\varphi(p + 2) = p + 1 = \sigma(p).
\]

Yes, if there are infinitely many Mersenne primes:

If \( 2^p - 1 \) is prime, then
\[
\varphi(2^{p+1}) = 2^p = \sigma(2^p - 1).
\]
Yes, if there are infinitely many twin primes:

If $p, p + 2$ are both prime, then

$$\varphi(p + 2) = p + 1 = \sigma(p).$$

Yes, if there are infinitely many Mersenne primes:

If $2^p - 1$ is prime, then

$$\varphi(2^{p+1}) = 2^p = \sigma(2^p - 1).$$

Yes, if the Extended Riemann Hypothesis holds.
It would seem a promising strategy to prove that there are at most finitely many solutions to $\sigma(m) = \varphi(n)$; it has some amazing and unexpected corollaries!

However, Ford, Luca, & P (2010): There are indeed infinitely many solutions to $\sigma(m) = \varphi(n)$.

We gave several proofs, but one proof uses a conditional result of Heath-Brown: If there are infinitely many Siegel zeros, then there are infinitely many twin primes.
Some further results:

**Garaev (2011):** For each fixed number $a$, the number $V_{\varphi,\sigma}(x)$ of common values of $\varphi$ and $\sigma$ in $[1, x]$ exceeds $\exp((\log \log x)^a)$ for $x$ sufficiently large.

**Ford & Pollack (2011):** Assuming a strong form of the prime $k$-tuples conjecture, $V_{\varphi,\sigma}(x) = x/(\log x)^{1+o(1)}$.

**Ford & Pollack (2012):** Most values of $\varphi$ are not values of $\sigma$ and vice versa.
The situation for Carmichael’s function $\lambda$ has only recently become clearer. Recall that $\lambda(p^a) = \varphi(p^a)$ unless $p = 2, a \geq 3$, when $\lambda(2^a) = 2^{a-2}$, and that

$$\lambda([m, n]) = [\lambda(m), \lambda(n)].$$

It is easy to see that the image of $\varphi$ has density 0, just playing with powers of 2 as did Pillai. But what can be done with $\lambda$? It’s not even obvious that $\lambda$-values that are 2 mod 4 have density 0.

The solution lies in the “anatomy of integers” and in particular of shifted primes. It is known (Erdős & Wagstaff) that most numbers do not have a large divisor of the form $p - 1$ with $p$ prime. But a $\lambda$-value has such a large divisor or it is “smooth”, so in either case, there are not many of them.
Using these thoughts, Erdős, P, & Schmutz (1991): *There is a positive constant $c$ such that $V_\lambda(x)$, the number of $\lambda$-values in $[1, x]$, is $O(x/(\log x)^c)$.*
Using these thoughts, Erdős, P, & Schmutz (1991): There is a positive constant $c$ such that $V_\lambda(x)$, the number of $\lambda$-values in $[1, x]$, is $O(x/(\log x)^c)$.

Friedlander & Luca (2007): A valid choice for $c$ is $1 - \frac{e}{2} \log 2 = 0.057\ldots$. 
Using these thoughts, Erdős, P, & Schmutz (1991): There is a positive constant \( c \) such that \( V_\lambda(x) \), the number of \( \lambda \)-values in \([1, x]\), is \( O(x/(\log x)^c) \).

Friedlander & Luca (2007): A valid choice for \( c \) is 
\[
1 - \frac{e}{2} \log 2 = 0.057 \ldots 
\]

Banks, Friedlander, Luca, Pappalardi, & Shparlinski (2006):
\[
V_\lambda(x) \geq \frac{x}{\log x} \exp \left( (A + o(1))(\log_3 x)^2 \right).
\]

So, \( V_\lambda(x) \) is somewhere between \( x/(\log x)^{1+o(1)} \) and \( x/(\log x)^c \), where \( c = 1 - \frac{e}{2} \log 2 \).
Recently, Luca & P (2013): $V_\lambda(x) \leq x/(\log x)^{\eta+o(1)}$, where 
$\eta = 1 - (1 + \log \log 2)/\log 2 = 0.086\ldots$.
Further, $V_\lambda(x) \geq x/(\log x)^{0.36}$ for all large $x$.

Late breaking news: The “correct” exponent is $\eta$ (Ford, Luca, & P, 2014).

The constant $\eta$ actually pops up in some other problems:

Erdős (1960): The number of distinct entries in the $N \times N$
multiplication table is $N^2/(\log N)^{\eta+o(1)}$.

The asymptotic density of integers with a divisor in the interval 
$[N, 2N]$ is $1/(\log N)^{\eta+o(1)}$. This result has its own history 
beginning with Besicovitch in 1934, some of the other players 
being Erdős, Hooley, Tenenbaum, and Ford.
Square values Banks, Friedlander, P, & Shparlinski (2004):
There are more than $x^{0.7}$ integers $n \leq x$ with $\varphi(n)$ a square. The same goes for $\sigma$ and $\lambda$. 
Square values Banks, Friedlander, P, & Shparlinski (2004): There are more than $x^{0.7}$ integers $n \leq x$ with $\varphi(n)$ a square. The same goes for $\sigma$ and $\lambda$.

Remark. There are only $x^{0.5}$ squares below $x$. (!)
**Square values** Banks, Friedlander, P, & Shparlinski (2004):  
*There are more than* $x^{0.7}$ *integers* $n \leq x$ *with* $\varphi(n)$ *a square.*  
*The same goes for* $\sigma$ *and* $\lambda$.

Remark. There are only $x^{0.5}$ squares below $x$. (!)

Might there be a positive proportion of integers $n$ with $n^2$ a value of $\varphi$?

Pollack & P (2013): No, the number of $n \leq x$ with $n^2$ a $\varphi$-value is $O(x/(\log x)^{0.0063})$. The same goes for $\sigma$.

Unsolved: Could possibly almost all even squares be $\lambda$-values??
Idea of the proof that a positive proportion of even numbers are values of \( s(n) = \sigma(n) - n \) (Luca & P, 2014):

Consider even numbers \( n \) with several constraints:

- \( n \) is deficient (means that \( s(n) < n \));
- \( n = pqrk \in [\frac{1}{2}x, x] \) with \( p > q > r > k \) and \( p, q, r \) primes;
- \( k \leq x^{1/60}, \quad r \in [x^{1/15}, x^{1/12}], \quad q \in [x^{7/20}, x^{11/30}] \);
- \( n \) is “normal”.  

If $n$ satisfies these conditions, then $s(n) \leq x$ is even.

Let $r(s)$ denote the number of representations of $s$ as $s(n)$ from such numbers $n$.

We have $\sum_s r(s) \gg x$.

The trick then is to show that $\sum_s r(s)^2 \ll x$.

For this, the sieve is useful. Stay tuned for details on my home page.
What’s next with $s(n)$?

Possibly a conjecture of Erdős, Granville, P, & Spiro (1990) is now tractable:

*If $A$ is a set of density 0, then $s^{-1}(A)$ has density 0.*

The same conjecture should hold for the function

$s_{\varphi}(n) := n - \varphi(n).$

(Our proof that the range of $s$ contains a positive proportion of evens, shows this as well for the range of $s_{\varphi}$, a fact not previously known.)
MERCI & THANK YOU