CORRECTION TO: ON ROBIN'S INEQUALITY

Florian Luca
Mathematics Division, Stellenbosch University, Stellenbosch, South Africa
`fluca@sun.ac.za`

Carl Pomerance
Dartmouth College, Hanover, New Hampshire, USA
`carlp@math.darmouth.edu`

Patrick Solé
I2M, (CNRS, Aix-Marseilles University) Marseille, France
`patrick.sole@telecom-paris.fr`

Abstract

Earlier this year, the first and third authors showed that there are at most $x^{o(1)}$ numbers $n \leq x$ which violate Robin's inequality: $\sigma(n)/n < e^\gamma \log\log n$. Unfortunately, the proof contains an oversight. This could be corrected and the theorem strengthened to give the bound $x^{O(1/\log\log x)}$ along the lines of the original argument. Instead, we give a short proof of this more explicit bound using a result of the second author from 1985.

1. Introduction

Robin [6] (also see Ramanujan [5]) conjectured that $\sigma(n) < e^\gamma n \log\log n$ holds for all positive integers $n > 7!$, where $\sigma(n)$ is the sum of divisors of n. And he proved that the validity of this inequality is equivalent to the Riemann Hypothesis. In the recent paper [3], the first and third authors let

$$\mathcal{NR}(x) := \{7! < n \leq x : \sigma(n) \geq e^\gamma n \log\log n\}$$

and proved various inequalities for $\#\mathcal{NR}(x)$. In particular, Theorem 3 in [3] claims that $\#\mathcal{NR}(x) \leq x^{o(1)}$ holds as $x \to \infty$. Unfortunately, that proof contains an oversight which we correct here. In addition, we make the $o(1)$ from the exponent explicit. It is possible to give a proof along the lines of [3] but here we give a much shorter proof using the second author's paper [4].

2. Main result

We have the following theorem.

Theorem 1. *For $x \geq 10$ we have*

$$\#\mathcal{NR}(x) = x^{O(1/\log\log x)}.$$

Let $p_1, p_2, \ldots$ denote the sequence of prime numbers. As usual, we let $\omega(n)$ denote the number of primes among the divisors of n, and we let $\log_j$ denote the j-times iterated log function. Let

$$y = y(x) = \log x / \log_2 x.$$

It is well-known that the maximal order of $\omega(n)$ for $n \leq x$ is $\sim y$ (see, for example, Section 5 of Ramanujan's paper [5]). We first show that this maximal order is exceeded by members of $\mathcal{NR}(x)$.

Lemma 1. *For x sufficiently large and $n \in (x/2, x] \cap \mathcal{NR}(x)$, we have $\omega(n) > y$.*

Proof. Suppose that $n \in (x/2, x]$ and $\omega(n) = k \leq y$. Then

$$\frac{\sigma(n)}{n} < \prod_{p|n}\left(1 - \frac{1}{p}\right)^{-1} \leq \prod_{j \leq k}\left(1 - \frac{1}{p_j}\right)^{-1}. \tag{1}$$

By a strong form of Mertens' theorem (cf. [7]), we have

$$\prod_{j \leq k}\left(1 - \frac{1}{p_j}\right)^{-1} = e^{\gamma}\log p_k + O\left(\frac{1}{(\log p_k)^2}\right). \tag{2}$$

We now estimate $\log p_k$. By a result of Cipolla [1], the prime number theorem with a modest error term implies that

$$p_k = k\left(\log k + \log_2 k - 1 + O\left(\frac{\log_2 k}{\log k}\right)\right). \tag{3}$$

Note that
$$\log k \leq \log y = \log_2 x - \log_3 x.$$

Then
$$\log_2 k \leq \log_3 x + O\left(\frac{\log_3 x}{\log_2 x}\right),$$

so that
$$\log k + \log_2 k \leq \log_2 x + O\left(\frac{\log_3 x}{\log_2 x}\right).$$

Taking the log of the equation in (3) and using the above inequalities we thus get

$$\log p_k \leq \log_2 x - \log_3 x + \log\left(\log_2 x - 1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right)$$

$$= \log_2 x - \frac{1}{\log_2 x} + O\left(\frac{\log_3 x}{(\log_2 x)^2}\right).$$

Using this with (1) and (2) we get

$$\frac{\sigma(n)}{n} \leq e^\gamma \log_2 x - \frac{e^\gamma}{\log_2 x} + O\left(\frac{\log_3 x}{(\log_2 x)^2}\right).$$

Since $\log_2 x - \log_2(x/2) \ll 1/\log x$ it follows that $n \notin \mathcal{NR}(x)$ for x large. This completes the proof. $\qquad\square$

We now prove the theorem. By [4, Theorem 6.1], for $k \geq y$ we have

$$\sum_{\substack{n \leq x \\ \omega(n)=k}} 1 \leq e^{O(y)},$$

uniformly. But since y is the maximal order of $\omega(n)$ for $n \leq x$, we have $\omega(n) \leq 2y$ for $n \leq x$ and x large. Thus,

$$\sum_{\substack{n \leq x \\ \omega(n)>y}} 1 \leq y e^{O(y)} = e^{O(y)}.$$

The lemma then gives the theorem.

3. Conclusion and open problems

It is interesting to consider the set $\mathcal{S}(x)$ of numbers $n \leq x$ with $\omega(n) > y$. In the lemma we showed that $(x/2, x] \cap \mathcal{NR}(x) \subset \mathcal{S}(x)$, and in the proof of the theorem, we showed that $\#\mathcal{S}(x) \leq e^{O(y)}$. How good is this estimate? A lower bound for $\#\mathcal{S}(x)$ can be found by letting n denote the product of the first $\lfloor y \rfloor + 1$ primes and then noting that $\omega(jn) > y$ for every integer j. So, $\#\mathcal{S}(x) \geq \lfloor x/n \rfloor$. A simple calculation not dissimilar from the above shows this quantity is $e^{(1+o(1))y}$ as $x \to \infty$.

An additional remark is that with slightly more effort a stronger lemma can be proved showing that if $\omega(n) \leq y + y/\log_2 x$ and $n \in (x/2, x]$, then $n \notin \mathcal{NR}(x)$. Presumably there are not many values of $n \leq x$ with $\omega(n) > y + y/\log_2 x$, and this may be a profitable line of attack to improve our theorem.

Let H_n denote the nth harmonic number, the reciprocal sum of the integers up to n. In [2] Lagarias leverages Robin's paper to prove that the Riemann Hypothesis

is equivalent to the inequality $\sigma(n) \leq H_n + \exp(H_n)\log(H_n)$ for all n. Since $H_n = \log n + \gamma + O(1/n)$,

$$\exp(H_n)\log(H_n) = e^\gamma n \log\log n + (\gamma e^\gamma + o(1))n/\log n,$$

so there are nominally fewer n's violating the Lagarias inequality than the Robin inequality; that is, our theorem pertains to exceptions to the Lagarias inequality. These thoughts also invite a possible improvement if one just aims to study exceptions to the Lagarias inequality.

References

[1] M. Cipolla, La determinazione assintotica dell nimo numero primo, *Matematiche Napoli* **3** (1902), 132–166.

[2] J. C. Lagarias, An elementary problem equivalent to the Riemann Hypothesis, *Amer. Math. Monthly* **109** (2002), 534–543.

[3] F. Luca and P. Solé, On Robin's inequality, *Integers* **25** (2025), #A8.

[4] C. Pomerance, On the distribution of round numbers, C. Pomerance, in *Number Theory Proceedings, Ootacamund, India 1984*, K. Alladi, ed., Lecture Notes in Math. 1122 (1985), 173–200.

[5] S. Ramanujan, Highly Composite Numbers, *Proc. London Math. Soc., Serie 2* **14** (1915), 347–400.

[6] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. Pures Appl.* **63** (1984), 187–213.

[7] I.M. Vinogradov, On the remainder in the Mertens' formula. (Russian), *Dokl. Akad. Nauk. SSSR* **148** (1963), 262–263.