CORRECTION TO: ON ROBIN'S INEQUALITY

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Abstract

Earlier this year, the first and third authors showed that there are at most $x^{o(1)}$ numbers $n \le x$ which violate Robin's inequality: $\sigma(n)/n < e^{\gamma} \log\log n$. Unfortunately, the proof contains an oversight. This could be corrected and the theorem strengthened to give the bound $x^{O(1/\log\log x)}$ along the lines of the original argument. Instead, we give a short proof of this more explicit bound using a result of the second author from 1985.

1. Introduction

Robin [6] (also see Ramanujan [5]) conjectured that $\sigma(n) < e^{\gamma} n \log\log n$ holds for all positive integers $n > 7!$, where $\sigma(n)$ is the sum of divisors of n. And he proved that the validity of this inequality is equivalent to the Riemann Hypothesis. In the recent paper [3], the first and third authors let

$$\mathcal{NR}(x) := \{7! < n \le x : \sigma(n) \ge e^{\gamma} n \log\log n\}$$

and proved various inequalities for $\#\mathcal{NR}(x)$. In particular, Theorem 3 in [3] claims that $\#\mathcal{NR}(x) \le x^{o(1)}$ holds as $x \to \infty$. Unfortunately, that proof contains an oversight which we correct here. In addition, we make the $o(1)$ from the exponent explicit. It is possible to give a proof along the lines of [3] but here we give a much shorter proof using the second author's paper [4].

2. Main result

We have the following theorem.

Theorem 1. *For $x > 7!$ we have*

$$\#\mathcal{NR}(x) = x^{O(1/\log\log x)}.$$

Let $p_1, p_2, \ldots$ denote the sequence of prime numbers. As usual, we let $\omega(n)$ denote the number of primes among the divisors of n, and we let $\log_j$ denote the j-times iterated log function. Let

$$y = y(x) = \log x / \log_2 x.$$

It is well-known that the maximal order of $\omega(n)$ for $n \le x$ is $\sim y$ (see, for example, Section 5 of Ramanujan's paper [5]). We first show that this maximal order is exceeded by members of $\mathcal{NR}(x)$.

Lemma 1. *For x sufficiently large and $n \in (x/2, x] \cap \mathcal{NR}(x)$, we have $\omega(n) > y$.*

Proof. Suppose that $n \in (x/2, x]$ and $\omega(n) = k \le y$. Then

$$\frac{\sigma(n)}{n} < \prod_{p|n} \Big(1 - \frac{1}{p}\Big)^{-1} \le \prod_{j \le k} \Big(1 - \frac{1}{p_j}\Big)^{-1}. \tag{1}$$

By a strong form of Mertens' theorem (cf. [7]), we have

$$\prod_{j \le k} \Big(1 - \frac{1}{p_j}\Big)^{-1} = e^{\gamma} \log p_k + O\Big(\frac{1}{(\log p_k)^2}\Big). \tag{2}$$

We now estimate $\log p_k$. By a result of Cipolla [1], the prime number theorem with a modest error term implies that

$$p_k = k\Big(\log k + \log_2 k - 1 + O\Big(\frac{\log_2 k}{\log k}\Big)\Big). \tag{3}$$

Note that

$$\log k \le \log y = \log_2 x - \log_3 x.$$

Then $\log_2 k \le \log_3 x$, so that

$$\log k + \log_2 k \le \log_2 x.$$

From $n \in (x/2, x] \cap \mathcal{NR}(x)$, (1), (2), and (3), we have $\log k \gg \log_2 x$, so that the error term in (3) can be replaced with $O(\log_3 x / \log_2 x)$. Taking the log of the equation in (3) and using the above inequalities we thus get

$$\begin{aligned}\log p_k &= \log k + \log\Big(\log k + \log_2 k - 1 + O\Big(\frac{\log_2 k}{\log k}\Big)\Big)\\ &\le \log_2 x - \log_3 x + \log\Big(\log_2 x - 1 + O\Big(\frac{\log_3 x}{\log_2 x}\Big)\Big)\\ &= \log_2 x - \frac{1}{\log_2 x} + O\Big(\frac{\log_3 x}{(\log_2 x)^2}\Big).\end{aligned}$$

Using this with (1) and (2) we get

$$\frac{\sigma(n)}{n} \le e^{\gamma}\log_2 x - \frac{e^{\gamma}}{\log_2 x} + O\Big(\frac{\log_3 x}{(\log_2 x)^2}\Big).$$

Since $\log_2 x - \log_2(x/2) \ll 1/\log x$ this contradicts $n \in \mathcal{NR}(x)$ for x large. This completes the proof. □

We now prove the theorem. By [4, Theorem 6.1], for $k \ge y$ we have

$$\sum_{\substack{n\le x\\ \omega(n)=k}} 1 \le e^{O(y)},$$

uniformly. But since y is the maximal order of $\omega(n)$ for $n \le x$, we have $\omega(n) \le 2y$ for $n \le x$ and x large. Thus,

$$\sum_{\substack{n\le x\\ \omega(n)>y}} 1 \le y e^{O(y)} = e^{O(y)}.$$

The lemma then gives the theorem.

3. Conclusion and open problems

It is interesting to consider the set $\mathcal{S}(x)$ of numbers $n \le x$ with $\omega(n) > y$. In the lemma we showed that $(x/2, x] \cap \mathcal{NR}(x) \subset \mathcal{S}(x)$, and in the proof of the theorem, we showed that $\#\mathcal{S}(x) \le e^{O(y)}$. How good is this estimate? A lower bound for $\#\mathcal{S}(x)$ can be found by letting n denote the product of the first $\lfloor y\rfloor + 1$ primes and then noting that $\omega(jn) > y$ for every integer j. So, $\#\mathcal{S}(x) \ge \lfloor x/n\rfloor$. A simple calculation not dissimilar from the above shows this quantity is $e^{(1+o(1))y}$ as $x \to \infty$.

An additional remark is that with slightly more effort a stronger lemma can be proved showing that if $\omega(n) \le y + y/\log_2 x$ and $n \in (x/2, x]$, then $n \notin \mathcal{NR}(x)$. Presumably there are not many values of $n \le x$ with $\omega(n) > y + y/\log_2 x$, and this may be a profitable line of attack to improve our theorem.

Let H_n denote the nth harmonic number, the reciprocal sum of the integers up to n. In [2] Lagarias leverages Robin's paper to prove that the Riemann Hypothesis is equivalent to the inequality $\sigma(n) \le H_n + \exp(H_n)\log(H_n)$ for all n. Since $H_n = \log n + \gamma + O(1/n)$, we have

$$\exp(H_n)\log(H_n) = e^\gamma n \log\log n + (\gamma e^\gamma + o(1))n/\log n.$$

Thus, if n violates the Lagarias inequality, then

$$\sigma(n) > e^\gamma n \log\log n + \text{something positive}.$$

So there are nominally fewer n's violating the Lagarias inequality than the Robin inequality; that is, our theorem pertains to exceptions to the Lagarias inequality. These thoughts also invite a possible improvement if one just aims to study exceptions to the Lagarias inequality.

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