On the parity of the number of small divisors of n

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To Professor Helmut Maier on his sixtieth birthday

Abstract For a positive integer j we look at the parity of the number of divisors of n that are at most j, proving that for large j, the count is even for most values of n.

1 Introduction

Let $\tau(n)$ denote the number of positive divisors of the positive integer n. It is easy to see that $\tau(n)$ is odd if and only if n is a square, so in the sense of asymptotic density, $\tau(n)$ is almost always even. In this note we consider the function $\tau_j(n) = \#\{d \mid n : d \leqslant j\}$, the number of positive divisors of n that are at most j. Here j is a positive integer. Can we say that $\tau_j(n)$ is usually even? Evidently not. This is patently false

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for $j = 1$, and it is false for all odd numbers n when $j \leqslant 2$. Here's another trivial case. Say n is not a square and $n/2 \leqslant j < n$. Then $\tau_j(n)$ is odd. In fact, if we list out all of the divisors of n: $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ and choose j at random in $[1,n]$, when n is not a square, more than half of the time $\tau_j(n)$ will be odd, since the top interval $[n/2,n)$ takes up half of the available values of j.

We are interested in the range $2 \leqslant j \leqslant \sqrt{n}$, showing that $\tau_j(n)$ tends to be even here. Let

$$\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607\ldots.$$

Theorem 1.1 *Let $N_j(x)$ denote the number of integers $n \leqslant x$ with $\tau_j(n)$ odd. Uniformly for $2 \leqslant j \leqslant \sqrt{x}$,*

$$N_j(x) = O\left(\frac{x}{(\log j)^{\delta/(1+\delta)}(\log\log(2j))^{1.5/(1+\delta)}}\right).$$

The theorem implies that when j is large and fixed, $\tau_j(n)$ is usually even as n varies.

It is interesting to look at this problem numerically. For a fixed number j, whether $\tau_j(n)$ is even or odd depends solely on the value of $\gcd(n,L_j)$, where L_j is the least common multiple of the integers in $[1,j]$. That is, $\tau_j(n) = \tau_j(\gcd(n,L_j))$. Thus, the set of integers n with $\tau_j(n)$ odd is a union of residue classes modulo L_j, so the asymptotic density of the set of such n exists; it is $N_j(L_j)/L_j$.

j	L_j	$N_j(L_j)$	$N_j(L_j)/L_j$
1	1	1	1
2	2	1	0.5
3	6	3	0.5
4	12	7	0.5833333333
5, 6	60	33	0.55
7	420	225	0.5357142857
8	840	405	0.4821428571
9	2520	1305	0.5178571429
10	2520	1235	0.4900793651
11, 12	27720	13635	0.4918831169
13	360360	177705	0.4931318681
14	360360	170775	0.4739010989
15	360360	170181	0.4722527473
16	720720	359073	0.4982142857
17	12252240	6106815	0.4984243697
18	12252240	5919705	0.4831528765
19	232792560	112887225	0.4849262580
20	232792560	109706355	0.4712622903
21	232792560	110362725	0.4740818392
22	232792560	107787735	0.4630205321
23, 24	5354228880	2496334995	0.4662361380
25	26771144400	12782443905	0.4774709558
26	26771144400	12538223775	0.4683484422
27	80313433200	37368330615	0.4652812005
28	80313433200	36653106105	0.4563757848

Our theorem implies that the right column approaches the limit 0 as $j \to \infty$ slightly faster, at the least, than $(\log j)^{-\delta/(1+\delta)}$.

In the following table we consider some larger values of j but only via some statistical experiments to approximate the density $N_j(L_j)/L_j$. The experiments involved taking the first 10^4 numbers following the kth prime, for $k = 10^5, 2\times 10^5, \dots, 6\times 10^5$. The numbers in the table are actual counts of the number of odd values of $\tau_j(n)$ among the 10^4 values of n. The numbers weakly suggest that $N_j(L_j)/L_j$ decays to 0 like $(\log j)^{-\theta}$ where θ is slightly above $1/2$. However, this too is misleading. Indeed, we will show below in Theorem 2.3 that $N(L_j)/L_j$ decays more slowly than about $1/(\log j)^{\delta}$. We do not resolve the issue of the "correct" exponent on $\log j$, but we do give a suggested plan for proving it is asymptotically δ.

j	10^5	2×10^5	3×10^5	4×10^5	5×10^5	6×10^5
100	4131	4121	4077	4099	4123	4109
200	4061	4107	4174	4181	4231	4050
300	3800	3850	3954	3980	4002	3969
400	3630	3703	3800	3744	3877	3875
500	3466	3587	3673	3710	3793	3772
600	3351	3512	3526	3594	3722	3682
700	3294	3435	3502	3543	3627	3593
800	3213	3301	3431	3475	3577	3574
900	2822	3245	3337	3411	3522	3477
1000	2358	3197	3248	3334	3459	3439

Throughout this note, the constants implied the by Landau symbol O and by the Vinogradov symbols $\ll$ and $\gg$ are absolute. We also use the notation $A \asymp B$ if $A \ll B \ll A$. We also write $a \| b$ for positive integers a, b if $a \mid b$ and $\gcd(a, b/a) = 1$.

2 Proof of Theorem 1.1

We begin with a criterion for $\tau_j(n)$ to be even.

Lemma 2.1 *Let $j \geqslant 2$. Suppose that there is a prime $p \| n$ and n has no divisor from the interval $(j, pj]$. Then $\tau_j(n)$ is even.*

Proof. Suppose the hypotheses hold. Let

$$A = \{d \mid n : d \leqslant j,\, p \nmid d\}, \quad B = \{d \mid n : d \leqslant j,\, p \mid d\},$$

so that $\{d \mid n : d \leqslant j\}$ is the disjoint union of A and B. It suffices to show that $\#A = \#B$. For each $d \in A$ consider pd. Then $pd \mid n$ and $pd \leqslant pj$. But by our hypothesis, we must then have $pd \leqslant j$, so that $pd \in B$. Thus, $\#A \leqslant \#B$. Now take $d \in B$. We may write $d = pd'$, where $d' \mid n$, $p \nmid d'$, and $d' \leqslant j/p \leqslant j$. Thus, $d' \in A$, which shows that $\#B \leqslant \#A$. So, $\#A = \#B$, completing the proof.

For real numbers $x \geqslant z \geqslant y \geqslant 1$, let $H(x,y,z)$ denote the number of integers $n \leqslant x$ which have a divisor in the interval $(y,z]$. From the main theorem in Ford [3], we have the following result.

Lemma 2.2 *Suppose that $y \geqslant 2$ and $2y \leqslant z \leqslant y^2 \leqslant x$. Let $u = \log z/\log y - 1$, so that $z = y^{1+u}$. Then,*

$$H(x,y,z) \asymp xu^{\delta}(\log(2/u))^{-3/2}.$$

Note that [3] states this result for $y \geqslant 100$ and $x \geqslant 100{,}000$, but by adjusting the implicit constants, the result can be seen to hold in the larger range asserted.

We now proceed to the proof of the theorem. Let $2 \leqslant k \leqslant j$ be a parameter to be chosen shortly. First note that the number of $n \leqslant x$ for which there is no prime $p \leqslant k$ with $p \| n$ is $O(x/\log k)$. Indeed this follows from sieve methods, in particular [4, Theorem 2.2]. Let $u = \log k/\log j$ and $z = kj = j^{1+u}$. By Lemma 2.2, the number of $n \leqslant x$ which have a divisor in $(j,z]$ is $O(xu^{\delta}(\log(2/u))^{-3/2})$. Let us equate these two O-estimates so as to fix our parameter k:

$$\frac{x}{\log k} = xu^{\delta}(\log(2/u))^{-3/2} = x\left(\frac{\log k}{\log j}\right)^{\delta}\left(\log\left(\frac{2\log j}{\log k}\right)\right)^{-3/2}.$$

After a small calculation this leads to a reasonable choice for k being

$$k = \exp\left((\log j)^{\delta/(1+\delta)}(\log\log(2j))^{1.5/(1+\delta)}\right).$$

With this choice of k we have that the number of $n \leqslant x$ for which it is not the case that both

- there is a prime $p \leqslant k$ with $p \| n$,
- n is free of divisors from the interval $(j,kj]$,

is $O(x/\log k)$. By Lemma 2.1, if both of these conditions hold, then $\tau_j(n)$ is even. With the choice of k given just above, this completes the proof.

It is clear that any integer n which has no prime factors in $[1,j]$ also has $\tau_j(n) = 1$; that is, $\tau_j(n)$ is odd. Thus,

$$N_j(x) \gg \frac{x}{\log j},$$

using [4, Theorem 2.5].

This "trivial" lower bound can be improved using ideas similar to those used to prove Theorem 1.1.

Theorem 2.3 *Let $c > 0$ be arbitrarily small. Uniformly for $j \leqslant x^{1/2-c}$ and x sufficiently large (in terms of c), we have*

$$N_j(x) \gg \frac{x}{(\log j)^{\delta}(\log\log(2j))^{3/2}}.$$

Proof. It follows from [3, Theorem 4] that for $j \leqslant x^{1/2-c}$, the number of integers $n \leqslant x$ with exactly 1 divisor in $(j/2,j]$ is of magnitude $x(\log j)^{-\delta}(\log\log(2j)^{-3/2}$.

Further, from the comments in the first paragraph of Section 1.3 in [3], the same is true if we ask in addition that n is odd. For such an odd number n, it follows by an argument akin to that of Lemma 2.1 that $\tau_j(2n)$ is odd. The claimed lower bound follows.

To close the gap between this lower bound and Theorem 1.1, the following strategy might be tried. It follows from Lemma 2.1 and its proof that if $\tau_j(n)$ is odd and if p is the least prime with $p\|n$, then n has a divisor in $(j, pj]$ that is divisible by p. (It is also possible that n has no prime factor p with $p\|n$, but such numbers are negligible.) Let $N_j(x,p)$ denote the number of integers $n \leqslant x$ such that (i) p is the least prime with $p\|n$ and (ii) n has a divisor in $(j, pj]$ divisible by p. Again following the thoughts in the first paragraph of Section 1.3 of [3], it may be possible to show that for each $p \leqslant \exp((\log j)^{1/2})$, $N_j(x,p)$ is uniformly bounded above by a constant times

$$\frac{1}{p\log p}\,\frac{x(\log p)^{\delta}}{(\log j)^{\delta}(\log\log(2j))^{3/2}}. \tag{1}$$

That is, a factor $p\log p$ is introduced in the denominator due to the condition that p is the least prime with $p\|n$. Summing this estimate for $p \leqslant \exp((\log j)^{1/2})$ yields the estimate $O(x(\log j)^{-\delta}(\log\log(2j))^{-3/2})$, with larger values of p being trivially negligible. Thus, we would have a match with the improved lower bound, at least for $j \leqslant x^{1/2-c}$.

The estimate (1) would follow if one could show that the number $N_j'(x,p)$ of integers $m \leqslant x/p$ having a divisor in $(j/p, j]$ and such that if $q\|m$ then $q > p$, is at most a constant times the expression in (1) for $p \leqslant \exp((\log j)^{1/2})$. Multiplying such a number m by p would cover all those n counted by $N_j(x,p)$, that is, $N_j(x,p) \leqslant N_j'(x,p)$. It would seem that upper bounding $N_j'(x,p)$ in this way is eminently provable using the ideas in [3], since integers m with such restrictions on their small prime divisors seem less likely to have a divisor in a given interval than integers in general.

3 A corollary

We saw at the start that if j is randomly chosen in $[1,n]$, it is more likely than not that $\tau_j(n)$ is odd, despite our theorem. This is because of the huge weight of the interval $[n/2,n)$. To equalize things, we might take a harmonic measure. For $y \in \mathbb{R}$, $y \geqslant 1$, let $\tau_y(n) = \tau_{\lfloor y\rfloor}(n)$. Let

$$S(n) = \{y \in [1,n] : \tau_y(n) \text{ is odd}\}, \quad f(n) = \frac{1}{\log n}\int_{S(n)} \frac{dy}{y}.$$

Then we always have $0 \leqslant f(n) \leqslant 1$. Further, if n is prime, then $f(n) = 1$, while if $n = 2p$ where p is prime, then $f(n) \to 0$ as $p \to \infty$. We can ask what is the normal value of the statistic $f(n)$. The following corollary of Theorem 1.1 addresses this question.

Corollary 3.1 *There is a set of integers $\mathscr{A}$ of asymptotic density* 1*, such that if $n \to \infty$ with $n \in \mathscr{A}$, then $f(n) \to 0$.*

Proof. Since we always have $f(n) \in [0,1]$, the assertion of the corollary is equivalent to

$$\sum_{n \leqslant x} f(n) = o(x), \quad x \to \infty,$$

and this in turn is equivalent to

$$\sum_{n \in (\frac{1}{2}x, x]} f(n) = o(x), \quad x \to \infty,$$

which is equivalent to

$$\sum_{n \in (\frac{1}{2}x, x]} \int_{S(n)} \frac{dy}{y} = o(x \log x), \quad x \to \infty. \tag{2}$$

For $n \in (\frac{1}{2}x, x]$, consider its divisors $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$, so that

$$\int_{S(n)} \frac{dy}{y} = \sum_{\substack{i < \tau(n) \\ i \text{ odd}}} \log\left(\frac{d_{i+1}}{d_i}\right).$$

The interval (d_i, d_{i+1}) has the companion interval $(n/d_{i+1}, n/d_i)$, which is the same as $(d_{\tau(n)-i}, d_{\tau(n)-i+1})$. Further if n is not a square, i is odd if and only if $\tau(n) - i$ is odd. Thus, for n not a square,

$$\int_{S(n)} \frac{dy}{y} = 2 \int_{S(n) \cap [1, \sqrt{n}]} \frac{dy}{y}.$$

Since the squares are negligible, to prove (2), it now suffices to prove that

$$\sum_{n \in (\frac{1}{2}x, x]} \int_{S(n) \cap [1, \sqrt{n}]} \frac{dy}{y} = o(x \log x), \quad x \to \infty. \tag{3}$$

This sum is equal to

$$\sum_{n \in (\frac{1}{2}x, x]} \sum_{\substack{j \leqslant \sqrt{n} \\ \tau_j(n) \text{ odd}}} \log \frac{j+1}{j},$$

except for a possible error of $o(1)$ as $x \to \infty$ caused by $j = \lfloor \sqrt{n} \rfloor$. Ignoring this triviality, the sum in (3) is now equal to

$$\sum_{j\leqslant\sqrt{x}}\log\frac{j+1}{j}\sum_{\substack{n\in(\frac{1}{2}x,x]\\ j\leqslant\sqrt{n}\\ \tau_j(n)\text{ odd}}}1\ll x\sum_{j\leqslant\sqrt{x}}\frac{1}{j(\log(2j))^{\delta/(1+\delta)}}\ll x(\log x)^{1/(1+\delta)},$$

using $\log((j+1)/j)<1/j$ and Theorem 1.1. Since $1/(1+\delta)<1$, it follows that (3) holds, and as we have seen, this is sufficient for the corollary. This completes the proof.

4 Final thoughts

Though the situation is much simpler in this note, the idea behind our Lemma 2.1 was inspired by the argument in Maier [5].

One might ask about other residue classes for $\tau_j(n)$. Our proof can show that for each fixed positive integer k, the set of numbers n such that $k\mid\tau(n)$ and $k\nmid\tau_j(n)$ has asymptotic density $o(1)$ as $j\to\infty$. For k not a power of 2, it might be interesting to investigate the density of those numbers n where $k\nmid\tau(n)$ and also $k\nmid\tau_j(n)$.

It is interesting to see several connections of this note to work of A. S. Besicovitch. First, Ford's theorem, cited in Lemma 2.2, is the latest chapter in a long story that began with work of Besicovitch [1] in 1934, when he showed that $\lim_{x\to\infty}H(x,y,2y)/x$ has liminf 0 as $y\to\infty$. And second, this note was motivated originally by looking for examples for sequences that perhaps violated a result known as the Besicovitch pseudometric, see [2]. In particular, it was thought if the densities $\lim_{x\to\infty}N_j(x)/x$ did not approach 0, then this would be a violation. It is interesting that we used a descendant of the 1934 Besicovitch result to show that these densites do approach 0 and so there is no counterexample.

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References

1. A. S. Besicovitch, "On the density of certain sequences of integers, *Math. Ann.* **110** (1934), 336–341.

2. F. Blanchard, E. Formenti, and P. Kůrka, “Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces”, *Complex Systems* **11** (1997), 107–123.
3. K. B. Ford, “The distribution of integers with a divisor in a given interval”, *Ann. of Math. (2)* **168** (2008), 367–433.
4. H. Halberstam and H.-E. Richert, “Sieve methods”, Academic Press, London, 1974.
5. H. Maier, “The size of the the coefficients of cyclotomic polynomials”, pp. 633–639 in B. Berndt, H. Diamond, and A. J. Hildebrand, eds., Analytic number theory, Proceedings of a conference in honor of Heini Halberstam, Vol. 2, Birkhäuser, Boston, 1996.