

# EXPLICIT ESTIMATES FOR THE DISTRIBUTION OF NUMBERS FREE OF LARGE PRIME FACTORS

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ABSTRACT. There is a large literature on the asymptotic distribution of numbers free of large prime factors, so-called *smooth* or *friable* numbers. But there is very little known about this distribution that is numerically explicit. In this paper we follow the general plan for the saddle point argument of Hildebrand and Tenenbaum, giving explicit and fairly tight intervals in which the true count lies. We give two numerical examples of our method, and with the larger one, our interval is so tight we can exclude the famous Dickman–de Bruijn asymptotic estimate as too small and the Hildebrand–Tenenbaum main term as too large.

## 1. INTRODUCTION

For a positive integer  $n > 1$ , denote by  $P(n)$  the largest prime factor of  $n$ , and let  $P(1) = 1$ . Let  $\Psi(x, y)$  denote the number of  $n \leq x$  with  $P(n) \leq y$ . Such integers  $n$  are known as  $y$ -smooth, or  $y$ -friable. Asymptotic estimates for  $\Psi(x, y)$  are quite useful in many applications, not least of which is in the analysis of factorization and discrete logarithm algorithms.

One of the earliest results is due to Dickman [7] in 1930, who gave an asymptotic formula for  $\Psi(x, y)$  in the case that  $x$  is a fixed power of  $y$ . Dickman showed that

$$(1.1) \quad \Psi(x, y) \sim x\rho(u) \quad (y \rightarrow \infty, x = y^u)$$

for every fixed  $u \geq 1$ , where  $\rho(u)$  is the “Dickman–de Bruijn” function, defined to be the continuous solution of the delay differential equation

$$\begin{aligned} u\rho'(u) + \rho(u-1) &= 0 & (u > 1), \\ \rho(u) &= 1 & (0 \leq u \leq 1). \end{aligned}$$

There remain the questions of the error in the approximation (1.1), and also the case when  $u = \log x / \log y$  is allowed to grow with  $x$  and  $y$ . In 1951, de Bruijn [4] proved that

$$\Psi(x, y) = x\rho(u) \left( 1 + O_\varepsilon \left( \frac{\log(1+u)}{\log y} \right) \right)$$

holds uniformly for  $x \geq 2$ ,  $\exp\{(\log x)^{5/8+\varepsilon}\} < y \leq x$ , for any fixed  $\varepsilon > 0$ . After improvements in the range of this result by Maier and Hensley, Hildebrand [12] showed that the de Bruijn estimate holds when  $\exp\{(\log \log x)^{5/3+\varepsilon}\} \leq y \leq x$ .

In 1986, Hildebrand and Tenenbaum [13] provided a uniform estimate for  $\Psi(x, y)$  for all  $x \geq y \geq 2$ , yielding an asymptotic formula when  $y$  and  $u$  tend to infinity.

The starting point for their method is an elementary argument of Rankin [17] from 1938, commonly known now as Rankin’s “trick”. For complex  $s$ , define

$$\zeta(s, y) = \sum_{\substack{n \geq 1 \\ P(n) \leq y}} n^{-s} = \prod_{p \leq y} (1 - p^{-s})^{-1}$$

(where  $p$  runs over primes) as the partial Euler product of the Riemann zeta function  $\zeta(s)$ . Then for  $0 < \sigma < 1$ , we have

$$(1.2) \quad \Psi(x, y) = \sum_{\substack{n \leq x \\ P(n) \leq y}} 1 \leq \sum_{P(n) \leq y} (x/n)^\sigma = x^\sigma \zeta(\sigma, y).$$

Then  $\sigma$  can be chosen optimally to minimize  $x^\sigma \zeta(\sigma, y)$ .

Let

$$\phi_j(s, y) = \frac{\partial^j}{\partial s^j} \log \zeta(s, y).$$

The function

$$\phi_1(s, y) = - \sum_{p \leq y} \frac{\log p}{p^s - 1}$$

is especially useful since the solution  $\alpha = \alpha(x, y)$  to  $\phi_1(\alpha, y) + \log x = 0$  gives the optimal  $\sigma$  in (1.2). We also denote  $\sigma_j(x, y) = |\phi_j(\alpha(x, y), y)|$ .

In this language, Hildebrand and Tenenbaum [13] proved that the estimate

$$\Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi \sigma_2(x, y)}} \left( 1 + O\left(\frac{1}{u} + \frac{\log y}{y}\right) \right)$$

holds uniformly for  $x \geq y \geq 2$ . As suggested by this formula, quantities  $\alpha(x, y)$  and  $\sigma_2(x, y)$  are of interest in their own right, and were given uniform estimates which imply the formulae

$$\alpha(x, y) \sim \frac{\log(1 + y/\log x)}{\log y}$$

and

$$\sigma_2(x, y) \sim \left( 1 + \frac{\log x}{y} \right) \log x \log y,$$

together which imply

$$\begin{aligned} \Psi(x, y) &\sim \frac{x^\alpha \zeta(\alpha, y)}{\sqrt{2\pi u \log(y/\log x)}} && \text{(if } y/\log x \rightarrow \infty), \\ \Psi(x, y) &\sim \frac{x^\alpha \zeta(\alpha, y)}{\sqrt{2\pi y/\log y}} && \text{(if } y/\log x \rightarrow 0). \end{aligned}$$

These formulae indicate that  $\Psi(x, y)$  undergoes a “phase change” when  $y$  is of order  $\log x$ , see [3]. This paper concentrates on the range where  $y$  is considerably larger, say  $y > (\log x)^4$ .

The primary aim of this paper is to make the Hildebrand–Tenenbaum method explicit and so effectively construct an algorithm for obtaining good bounds for  $\Psi(x, y)$ .

1.1. **Explicit Results.** Beyond the Rankin upper bound  $\Psi(x, y) \leq x^\alpha \zeta(\alpha, y)$ , we have the explicit lower bound

$$\Psi(x, y) \geq x^{1-\log \log x / \log y} = \frac{x}{(\log x)^u}$$

due to Konyagin and Pomerance [11]. Recently Granville and Soundararajan [10] found an elementary improvement of Rankin’s upper bound, which they have graciously permitted us to include in an appendix in this paper. In particular, they show that

$$\Psi(x, y) \leq 1.39y^{1-\sigma} x^\sigma \zeta(\sigma, y) / \log x$$

for every value of  $\sigma \in [1/\log y, 1]$ , see Theorem 5.1.

In another direction, by relinquishing the goal of a compact formula, several authors have devised algorithms to compute bounds on  $\Psi(x, y)$  for given  $x, y$  as inputs. For example, using an accuracy parameter  $c$ , Bernstein [2] created an algorithm to generate bounds  $B^-(x, y) \leq \Psi(x, y) \leq B^+(x, y)$  with

$$\frac{B^-}{\Psi} \geq 1 - \frac{\log x}{c \log 3 / \log 2} \quad \text{and} \quad \frac{B^+}{\Psi} \leq 1 + \frac{2 \log x}{c \log 3 / \log 2},$$

running in

$$O\left(\frac{y}{\log_2 y} + \frac{y \log x}{\log^2 y} + c \log x \log c\right)$$

time. Parsell and Sorenson [15] refined this algorithm to run in

$$O\left(c \frac{y^{2/3}}{\log y} + c \log x \log c\right)$$

time, as well as obtaining faster and tighter bounds assuming the Riemann Hypothesis. The largest example computed by this method was an approximation of  $\Psi(2^{255}, 2^{28})$ .

As seen in Figure 1, the bounds presented in this paper far outshine best-known upper and lower bounds for the two examples presented. We also provide the main term estimates  $x^\alpha \zeta(\alpha, y) / \alpha \sqrt{2\pi\sigma_2}$  from [13] and  $\rho(u)x$  from [7] as points of reference. It is interesting that our estimates in the second example are closer to the truth than are the Dickman–de Bruijn and Hildebrand–Tenenbaum main terms. The second-named author has asked if  $\Psi(x, y) \geq x\rho(u)$  holds in general for  $x \geq 2y \geq 2$ , see [9, (1.25)]. This inequality is known for  $u$  bounded and  $x$  sufficiently large, see the discussion in [14, Section 9].

FIGURE 1. Examples.

$x$	$10^{100}$	$10^{500}$
$y$	$10^{15}$	$10^{35}$
KP	$1.786 \cdot 10^{84}$	$1.857 \cdot 10^{456}$
R	$4.599 \cdot 10^{96}$	$9.639 \cdot 10^{484}$
GS	$5.350 \cdot 10^{95}$	$6.596 \cdot 10^{483}$
DD	$2.523 \cdot 10^{94}$	$1.472 \cdot 10^{482}$
HT	$2.652 \cdot 10^{94}$	$1.5127 \cdot 10^{482}$
$\Psi^-$	$2.330 \cdot 10^{94}$	$1.4989 \cdot 10^{482}$
$\Psi^+$	$2.923 \cdot 10^{94}$	$1.5118 \cdot 10^{482}$

Here,

KP is the Konyagin–Pomerance lower bound  $x/(\log x)^u$ ,

R is the Rankin upper bound  $x^\alpha \zeta(\alpha, y)$ ,

GS is the Granville–Soundararajan upper bound  $1.39y^{1-\alpha} x^\alpha \zeta(\alpha, y)/\log x$ ,

DD is the Dickman–de Bruijn main term  $\rho(u)x$ , and

HT is the Hildebrand–Tenenbaum main term  $x^\alpha \zeta(\alpha, y)/(\alpha\sqrt{2\pi\sigma_2})$ .

Our principal result, which benefits from some notation developed over the course of the paper, is Theorem 3.11. It is via this theorem that we were able to estimate  $\Psi(10^{100}, 10^{15})$  and  $\Psi(10^{500}, 10^{35})$  as in the table above.

## 2. PLAN FOR THE PAPER

The basic strategy of the saddle-point method relies on Perron’s formula, which implies the identity

$$\Psi(x, y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s, y) \frac{x^s}{s} ds,$$

for any  $\sigma > 0$ . It turns out that the best value of  $\sigma$  to use is  $\alpha = \alpha(x, y)$  discussed in the Introduction. We are interested in abridging the integral at a certain height  $T$  and then approximating the contribution given by the tail. To this end, we have

$$(2.1) \quad \Psi(x, y) = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \zeta(s, y) \frac{x^s}{s} ds + \text{Error}.$$

There is a change in behavior occurring in  $\zeta(s, y)$  when  $t$  is on the order  $1/\log y$ . In [13] it is shown that

$$(2.2) \quad \begin{aligned} \left| \frac{\zeta(s, y)}{\zeta(\alpha, y)} \right| &= \prod_{p \leq y} \left| \frac{1-p^{-\alpha}}{1-p^{-s}} \right| = \prod_{p \leq y} \left( 1 + \frac{2(1-\cos(t \log p))}{p^\alpha(1-p^{-\alpha})^2} \right)^{-1/2} \\ &\leq \exp \left\{ - \sum_{p \leq y} \frac{1-\cos(t \log p)}{p^\alpha} \right\}. \end{aligned}$$

Thus when  $t$  is small (compared to  $1/\log y$ ) the oscillatory terms are in resonance, and when  $t$  is large the oscillatory terms should exhibit cancellation. This behavior

suggests we should divide our range of integration into  $|t| \leq T_0$  and  $T_0 < |t| < T$ , where  $T_0 \approx 1/\log y$  is a parameter to be optimized.

The contribution for  $|t| \leq T_0$  will constitute a “main term”, and so we will try to estimate this part very carefully. In this range we forgo (2.2) and attack the integrand  $\zeta(s, y)x^s/s$  directly. The basic idea is to expand  $\phi(s, y) = \log \zeta(s, y)$  as a Taylor series in  $t$ . This approach, when carefully done, gives us fairly close upper and lower bounds for the integral. In our smaller example, the upper bound is less than 1% higher than the lower bound, and in the larger example, this is better by a factor of 20. Considerably more noise is encountered beyond  $T_0$  and in the Error in (2.1).

For the second range  $T_0 < |t| < T$ , we focus on obtaining a satisfactory lower bound on the sum over primes,

$$\sum_{p \leq y} \frac{1 - \cos(t \log p)}{p^\alpha}.$$

Our strategy is to sum the first  $L$  terms directly, and then obtain an analytic formula  $W(y, w)$  to lower bound the remaining terms starting at some  $w \geq L$ , where essentially

$$W(y, w) = \frac{y^{1-\alpha} - w^{1-\alpha}}{1 - \alpha} + \text{error}.$$

With an explicit version of Perron’s formula, the Error in (2.1) may be handled by

$$\begin{aligned} |\text{Error}| &\leq x^\alpha \sum_{P(n) \leq y} \frac{1}{n^\alpha} \min \left( 1, \frac{1}{\pi T |\log(x/n)|} \right) \\ &\leq x^\alpha \sum_{\substack{P(n) \leq y \\ T |\log(x/n)| > T^d}} \frac{1}{n^\alpha} \frac{1}{\pi T |\log(x/n)|} + \sum_{\substack{P(n) \leq y \\ T |\log(x/n)| \leq T^d}} \left( \frac{x}{n} \right)^\alpha \\ &\leq \frac{x^\alpha \zeta(\alpha, y)}{\pi T^d} + e^{\alpha T^{d-1}} \left[ \Psi(xe^{T^{1-d}}, y) - \Psi(xe^{-T^{1-d}}, y) \right]. \end{aligned}$$

Here  $d \approx \frac{1}{2}$  is a parameter of our choosing, which we set to balance the two terms above. Thus the problem of bounding  $|\text{Error}|$  is reduced to estimating the number of  $y$ -smooth integers in the “short” interval  $(xe^{-T^{1-d}}, xe^{T^{1-d}}]$ .

This latter portion is better handled when  $T$  is large, but the earlier portion in the range  $[T_0, T]$  is better handled when  $T$  is small. Thus,  $T$  is numerically set to balance these two forces.

In our proofs we take full advantage of some recent calculations involving the prime-counting function  $\pi(x)$  and the Chebyshev functions

$$\psi(x) = \sum_{p^m \leq x} \log p, \quad \vartheta(x) = \sum_{p \leq x} \log p,$$

with  $p$  running over primes and  $m$  running over positive integers. As a corollary of the papers [5], [6] of Büthe we have the following excellent result.

**Proposition 2.1.** *For  $1427 \leq x \leq 10^{19}$  we have*

$$.05\sqrt{x} \leq x - \vartheta(x) \leq 1.95\sqrt{x}.$$

We have

$$\frac{|\vartheta(x) - x|}{x} < \begin{cases} 2.3 \cdot 10^{-8}, & \text{when } x > 10^{19}, \\ 1.2 \cdot 10^{-8}, & \text{when } x > e^{45}, \\ 1.2 \cdot 10^{-9}, & \text{when } x > e^{50}, \\ 2.9 \cdot 10^{-10}, & \text{when } x > e^{55}. \end{cases}$$

*Proof.* The first assertion is one of the main results in Büthe [6]. Let  $H$  be a number such that all zeros of the Riemann zeta-function with imaginary parts in  $[0, H]$  lie on the  $1/2$ -line. Inequality (7.4) in Büthe [5] asserts that if  $x/\log x \leq H^2/4.92^2$  and  $x \geq 5000$ , then

$$\frac{|\vartheta(x) - x|}{x} < \frac{(\log x - 2) \log x}{8\pi\sqrt{x}}.$$

We can take  $H = 3 \cdot 10^{10}$ , see Platt [16]. Thus, we have the result in the range  $10^{19} \leq x \leq e^{45}$ . For  $x \geq e^{45}$  we have from Büthe [5] that  $|\psi(x) - x|/x \leq 1.118 \cdot 10^{-8}$ . Further, we have (see [18, (3.39)]) for  $x > 0$ ,

$$\psi(x) \geq \vartheta(x) > \psi(x) - 1.02x^{1/2} - 3x^{1/3}.$$

(This result can be improved, but it is not important to us.) Thus, for  $x \geq e^{45}$  we have  $|\vartheta(x) - x|/x \leq 1.151 \cdot 10^{-8}$ , establishing our result in this range. For the latter two ranges we argue similarly, using  $|\psi(x) - x| \leq 1.165 \cdot 10^{-9}$  when  $x \geq e^{50}$  and  $|\psi(x) - x| \leq 2.885 \cdot 10^{-10}$  for  $x \geq e^{55}$ , both of these inequalities coming from [5].  $\square$

We remark that there are improved inequalities at higher values of  $x$ , found in [5] and [8], which one would want to use if estimating  $\Psi(x, y)$  for larger values of  $y$  than we have done here.

### 3. THE MAIN ARGUMENT

As in the Introduction, for complex  $s$ , define

$$\zeta(s, y) = \sum_{\substack{n \geq 1 \\ P(n) \leq y}} n^{-s} = \prod_{p \leq y} (1 - p^{-s})^{-1},$$

which is the Riemann zeta function restricted to  $y$ -smooth numbers, and for  $j \geq 0$ , let

$$\phi_j(s, y) = \frac{\partial^j}{\partial s^j} \log \zeta(s, y).$$

We have the explicit formulae,

$$\begin{aligned}\phi_1(s, y) &= -\sum_{p \leq y} \frac{\log p}{p^s - 1}, \\ \phi_2(s, y) &= \sum_{p \leq y} \frac{p^s \log^2 p}{(p^s - 1)^2}, \\ \phi_3(s, y) &= -\sum_{p \leq y} \frac{(p^{2s} + p^s) \log^3 p}{(p^s - 1)^3}, \\ \phi_4(s, y) &= \sum_{p \leq y} \frac{(p^{3s} + 4p^{2s} + p^s) \log^4 p}{(p^s - 1)^4}, \\ \phi_5(s, y) &= -\sum_{p \leq y} \frac{(p^{4s} + 11p^{3s} + 11p^{2s} + p^s) \log^5 p}{(p^s - 1)^5}.\end{aligned}$$

Note that for  $y \geq 2$ ,  $\sigma > 0$ ,  $\phi_1(\sigma, y)$  is strictly increasing from 0, so there is a unique solution  $\alpha = \alpha(x, y) > 0$  to the equation

$$\log x + \phi_1(\alpha, y) = 0.$$

Since we cannot exactly solve this equation, we shall assume any choice of  $\alpha$  that we use is a reasonable approximation to the exact solution, and we must take into account an upper bound for the difference between our value and the exact value. We denote

$$\phi_j = \phi_j(\alpha, y), \quad \sigma_j = |\phi_j| = (-1)^j \phi_j, \quad B_j = B_j(t) = \sigma_j t^j / j!$$

so that the Taylor series of  $\phi(s, y) = \log \zeta(s, y)$  about  $s = \alpha$  is

$$\phi(\alpha + it, y) = \sum_{j \geq 0} \frac{\sigma_j}{j!} (-it)^j = \sum_{j \geq 0} (-i)^j B_j.$$

Our first result, which is analogous to Lemma 10 in [13], sets the stage for our estimates.

**Lemma 3.1.** *Let  $0 < d < 1$  and  $T > 1$ . We have that*

$$\begin{aligned}\left| \Psi(x, y) - \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \zeta(s, y) \frac{x^s}{s} ds \right| \\ \leq \frac{x^\alpha \zeta(\alpha, y)}{\pi T^d} + e^{\alpha T^{d-1}} \left[ \Psi(xe^{T^{d-1}}, y) - \Psi(xe^{-T^{d-1}}, y) \right].\end{aligned}$$

*Proof.* We have

$$\begin{aligned}\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \zeta(s, y) \frac{x^s}{s} ds &= \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \sum_{P(n) \leq y} \left( \frac{x}{n} \right)^s \frac{ds}{s} \\ &= \sum_{P(n) \leq y} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \left( \frac{x}{n} \right)^s \frac{ds}{s},\end{aligned}$$

where the interchange of sum and integral is justified since  $\zeta(s, y)$  is a finite product, hence uniformly convergent as a sum.

By Perron's formula (see [1, §11.12]), we have

$$\left| \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} \right| \leq \frac{(x/n)^\alpha}{\min(1, \pi T |\log(x/n)|)} \quad \text{if } n > x,$$

$$\left| 1 - \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \left(\frac{x}{n}\right)^s \frac{ds}{s} \right| \leq \frac{(x/n)^\alpha}{\min(1, \pi T |\log(x/n)|)} \quad \text{if } n \leq x.$$

Together these imply

$$\begin{aligned} \left| \Psi(x, y) - \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \zeta(s, y) \frac{x^s}{s} ds \right| &\leq x^\alpha \sum_{P(n) \leq y} \frac{1}{n^\alpha} \min\left(1, \frac{1}{\pi T |\log(x/n)|}\right) \\ &\leq x^\alpha \sum_{\substack{P(n) \leq y \\ |\log(x/n)| > T^{d-1}}} \frac{1}{n^\alpha} \frac{1}{\pi T |\log(x/n)|} + x^\alpha \sum_{\substack{P(n) \leq y \\ |\log(x/n)| \leq T^{d-1}}} \frac{1}{n^\alpha} \\ &\leq \frac{x^\alpha \zeta(\alpha, y)}{\pi T^d} + e^{\alpha T^{d-1}} \left[ \Psi(xe^{T^{d-1}}, y) - \Psi(xe^{-T^{d-1}}, y) \right]. \end{aligned}$$

This completes the proof.  $\square$

In using this result we have the problems of performing the integration from  $\alpha - iT$  to  $\alpha + iT$  and estimating the number of  $y$ -smooth integers in the interval  $(xe^{-T^{d-1}}, xe^{T^{d-1}}]$ . We turn first to the integral evaluation.

Recall that  $B_j = B_j(t) = \sigma_j(x, y)t^j/j!$  and let  $B_1^* = B_1^*(t) = t \log x - B_1(t)$ . Note that  $B_1^* = 0$  if  $\alpha$  is chosen perfectly.

**Lemma 3.2.** *For  $s = \alpha + it$ , we have*

$$\Re \left\{ \zeta(s, y) \frac{x^s}{s} \right\} = \frac{x^\alpha \zeta(\alpha, y)}{\alpha^2 + t^2} (\alpha \cos(B_3 + B_1^* + b_5) + t \sin(B_3 + B_1^* + b_5)) \exp \{ -B_2 + B_4 + a_5 \},$$

where  $a_5, b_5$  are real numbers, depending on the choice of  $t$ , with  $|a_5 + ib_5| \leq B_5(t)$ .

*Proof.* We expand  $\phi(\alpha + it, y) = \log \zeta(\alpha + it, y)$  in a Taylor series around  $t = 0$ . There exists some real  $\xi$  between 0 and  $t$  such that

$$\begin{aligned} \phi(\alpha + it, y) &= \phi(\alpha, y) + it\phi_1 - \frac{t^2}{2}\phi_2 - \frac{it^3}{3!}\phi_3 + \frac{t^4}{4!}\phi_4 - i\frac{t^5}{5!}(\alpha + i\xi, y) \\ &= B_0 - iB_1 - B_2 + iB_3 + B_4 - i\frac{t^5}{5!}\phi_5(\alpha + i\xi, y). \end{aligned}$$

Since  $\zeta(s, y) = \exp(\phi(s, y))$ , we obtain

$$\begin{aligned} \zeta(s, y) \frac{x^s}{s} &= \frac{\zeta(\alpha, y)x^\alpha}{\alpha + it} \exp \left\{ it \log x - iB_1 - B_2 + iB_3 + B_4 + i\frac{t^5}{5!}\phi_5(\alpha + i\xi, y) \right\} \\ &= \frac{x^\alpha \zeta(\alpha, y)}{\alpha + it} \exp \left\{ -B_2 + B_4 + i(B_1^* + B_3) + i\frac{t^5}{5!}\phi_5(\alpha + i\xi, y) \right\}. \end{aligned}$$

Letting  $i\phi_5(\alpha + i\xi)t^5/5! = a_5 + b_5i$ , we have

$$\begin{aligned} \zeta(s, y) \frac{x^s}{s} &= \frac{x^\alpha \zeta(\alpha, y)}{\alpha^2 + t^2} (\alpha - it) (\cos(B_1^* + B_3 + b_5) + i \sin(B_1^* + B_3 + b_5)) \exp \{ -B_2 + B_4 + a_5 \}, \end{aligned}$$



and taking the real part gives the result.  $\square$

The main contribution to the integral in Lemma 3.1 turns out to come from the interval  $[-T_0, T_0]$ , where  $T_0$  is fairly small. We have

$$\frac{1}{2\pi i} \int_{\alpha-iT_0}^{\alpha+iT_0} \zeta(s, y) \frac{x^s}{s} ds = \frac{1}{2\pi} \int_{-T_0}^{T_0} \zeta(\alpha + it, y) \frac{x^{\alpha+it}}{\alpha + it} dt.$$

Note that the integrand, written as a Taylor series around  $s = \alpha$ , has real coefficients, so the real part is an even function of  $t$  and the imaginary part is an odd function. Thus, the integral is real, and its value is double the value of the integral on  $[0, T_0]$ .

Consider the cosine, sine combination in Lemma 3.2:

$$f(t, v) := \alpha \cos(B_3(t) + v) + t \sin(B_3(t) + v),$$

and let

$$v_0(t) = |B_1^*(t)| + B_5(t).$$

We have, for each value of  $t$ , the constraint that  $|v| \leq v_0(t)$ . The partial derivative of  $f(t, v)$  with respect to  $v$  is zero when  $\arctan(t/\alpha) - B_3(t) \equiv 0 \pmod{\pi}$ . Let

$$u(t) = \arctan(t/\alpha) - B_3(t).$$

If  $u(t) \notin [-v_0(t), v_0(t)]$ , then  $f(t, v)$  is monotone in  $v$  on that interval; otherwise it has a min or max at  $u(t)$ . Let  $T_3, T_2, T_1, T_0$  be defined, respectively, as the least positive solutions of the equations

$$u(t) = v_0(t), \quad u(t) = -v_0(t), \quad u(t) + \pi = v_0(t), \quad u(t) + \pi = -v_0(t).$$

Then  $0 < T_3 < T_2 < T_1 < T_0$ . We have the following properties for  $f(t, v)$ :

- (1) For  $t$  in the interval  $[0, T_3]$  we have  $f(t, v)$  increasing for  $v \in [-v_0(t), v_0(t)]$ , so that

$$f(t, -v_0(t)) \leq f(t, v) \leq f(t, v_0(t)).$$

- (2) For  $t$  in the interval  $[T_3, T_2]$ , we have  $f(t, v)$  increasing for  $-v_0(t) \leq v \leq u(t)$  and then decreasing for  $u(t) \leq v \leq v_0(t)$ . Thus,

$$\min\{f(t, -v_0(t)), f(t, v_0(t))\} \leq f(t, v) \leq f(t, u(t)).$$

- (3) For  $t \in [T_2, T_1]$ ,  $f(t, v)$  is decreasing for  $v \in [-v_0(t), v_0(t)]$ , so that

$$f(t, v_0(t)) \leq f(t, v) \leq f(t, -v_0(t)).$$

- (4) For  $t \in [T_1, T_0]$ , we have  $f(t, v)$  decreasing for  $v \in [-v_0(t), u(t) + \pi]$  and increasing for  $v \in [u(t) + \pi, v_0(t)]$ ; that is,

$$f(t, u(t) + \pi) \leq f(t, v) \leq \max\{f(t, -v_0(t)), f(t, v_0(t))\}.$$

Note too that  $f(t, v)$  has a sign change from positive to negative in the interval  $[T_2, T_1]$ . Let  $Z^-, Z^+$  be, respectively, the least positive roots of  $f(t, v(t)) = 0$ ,  $f(t, -v(t)) = 0$ .

Let  $I_0^+$  be an upper bound for the function appearing in Lemma 3.2 on  $[0, T_0]$  using  $|a_5|, |b_5| \leq B_5$  and the above facts about  $f(t, v)$ , and let  $I_0^-$  be the corresponding lower bound. We choose  $a_5 = B_5$  in  $I_0^+$  when the cos, sin combination is positive, and  $a_5 = -B_5$  when it is negative. For  $I_0^-$ , we choose  $a_5$  in the reverse way.

Let

$$(3.1) \quad J_0^+ = \int_0^{T_0} I_0^+(t) dt, \quad J_0^- = \int_0^{T_0} I_0^-(t) dt.$$

We thus have the following result, which is our analogue of Lemma 11 in [13].

**Lemma 3.3.** *We have*

$$\frac{x^\alpha \zeta(\alpha, y)}{\pi} J_0^- \leq \frac{1}{2\pi i} \int_{\alpha - iT_0}^{\alpha + iT_0} \zeta(s, y) \frac{x^s}{s} ds \leq \frac{x^\alpha \zeta(\alpha, y)}{\pi} J_0^+.$$

In order to estimate the integral in Lemma 3.1 when  $|t| > T_0$  we must know something about prime sums to  $y$ .

**Lemma 3.4.** *We have*

$$\left| \int_{\alpha + iT_0}^{\alpha + iT} \zeta(s, y) \frac{x^s}{s} ds \right| \leq x^\alpha \zeta(\alpha, y) J_1,$$

where

$$J_1 := \int_{T_0}^T \exp(-W(y, 1, t)) \frac{dt}{\sqrt{\alpha^2 + t^2}}$$

and

$$(3.2) \quad W(v, w, t) := \sum_{w < p \leq v} \frac{1 - \cos(t \log p)}{p^\alpha}.$$

*Proof.* For  $0 \leq v \leq 1 < t$ , equation (3.14) in [13] states that

$$(1 + 4vt/(t-1)^2)^{-1} \leq \exp\{-4v/t\}.$$

Applied to (3.17) in [13] with  $v = (1 - \cos(t \log p))/2$ , we have that

$$(3.3) \quad \begin{aligned} \left| \frac{\zeta(s, y)}{\zeta(\alpha, y)} \right| &= \prod_{p \leq y} \left| \frac{1 - p^{-\alpha}}{1 - p^{-s}} \right| = \prod_{p \leq y} \left( 1 + \frac{2(1 - \cos(t \log p))}{p^\alpha (1 - p^{-\alpha})^2} \right)^{-1/2} \\ &\leq \exp \left\{ - \sum_{p \leq y} \frac{1 - \cos(t \log p)}{p^\alpha} \right\}. \end{aligned}$$

This completes the proof.  $\square$

Our goal now is to find a way to estimate  $W(v, w, t)$ . The following result is analogous to Lemma 6 in [13].

**Lemma 3.5.** *Let  $s$  be a complex number, let  $1 < w < v$ , and define*

$$F_s(v, w) := \sum_{w < p \leq v} \frac{\log p}{p^s} - \frac{v^{1-s} - w^{1-s}}{1-s}.$$

(i) *If  $v \leq 10^{19}$  we have*

$$|F_s(v, w)| \leq 2(v^{1/2-\alpha} + w^{1/2-\alpha}) + 2|s| \frac{w^{1/2-\alpha} - v^{1/2-\alpha}}{\alpha - 1/2}.$$

(ii) *If  $10^{19} \leq w \leq v$  we have*

$$|F_s(v, w)| \leq \varepsilon_w \left( v^\beta + w^\beta + |s| \frac{v^\beta - w^\beta}{\beta} \right),$$

where  $\beta = 1 - \alpha$  and

$$\varepsilon_w = \begin{cases} 2.3 \cdot 10^{-8}, & w \in (10^{19}, e^{45}], \\ 1.2 \cdot 10^{-8}, & w \in (e^{50}, e^{55}], \\ 1.2 \cdot 10^{-9}, & w \in (e^{50}, e^{55}], \\ 2.9 \cdot 10^{-10}, & w > e^{55}. \end{cases}$$

*Proof.* (i) By partial summation,

$$\begin{aligned} \sum_{w < p \leq v} \frac{\log p}{p^s} &= \frac{\vartheta(v)}{v^s} - \frac{\vartheta(w)}{w^s} + \int_w^v s \frac{\vartheta(t)}{t^{s+1}} dt \\ &= \frac{v^{1-s} - w^{1-s}}{1-s} - \frac{E(v)}{v^s} + \frac{E(w)}{w^s} - \int_w^v s \frac{E(t)}{t^{s+1}} dt, \end{aligned}$$

so by the first part of Proposition 2.1,

$$\begin{aligned} |F_s(v, w)| &\leq \frac{|E(v)|}{v^\alpha} + \frac{|E(w)|}{w^\alpha} + |s| \int_w^v \frac{E(t)}{t^{1+\alpha}} dt \\ &\leq 2v^{1/2-\alpha} + 2w^{1/2-\alpha} + 2|s| \frac{v^{1/2-\alpha} - w^{1/2-\alpha}}{1/2 - \alpha}. \end{aligned}$$

(ii) Similarly, by the second part of Proposition 2.1,

$$\begin{aligned} |F_s(v, w)| &\leq \frac{|E(v)|}{v^\alpha} + \frac{|E(w)|}{w^\alpha} + |s| \int_w^v \frac{E(t)}{t^{1+\alpha}} dt \leq \varepsilon_w \left( v^{1-\alpha} + w^{1-\alpha} + |s| \int_w^v \frac{dt}{t^\alpha} \right) \\ &= \varepsilon_w \left( v^{1-\alpha} + w^{1-\alpha} + |s| \frac{v^{1-\alpha} - w^{1-\alpha}}{1-\alpha} \right). \end{aligned}$$

□

The following result plays the role of Corollary 6.1 in [13].

**Lemma 3.6.** *For  $t \in \mathbb{R}$ ,  $z > 1$ , and  $\beta = 1 - \alpha$ , let*

$$\delta_z := t \log z - \arctan(t/\beta).$$

(i) *For  $1427 \leq w < v \leq 10^{19}$  we have that  $W(v, w, t) \geq W_0(v, w, t)$ , where*

$$\begin{aligned} W_0(v, w, t) \log v &= \frac{v^\beta - w^\beta}{\beta} - \frac{v^\beta \cos \delta_v - w^\beta \cos \delta_w}{\sqrt{\beta^2 + t^2}} \\ &\quad - 4(v^{1/2-\alpha} + w^{1/2-\alpha}) - 2(\alpha + |s|) \frac{w^{1/2-\alpha} - v^{1/2-\alpha}}{\alpha - 1/2}. \end{aligned}$$

(ii) *For  $10^{19} \leq w < v$  we have that  $W(v, w, t) \geq W_0(v, w, t)$ , where*

$$\begin{aligned} W_0(v, w, t) \log v &= \frac{v^\beta - w^\beta}{\beta} - \frac{v^\beta \cos \delta_v - w^\beta \cos \delta_w}{\sqrt{\beta^2 + t^2}} \\ &\quad - 2\varepsilon_w(v^\beta + w^\beta) - \varepsilon_w(\alpha + |s|) \left( \frac{v^\beta - w^\beta}{\beta} \right). \end{aligned}$$

*Proof.* We apply Lemma 3.5 with  $s = 1 - \beta$  and  $s = 1 - \beta + it$ , and take the real part of the difference. Letting the difference of the sums be  $S$ , we have that

$$S := \sum_{w < p \leq v} \left( \frac{\log p}{p^{1-\beta}} - \frac{\log p}{p^{1-\beta+it}} \right) = \sum_{w < p \leq v} \frac{\log p}{p^{1-\beta}} (1 - p^{-it}), \text{ so}$$

$$\Re(S) = \sum_{w < p \leq v} \frac{\log p}{p^{1-\beta}} (1 - \cos(t \log p)),$$

which is the sum we wish to bound.

For a positive real number  $z$ , let  $S_z := \frac{z^\beta}{\beta} - \frac{z^{\beta-it}}{\beta-it}$ . We have that

$$S_z = \frac{z^\beta}{\beta} \left( 1 - \frac{\beta}{\beta-it} z^{-it} \right) = \frac{z^\beta}{\beta} \left( 1 - \beta \frac{\beta+it}{\beta^2+t^2} e^{-it \log z} \right)$$

$$= \frac{z^\beta}{\beta} \left( 1 - \beta \frac{\beta+it}{\beta^2+t^2} [\cos(t \log z) - i \sin(t \log z)] \right),$$

so by Lemma 3.6,

$$\begin{aligned} \Re(S_z) &= \frac{z^\beta}{\beta} \left( 1 - \frac{\beta}{\beta^2+t^2} [\beta \cos(t \log z) + t \sin(t \log z)] \right) \\ &= \frac{z^\beta}{\beta} \left( 1 - \frac{\beta}{\sqrt{\beta^2+t^2}} \left[ \frac{\beta \cos(t \log z)}{\sqrt{\beta^2+t^2}} + \frac{t \sin(t \log z)}{\sqrt{\beta^2+t^2}} \right] \right) \\ &= \frac{z^\beta}{\beta} \left( 1 - \frac{\beta}{\sqrt{\beta^2+t^2}} \cos(t \log z + \arctan(\beta/t)) \right) = \frac{z^\beta}{\beta} \left( 1 - \frac{\beta \cos \delta_z}{\sqrt{\beta^2+t^2}} \right). \end{aligned}$$

Thus,

$$(3.4) \quad \Re(S_v - S_w) = \frac{v^\beta - w^\beta}{\beta} - \frac{v^\beta \cos \delta_v - w^\beta \cos \delta_w}{\sqrt{\beta^2+t^2}}.$$

Recalling the definition of  $F_s(v, w)$ , we have

$$\begin{aligned} \Re(S) &= \Re(S_v - S_w + F_\alpha(v, w) - F_s(v, w)) \\ &\geq \Re(S_v - S_w) - |F_\alpha(v, w)| - |F_s(v, w)| \end{aligned}$$

which gives the desired result by (3.4) and Lemma 3.5.  $\square$

From Lemma 3.4, we see that a goal is to bound  $W(y, 1, t)$  from below, and pieces of this sum are bounded by Lemma 3.6. Ideally, if  $y$  were sufficiently small  $W$  could be computed directly and the problem settled. In practice  $W$  might only be computed up to some convenient number  $L$ , suitable for numerical integration, after which the analytic bound  $W_0(y, w, t)$  may be used. Still, there are further refinements to be made. Just as  $x/\log x$  loses out to  $\text{li}(x)$ ,  $W_0$  on a long interval is smaller than  $W_0$  summed on a partition of the interval into shorter parts. This plan is reflected in the following lemma.

**Lemma 3.7.** *If  $v, w$  satisfy the hypotheses of Lemma 3.5, let*

$$W_*(v, w, t) := W_0(v/e^{\lfloor \log(y/w) \rfloor}, w, t) + \sum_{j=0}^{\lfloor \log(v/w) \rfloor - 1} W_0(v/e^j, v/e^{j+1}, t).$$

Suppose that  $w, L$  satisfy 1427,  $L \leq w$ . If  $y \leq 10^{19}$ , then

$$J_1 \leq \int_{T_0}^T \exp(-W_*(y, w, t) - W(L, 1, t)) \frac{dt}{\sqrt{\alpha^2 + t^2}}.$$

If  $y > e^{55}$  and 1427,  $L \leq w \leq 10^{19}$ , let

$$\begin{aligned} W_1 &= W_*(10^{19}, w, t), \quad W_2 = W_*(e^{45}, 10^{19}, t), \quad W_3 = W_*(e^{50}, e^{45}, t), \\ W_4 &= W_*(e^{55}, e^{50}, t), \quad W_5 = W_*(y, e^{55}, t). \end{aligned}$$

Then

$$J_1 \leq \int_{T_0}^T \exp(-W_1 - W_2 - W_3 - W_4 - W_5 - W(L, 1, t)) \frac{dt}{\sqrt{\alpha^2 + t^2}}.$$

We remark that if  $10^{19} < y \leq e^{55}$ , then there is an appropriate inequality for  $J_1$  involving fewer  $W_j$ 's. If  $y$  is much larger than our largest example of  $y = 10^{35}$ , one might wish to use better approximations to  $\vartheta(y)$  than were used in Proposition 2.1.

*Proof.* If  $1427 \leq w < v$  and  $[w, v]$  satisfy the hypotheses of Lemma 3.5, we have

$$\begin{aligned} W(v, w, t) &= W(v/e^{\lfloor \log(v/w) \rfloor}, w, t) + \sum_{j=0}^{\lfloor \log(v/w) \rfloor - 1} W(v/e^j, v/e^{j+1}, t) \\ &\geq W_0(v/e^{\lfloor \log(v/w) \rfloor}, w, t) + \sum_{j=0}^{\lfloor \log(v/w) \rfloor - 1} W_0(v/e^j, v/e^{j+1}, t). \end{aligned}$$

The result then follows from Lemma 3.4.  $\square$

*Remark 3.8.* We implement Lemma 3.7 by choosing  $L$  as large as possible so as not to interfere overly with numerical integration. We have found that  $L = 10^6$  works well. The ratio  $e$  in the definition of  $W_*$  is convenient, but might be tweaked for slightly better results. The individual terms in the sum  $W(L, 1, t)$  are as in (3.2), except for the first 30 primes, where instead we forgo using the inequality in (3.3), using instead the slightly larger expression

$$\frac{1}{2} \log \left( 1 + \frac{2(1 - \cos(t \log p))}{p^\alpha(1 - p^{-\alpha})^2} \right).$$

We choose  $w$  as a function  $w(t)$  in such a way that the bound in Lemma 3.6 is minimized. For simplicity, we ignore the oscillating terms, i.e., we set

$$\begin{aligned} \frac{\partial}{\partial w} \left[ -w^\beta/\beta - 4w^{1/2-\alpha} + 2(\alpha + |s|)w^{1/2-\alpha}/(1/2 - \alpha) \right] \\ = -w^{\beta-1} - 4w^{-1/2-\alpha}/(1/2 - \alpha) + 2(\alpha + |s|)w^{-1/2-\alpha} \end{aligned}$$

equal to 0. Multiplying by  $w^{1/2+\alpha}$  and solving for  $w$  gives

$$w(\alpha, t) := \left( \frac{4}{\alpha - 1/2} + 2\alpha + 2\sqrt{\alpha^2 + t^2} \right)^2.$$

We let

$$w(t) := \max\{L, w(\alpha, t)\}.$$

Our next result, based on [13, Lemma 9], gives a bound on the number of  $y$ -smooth integers in a short interval.

**Lemma 3.9.** *Let  $0 < d < 1$ ,  $T > 1$  be such that  $z := (e^{2T^{d-1}} - 1)^{-1} > 1$ . We have*

$$\Psi(xe^{T^{1-d}}, y) - \Psi(xe^{-T^{1-d}}, y) \leq e^{\alpha^2/2z^2 - \alpha T^{d-1}} x^\alpha \zeta(\alpha, y) \sqrt{\frac{2e}{\pi}} \frac{J_2}{z}.$$

where, with  $W(y, w, t)$  as in Lemma 3.6,

$$J_2 := \int_0^\infty \exp\left\{-\frac{t^2}{2z^2} - W(y, 1, t)\right\} dt.$$

*Proof.* Let  $\xi = xe^{-T^{d-1}}$ , so that

$$(3.5) \quad \Psi(xe^{T^{d-1}}, y) - \Psi(xe^{-T^{d-1}}, y) = \Psi(\xi + \xi/z, y) - \Psi(\xi, y).$$

For  $\xi < n \leq \xi + \frac{\xi}{z}$ , we have that

$$1 > \frac{\xi}{n} \geq \left(1 + \frac{1}{z}\right)^{-1},$$

so  $0 > \log(\xi/n) \geq -\log(1 + 1/z) \geq -\frac{1}{z}$ , which implies that  $0 < [z \log(\xi/n)]^2 \leq 1$ . Thus,

$$\Psi(\xi + \xi/z, y) - \Psi(\xi, y) = \sum_{\substack{P(n) \leq y \\ \xi < n \leq \xi + \xi/z}} 1 \leq \sqrt{e} \sum_{\substack{P(n) \leq y \\ \xi < n \leq \xi + \xi/z}} \exp\left\{-\frac{1}{2}[z \log(\xi/n)]^2\right\}.$$

For  $\sigma, v \in \mathbb{R}$ , we have the formula

$$e^{-v^2/2} = \frac{1}{\sqrt{2\pi}} e^{\sigma^2/2 - \sigma v} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}t^2 + it(\sigma - v)\right\} dt$$

Letting  $\sigma = \alpha/z$ ,  $v = -z \log(\xi/n)$ , we obtain

$$\begin{aligned} & \Psi(\xi + \xi/z, y) - \Psi(\xi, y) \\ & \leq \sqrt{\frac{e}{2\pi}} \sum_{\substack{P(n) \leq y \\ \xi < n \leq \xi + \xi/z}} e^{\sigma^2/2 - \sigma v} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}t^2 + it(\sigma - v)\right\} dt \\ & = e^{\alpha^2/2z^2} \sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}t^2 + it\alpha/z\right\} \sum_{\substack{P(n) \leq y \\ \xi < n \leq \xi + \xi/z}} \left(\frac{\xi}{n}\right)^{\alpha + itz} dt. \end{aligned}$$

Since  $\alpha \leq 1 \leq z$ , changing variables  $t \mapsto t/z$  and taking the modulus gives

$$\begin{aligned} & \Psi(\xi + \xi/z, y) - \Psi(\xi, y) \\ & \leq z^{-1} e^{\alpha^2/2z^2} \sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{t^2}{2z^2} + it\alpha/z\right\} \sum_{\substack{P(n) \leq y \\ \xi < n \leq \xi + \xi/z}} \left(\frac{\xi}{n}\right)^{\alpha + it} dt \\ & \leq \frac{\xi^\alpha}{z} e^{\alpha^2/2z^2} \sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2z^2} |\zeta(\alpha + it, y)| dt \\ & = \frac{\xi^\alpha}{z} e^{\alpha^2/2z^2} \sqrt{\frac{2e}{\pi}} \int_0^\infty e^{-t^2/2z^2} |\zeta(\alpha + it, y)| dt. \end{aligned}$$

This last integral may be estimated by the method of Lemma 3.4, giving

$$\int_0^\infty e^{-t^2/2z^2} |\zeta(\alpha + it, y)| dt \leq \zeta(\alpha, y) \int_0^\infty \exp\left(-\frac{t^2}{2z^2} - W(y, 1, t)\right) dt = \zeta(\alpha, y) J_2.$$

We have

$$\Psi(\xi + \xi/z, y) - \Psi(\xi, y) \leq \xi^\alpha \zeta(\alpha, y) e^{\alpha^2/2z^2} \sqrt{\frac{2e}{\pi}} \frac{J_2}{z},$$

and the lemma now follows from (3.5) and the definition of  $\xi$ .  $\square$

*Remark 3.10.* For  $t$  large, say  $t > 2z \log z$ , we can ignore the term  $W(y, 1, t)$  in  $J_2$ , getting a suitably tiny numerical estimate for the tail of this rapidly converging integral. The part for  $t$  small may be integrated numerically with  $w(t), L$  as in Remark 3.8.

With these lemmas, we now have our principal result.

**Theorem 3.11.** *Let  $d, T, z$  be as in Lemma 3.9, let  $J_0^\pm$  be as in (3.1),  $J_1$  as in Lemma 3.4, and  $J_2$  as in Lemma 3.9. We have*

$$\Psi(x, y) \geq \frac{x^\alpha \zeta(\alpha, y)}{\pi} \left( J_0^- - J_1 - T^{-d} - e^{\alpha^2/2z^2} \sqrt{2\pi} e \frac{J_2}{z} \right)$$

and

$$\Psi(x, y) \leq \frac{x^\alpha \zeta(\alpha, y)}{\pi} \left( J_0^+ + J_1 + T^{-d} + e^{\alpha^2/2z^2} \sqrt{2\pi} e \frac{J_2}{z} \right).$$

#### 4. COMPUTATIONS

In this section we give some guidance on how, for a given pair  $x, y$ , the numbers  $\alpha, \zeta(\alpha, y)$ , and  $\sigma_j$  for  $j \leq 5$  may be numerically approximated. Further, we discuss how these data may be used to numerically approximate  $\Psi(x, y)$  via Theorem 3.11.

**4.1. Computing  $\alpha$ .** Given a number  $a \in (0, 1)$  and a large number  $y$  we may obtain upper and lower bounds for the sum

$$\sigma_1(a, y) = \sum_{p \leq y} \frac{\log p}{p^a - 1}.$$

First, we choose a moderate bound  $w_0 \leq y$  where we can compute the sum  $\sigma_1(a, w_0)$  relatively easily, such as  $w_0 = 179,424,673$ , the ten-millionth prime. The sum

$$(4.1) \quad \sum_{w_0 < p \leq y} \frac{\log p}{p^a}$$

may be approximated easily with Proposition 2.1 and partial summation. Let  $l^-(a, w_0, y)$  be a lower bound for this sum and let  $l^+(a, w_0, y)$  be an upper bound. Then

$$l^-(a, w_0, y) + \sigma_1(a, w_0) \leq \sigma_1(a, y) \leq \frac{w_0^a}{w_0^a - 1} l^+(a, w_0, y) + \sigma_1(a, w_0).$$

We choose  $\alpha$  as a number  $a$  where  $\log x$  lies between these two bounds. If a given trial for  $a$  is too small, this is detected by our lower bound for  $\sigma_1(a, y)$  lying above  $\log x$ , and if  $a$  is too large, we see this if our upper bound for  $\sigma_1(a, y)$  lies below  $\log x$ . It does not take long via linear interpolation to find a reasonable choice for  $\alpha$ . While narrowing in, one might use a less ambitious choice for  $w_0$ .

The partial summation used to estimate (4.1) and similar sums may be summarized in the following result.

**Lemma 4.1.** *Suppose  $f(t)$  is positive and  $f'(t)$  is negative on  $[w_0, w_1]$ . Suppose too that  $t - 2\sqrt{t} < \vartheta(t) \leq t$  on  $[w_0, w_1]$ . Then*

$$\begin{aligned} \int_{w_0}^{w_1} (1 - 1/\sqrt{t})f(t) dt + (w_0 - \vartheta(w_0) - 2\sqrt{w_0})f(w_0) \\ \leq \sum_{w_0 < p \leq w_1} f(p) \log p \leq \int_{w_0}^{w_1} f(t) dt + (w_0 - \vartheta(w_0))f(w_0). \end{aligned}$$

Because of Proposition 2.1, the condition on  $\vartheta$  holds if  $[w_0, w_1] \subset [1427, 10^{19}]$ . For intervals beyond  $10^{19}$ , it is easy to fashion an analogue of Lemma 4.1 using the other estimates of Proposition 2.1.

**4.2. Computing  $\sigma_0 = \log \zeta(\alpha, y)$  and the other  $\sigma_j$ 's.** Once a choice for  $\alpha$  is computed it is straightforward to compute  $\sigma_0$  and the other  $\sigma_j$ 's.

We have

$$\sigma_0(\alpha, y) = \sum_{p \leq y} -\log(1 - p^{-\alpha}).$$

We may compute this sum up to some moderate  $w_0$  as with the  $\alpha$  computation. For the range  $w_0 < p \leq y$  we may approximate the summand by  $p^{-\alpha}$  and sum this over  $(w_0, y]$  using partial summation (Lemma 4.1) and Proposition 2.1, say a lower bound is  $l_0^-$  and an upper bound is  $l_0^+$ . Then

$$l_0^- + \sigma_0(\alpha, w_0) \leq \sigma_0(\alpha, y) \leq \frac{-\log(1 - w_0^{-\alpha})}{w_0^{-\alpha}} l_0^+ + \sigma_0(\alpha, w_0).$$

The other  $\sigma_j$ 's are computed in a similar manner.

**4.3. Data.** We record our calculations of  $\alpha$  and the numbers  $\sigma_j$  for two examples. Note that we obtain bounds for  $\zeta$  via  $\sigma_0 = \log \zeta$ .

FIGURE 2. Data.

$x$	$10^{100}$	$10^{500}$
$y$	$10^{15}$	$10^{35}$
$\alpha$	.9111581	.94932677
$\zeta$	$352,189 \pm 16$	$2.09222 \cdot 10^{10} \pm 5 \cdot 10^5$
$\sigma_1^*$	$4.3 \cdot 10^{-4}$	$5.6 \cdot 10^{-4}$
$\sigma_2$	$5,763.47 \pm 0.03$	$71,689.2 \pm 0.02$
$\sigma_3$	$159,066.8 \pm 0.5$	$4,779,948.5 \pm 0.5$
$\sigma_4$	$4,604,079 \pm 8$	$330,260,722 \pm 21$
$\sigma_5^+$	$1.3725 \cdot 10^8$	$2.3353 \cdot 10^{10}$

Note that  $\sigma_1^*$  is an upper bound for  $|\sigma_1 - \log x|$ , and  $\sigma_5^+$  is an upper bound for  $\sigma_5$ .

The functions  $\alpha(x, y)$  and  $\sigma_j(x, y)$  are of interest in their own right. A simple observation from their definitions allows for more general bounds on  $\alpha$  and  $\sigma_j$  using the data in Figure 2, as described in the following remark.

*Remark 4.2.* For pairs  $x, y$  and  $x', y'$ , if  $x \geq x'$  and  $y \leq y'$  then  $\alpha(x, y) \leq \alpha(x', y')$ . Similarly, if  $\alpha(x, y) \geq \alpha(x', y')$  and  $y \leq y'$  then  $\sigma_j(x, y) \leq \sigma_j(x', y')$ .



**4.4. A word on numerical integration.** The numerical integration needed to estimate  $J_1, J_2$  is difficult, especially when we choose a large value of  $L$ , like  $L = 10^6$ . We performed these integrals independently on both Mathematica and Sage platforms. It helps to segment the range of integration, but even so, the software can report an error bound in addition to the main estimate. In such cases we have always added on this error bound and then rounded up, since we seek upper bounds for these integrals. In a case where one wants to be assured of a rigorous estimate, there are several options, each carrying some costs. One can use a Simpson or midpoint quadrature with a mesh say of 0.1 together with a careful estimation of the higher derivatives needed to estimate the error. An alternative is to do a Riemann sum with mesh 0.1, where on each interval and for each separate cosine term appearing, the maximum contribution is calculated. If this is done with  $T = 4 \cdot 10^5$  and  $L = 10^6$ , there would be magnitude  $10^{11}$  of these calculations. The extreme value of the cosine contribution would either be at an endpoint of an interval or  $-1$  if the argument straddles a number that is  $\pi \pmod{2\pi}$ . We have done a mild form of this method in our estimation of the integrals  $J_0^\pm$ .

**4.5. Example estimates.** We list some example values of  $x, y$  and the corresponding estimates in the figure below.

FIGURE 3. Results.

$x$	$10^{100}$	$10^{500}$
$y$	$10^{15}$	$10^{35}$
$T_3$	.00642708	.00114940
$T_2$	.00644109	.00115038
$Z^-$	.0385260	.0124202
$Z^+$	.0403125	.0127461
$T_1$	.0478624	.0155272
$T_0$	.0514483	.0161799
$T$	$4 \cdot 10^5$	$10^9$
$d$	0.57	0.58
$J_0^-$	$1.78554 \cdot 10^{-2}$	$4.90043 \cdot 10^{-3}$
$J_0^+$	$1.80312 \cdot 10^{-2}$	$4.92738 \cdot 10^{-3}$
$J_1$	$7.236 \cdot 10^{-4}$	$1.717 \cdot 10^{-6}$
$J_2$	$1.758 \cdot 10^{-2}$	$4.745 \cdot 10^{-3}$
$\Psi^-$	$2.3302 \cdot 10^{94}$	$1.4989 \cdot 10^{482}$
$\Psi^+$	$2.9227 \cdot 10^{94}$	$1.5118 \cdot 10^{482}$

5. APPENDIX

We prove the following theorem.

**Theorem 5.1** (Granville and Soundararajan). *If  $3 \leq y \leq x$  and  $1/\log y \leq \sigma \leq 1$ , then*

$$\Psi(x, y) \leq 1.39 \frac{y^{1-\sigma}}{\log x} x^\sigma \zeta(\sigma, y).$$

*Proof.* By the identity  $\log n = \sum_{d|n} \Lambda(d)$ , we have

$$\begin{aligned} \sum_{\substack{n \leq x \\ P(n) \leq y}} \log n &= \sum_{\substack{m \leq x \\ P(m) \leq y}} \sum_{\substack{d \leq x/m \\ P(d) \leq y}} \Lambda(d) = \sum_{\substack{m \leq x \\ P(m) \leq y}} \sum_{p \leq \min\{y, x/m\}} \log p \left\lfloor \frac{\log(x/m)}{\log p} \right\rfloor \\ &\leq \sum_{\substack{m \leq x \\ P(m) \leq y}} \pi(\min\{y, x/m\}) \log(x/m). \end{aligned}$$

Thus,

$$\Psi(x, y) \log x = \sum_{\substack{n \leq x \\ P(n) \leq y}} (\log n + \log(x/n)) \leq \sum_{\substack{n \leq x \\ P(n) \leq y}} (1 + \pi(\min\{y, x/n\})) \log(x/n).$$

Using the estimates in [18] we see that the maximum of  $(1 + \pi(t))/(t/\log t)$  occurs at  $t = 7$ , so that

$$1 + \pi(t) < 1.39t/\log t$$

for all  $t > 1$ . The above estimate then gives

$$\Psi(x, y) \log x < 1.39 \sum_{\substack{x/y < n \leq x \\ P(n) \leq y}} x/n + 1.39 \sum_{\substack{n \leq x/y \\ P(n) \leq y}} y \log(x/n)/\log y.$$

We now note that if  $1/\log y \leq \sigma \leq 1$ , then

$$y^{1-\sigma}(x/n)^\sigma \geq \begin{cases} x/n, & \text{if } x/y < n \leq x, \\ y \log(x/n)/\log y, & \text{if } n \leq x/y. \end{cases}$$

Indeed, in the first case, since  $t^{1-\sigma}$  is non-decreasing in  $t$ , we have  $(x/n)^{1-\sigma} \leq y^{1-\sigma}$ . And in the second case, since  $t^{-\sigma} \log t$  is decreasing in  $t$  for  $t \geq y$ , we have  $(x/n)^{-\sigma} \log(x/n) \leq y^{-\sigma} \log y$ .

We thus have

$$\Psi(x, y) \log x < 1.39 \sum_{\substack{n \leq x \\ P(n) \leq y}} y^{1-\sigma}(x/n)^\sigma < 1.39y^{1-\sigma}x^\sigma\zeta(\sigma, y).$$

This completes the proof.  $\square$

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