EXPLICIT ESTIMATES FOR THE DISTRIBUTION OF NUMBERS FREE OF LARGE PRIME FACTORS

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Abstract. There is a large literature on the asymptotic distribution of numbers free of large prime factors, so-called *smooth* or *friable* numbers. But there is very little known about this distribution that is numerically explicit. In this paper we follow the general plan for the saddle point argument of Hildebrand and Tenenbaum, giving explicit and fairly tight intervals in which the true count lies. We give two numerical examples of our method, and with the larger one, our interval is so tight we can exclude the famous Dickman–de Bruijn asymptotic estimate as too small and the Hildebrand–Tenenbaum main term as too large.

1. Introduction

For a positive integer $n > 1$, denote by $P(n)$ the largest prime factor of n, and let $P(1) = 1$. Let $\Psi(x, y)$ denote the number of $n \le x$ with $P(n) \le y$. Such integers n are known as y-smooth, or y-friable. Asymptotic estimates for $\Psi(x, y)$ are quite useful in many applications, not least of which is in the analysis of factorization and discrete logarithm algorithms.

One of the earliest results is due to Dickman [6] in 1930, who gave an asympotic formula for $\Psi(x, y)$ in the case that x is a fixed power of y. Dickman showed that

$$\Psi(x, y) \sim x\rho(u) \qquad (y \to \infty,\ x = y^u) \tag{1.1}$$

for every fixed $u \ge 1$, where $\rho(u)$ is the "Dickman–de Bruijn" function, defined to be the continuous solution of the delay differential equation

$$\begin{aligned} u\rho'(u) + \rho(u-1) &= 0 \qquad (u > 1), \\ \rho(u) &= 1 \qquad (0 \le u \le 1). \end{aligned}$$

There remain the questions of the error in the approximation (1.1), and also the case when $u = \log x/\log y$ is allowed to grow with x and y. In 1951, de Bruijn [3] proved that

$$\Psi(x, y) = x\rho(u)\Big(1 + O_\varepsilon\Big(\frac{\log(1+u)}{\log y}\Big)\Big)$$

holds uniformly for $x \ge 2$, $\exp\{(\log x)^{5/8+\varepsilon}\} < y \le x$, for any fixed $\varepsilon > 0$. After improvements in the range of this result by Maier and Hensley, Hildebrand [10] showed that the de Bruijn estimate holds when $\exp(\{(\log\log x)^{5/3+\varepsilon}\}) \le y \le x$.

In 1986, Hildebrand and Tenenbaum [11] provided a uniform estimate for $\Psi(x, y)$ for all $x \ge y \ge 2$, yielding an asymptotic formula when y and u tend to infinity.

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The starting point for their method is an elementary argument of Rankin [17] from 1938, commonly known now as Rankin's "trick". For complex s, define

$$\zeta(s,y) = \sum_{\substack{n\ge 1\\ P(n)\le y}} n^{-s} = \prod_{p\le y}(1-p^{-s})^{-1}$$

(where p runs over primes) as the partial Euler product of the Riemann zeta function $\zeta(s)$. In the case that $s=\sigma$ is real and $0<\sigma<1$, we have

$$\Psi(x,y) = \sum_{\substack{n\le x\\ P(n)\le y}} 1 \le \sum_{P(n)\le y} (x/n)^{\sigma} = x^{\sigma}\zeta(\sigma,y). \tag{1.2}$$

Then σ can be chosen optimally to minimize $x^{\sigma}\zeta(\sigma,y)$.

Let

$$\phi_j(s,y) = \frac{\partial^j}{\partial s^j}\log\zeta(s,y).$$

The function

$$\phi_1(s,y) = -\sum_{p\le y}\frac{\log p}{p^s-1}$$

is especially useful since the real solution $\alpha=\alpha(x,y)$ to $\phi_1(\alpha,y)+\log x=0$ gives the optimal σ in (1.2). We also denote $\sigma_j(x,y)=|\phi_j(\alpha(x,y),y)|$.

In this language, Hildebrand and Tenenbaum [11] proved that the estimate

$$\Psi(x,y) = \frac{x^{\alpha}\zeta(\alpha,y)}{\alpha\sqrt{2\pi\sigma_2(x,y)}}\Big(1+O\Big(\frac{1}{u}+\frac{\log y}{y}\Big)\Big)$$

holds uniformly for $x\ge y\ge 2$. As suggested by this formula, quantities $\alpha(x,y)$ and $\sigma_2(x,y)$ are of interest, and were given uniform estimates which imply the formulae

$$\alpha(x,y) \sim \frac{\log(1+y/\log x)}{\log y}$$

and

$$\sigma_2(x,y) \sim \Big(1+\frac{\log x}{y}\Big)\log x\log y,$$

together which imply

$$\Psi(x,y) \sim \frac{x^{\alpha}\zeta(\alpha,y)}{\sqrt{2\pi u}\,\log(y/\log x)} \qquad (\text{if } y/\log x\to\infty),$$
$$\Psi(x,y) \sim \frac{x^{\alpha}\zeta(\alpha,y)}{\sqrt{2\pi y/\log y}} \qquad (\text{if } y/\log x\to 0).$$

These formulae indicate that $\Psi(x,y)$ undergoes a "phase change" when y is of order $\log x$, see [2]. This paper concentrates on the range where y is considerably larger, say $y>(\log x)^4$.

The primary aim of this paper is to make the Hildebrand–Tenenbaum method explicit and so effectively construct an algorithm for obtaining good bounds for $\Psi(x,y)$.

1.1. **Explicit Results.** Beyond the Rankin upper bound $\Psi(x,y) \le x^\alpha \zeta(\alpha,y)$, we have the explicit lower bound

$$\Psi(x,y) \ge x^{1-\log\log x/\log y} = \frac{x}{(\log x)^u}$$

due to Konyagin and Pomerance [13]. Recently Granville and Soundararajan [9] found an elementary improvement of Rankin's upper bound, which they have graciously permitted us to include in an appendix in this paper. In particular, they show that

$$\Psi(x,y) \le 1.39 y^{1-\sigma} x^\sigma \zeta(\sigma,y)/\log x$$

for every value of $\sigma \in [1/\log y, 1]$, see Theorem 5.1.

In another direction, by relinquishing the goal of a compact formula, several authors have devised algorithms to compute bounds on $\Psi(x,y)$ for given x,y as inputs. For example, using an accuracy parameter c, Bernstein [1] created an algorithm to generate bounds $B^-(x,y) \le \Psi(x,y) \le B^+(x,y)$ with

$$\frac{B^-}{\Psi} \ge 1 - \frac{\log x}{c\log 3/\log 2} \quad \text{and} \quad \frac{B^+}{\Psi} \le 1 + \frac{2\log x}{c\log 3/\log 2},$$

running in

$$O\Big(\frac{y}{\log_2 y} + \frac{y\log x}{\log^2 y} + c\log x\log c\Big)$$

time. Parsell and Sorenson [15] refined this algorithm to run in

$$O\Big(c\frac{y^{2/3}}{\log y} + c\log x\log c\Big)$$

time, as well as obtaining faster and tighter bounds assuming the Riemann Hypothesis. The largest example computed by this method was an approximation of $\Psi(2^{255}, 2^{28})$.

FIGURE 1. Examples.

x y	10^{100} 10^{15}	10^{500} 10^{35}
KP	$1.786 \cdot 10^{84}$	$1.857 \cdot 10^{456}$
R	$4.599 \cdot 10^{96}$	$9.639 \cdot 10^{484}$
GS	$5.350 \cdot 10^{95}$	$6.596 \cdot 10^{483}$
DD	$2.523 \cdot 10^{94}$	$1.472 \cdot 10^{482}$
HT	$2.652 \cdot 10^{94}$	$1.5127 \cdot 10^{482}$
Ψ^-	$2.330 \cdot 10^{94}$	$1.4989 \cdot 10^{482}$
Ψ^+	$2.923 \cdot 10^{94}$	$1.5118 \cdot 10^{482}$

In Fig. 1,

KP is the Konyagin–Pomerance lower bound $x/(\log x)^u$,
R is the Rankin upper bound $x^\alpha \zeta(\alpha, y)$,
GS is the Granville–Soundararajan upper bound $1.39y^{1-\alpha}x^\alpha\zeta(\alpha, y)/\log x$,
DD is the Dickman–de Bruijn main term $\rho(u)x$,
HT is the Hildebrand–Tenenbaum main term $x^\alpha\zeta(\alpha, y)/(\alpha\sqrt{2\pi\sigma_2})$, and
Ψ^-, Ψ^+ are the lower and upper bounds obtained in this paper.

As seen in Fig. 1, the lower bound presented in this paper does better than the Konyagin–Pomerance lower bound by 10 orders of magnitude in the smaller example and 26 orders of magnitude in the larger example. The upper bound presented is about 2 to 3 orders of magnitude better than the Rankin estimate and about 1.5 orders of magnitude better than the new Granville–Soundararajan estimate.

As a point of reference we also give the main-term estimates $x^\alpha\zeta(\alpha, y)/\alpha\sqrt{2\pi\sigma_2}$ from [11] and $\rho(u)x$ from [6]. It is interesting that our lower and upper estimates in the second example create an interval for the true count that is tight enough to exclude both the Dickman–de Bruijn and Hildebrand–Tenenbaum main terms. The second-named author has asked if $\Psi(x, y) \geq x\rho(u)$ holds in general for $x \geq 2y \geq 2$, see [8, (1.25)]. This inequality is known for u bounded and x sufficiently large, see the discussion in [14, Section 9].

Our principal result, which benefits from some notation developed over the course of the paper, is Theorem 3.11. It is via this theorem that we were able to estimate $\Psi(10^{100}, 10^{15})$ and $\Psi(10^{500}, 10^{35})$ as in the table above.

2. Plan for the paper

The basic strategy of the saddle-point method relies on Perron's formula, which implies the identity[1]

$$\Psi(x, y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(s, y)\frac{x^s}{s}\, ds,$$

for any $\sigma > 0$. A convenient value of σ to use is the saddle point $\alpha = \alpha(x, y)$ discussed in the Introduction: For any $\sigma > 0$, the integrand is maximized on the vertical line with real part σ at $s = \sigma$, and this maximum is minimized for $\sigma > 0$ at α.

We are interested in abridging the integral at a certain height T and then approximating the contribution given by the tail. To this end, we have

$$\Psi(x, y) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \zeta(s, y)\frac{x^s}{s}\, ds + \text{Error}. \tag{2.1}$$

[1] The right side should be increased by $\frac{1}{2}$ in the case that x itself is a y-smooth integer.

There is a change in behavior occurring in $\zeta(s,y)$ when $t = \mathfrak{Im}(s)$ is on the order $1/\log y$. In [11] it is shown that

$$\begin{aligned}\Big|\frac{\zeta(s,y)}{\zeta(\alpha,y)}\Big| = \prod_{p\le y}\Big|\frac{1-p^{-\alpha}}{1-p^{-s}}\Big| &= \prod_{p\le y}\Big(1+\frac{2(1-\cos(t\log p))}{p^{\alpha}(1-p^{-\alpha})^2}\Big)^{-1/2}\\ &\le \exp\Big\{-\sum_{p\le y}\frac{1-\cos(t\log p)}{p^{\alpha}}\Big\}. \qquad (2.2)\end{aligned}$$

Thus, when t is small (compared to $1/\log y$) the oscillatory terms are in resonance, and when t is large the oscillatory terms should exhibit cancellation. This behavior suggests we should divide our range of integration into $|t| \le T_0$ and $T_0 < |t| < T$, where $T_0 \approx 1/\log y$ is a parameter to be optimized.

The contribution for $|t| \le T_0$ will constitute a "main term", and so we will try to estimate this part very carefully. In this range we forgo (2.2) and attack the integrand $\zeta(s,y)x^s/s$ directly. The basic idea is to expand $\phi(s,y) = \log\zeta(s,y)$ as a Taylor series in t. This approach, when carefully done, gives us fairly close upper and lower bounds for the integral. In our smaller example, the upper bound is less than 1% higher than the lower bound, and in the larger example, this is better by a factor of 20. Considerably more noise is encountered beyond T_0 and in the Error in (2.1).

For the second range $T_0 < |t| < T$, we focus on obtaining a satisfactory lower bound on the sum over primes,

$$\sum_{p\le y}\frac{1-\cos(t\log p)}{p^{\alpha}}.$$

Our strategy is to sum the first L terms directly, and then obtain an analytic formula $W(y,w)$ to lower bound the remaining terms starting at some $w \ge L$, where essentially

$$W(y,w) = \frac{y^{1-\alpha}-w^{1-\alpha}}{1-\alpha} + \text{error}.$$

With an explicit version of Perron's formula, the Error in (2.1) may be handled by

$$\begin{aligned}\big|\text{Error}\big| &\le x^{\alpha}\sum_{\substack{P(n)\le y\\ T|\log(x/n)|>T^d}}\frac{1}{n^{\alpha}}\frac{1}{\pi T|\log(x/n)|} + \sum_{\substack{P(n)\le y\\ T|\log(x/n)|\le T^d}}\Big(\frac{x}{n}\Big)^{\alpha}\\ &\le \frac{x^{\alpha}\zeta(\alpha,y)}{\pi T^d} + e^{\alpha T^{d-1}}\Big[\Psi(xe^{T^{1-d}},y)-\Psi(xe^{-T^{1-d}},y)\Big].\end{aligned}$$

Here $d \approx \frac{1}{2}$ is a parameter of our choosing, which we set to balance the two terms above. Thus the problem of bounding $|\text{Error}|$ is reduced to estimating the number of y-smooth integers in the "short" interval $\big(xe^{-T^{1-d}}, xe^{T^{1-d}}\big]$.

This latter portion is better handled when T is large, but the earlier portion in the range $[T_0, T]$ is better handled when T is small. Thus, T is numerically set to balance these two forces.

In our proofs we take full advantage of some recent calculations involving the prime-counting function $\pi(x)$ and the Chebyshev functions

$$\psi(x) = \sum_{p^m \le x} \log p, \quad \vartheta(x) = \sum_{p \le x} \log p,$$

with p running over primes and m running over positive integers. As a corollary of the papers [4], [5] of Büthe we have the following excellent result.

Proposition 2.1. *For* $1427 \le x \le 10^{19}$ *we have*

$$.05\sqrt{x} \le x - \vartheta(x) \le 1.95\sqrt{x}.$$

We have

$$\frac{|\vartheta(x) - x|}{x} < \begin{cases} 2.3 \cdot 10^{-8}, & \text{when } x > 10^{19}, \\ 1.2 \cdot 10^{-8}, & \text{when } x > e^{45}, \\ 1.2 \cdot 10^{-9}, & \text{when } x > e^{50}, \\ 2.9 \cdot 10^{-10}, & \text{when } x > e^{55}. \end{cases}$$

Proof. The first assertion is one of the main results in Büthe [5]. Let H be a number such that all zeros of the Riemann zeta-function with imaginary parts in $[0, H]$ lie on the 1/2-line. Inequality (7.4) in Büthe [4] asserts that if $x/\log x \le H^2/4.92^2$ and $x \ge 5000$, then

$$\frac{|\vartheta(x) - x|}{x} < \frac{(\log x - 2)\log x}{8\pi\sqrt{x}}.$$

We can take $H = 3 \cdot 10^{10}$, see Platt [16]. Thus, we have the result in the range $10^{19} \le x \le e^{45}$. For $x \ge e^{45}$ we have from Büthe [4] that $|\psi(x) - x|/x \le 1.118 \cdot 10^{-8}$. Further, we have (see [18, (3.39)]) for $x > 0$,

$$\psi(x) \ge \vartheta(x) > \psi(x) - 1.02x^{1/2} - 3x^{1/3}.$$

(This result can be improved, but it is not important to us.) Thus, for $x \ge e^{45}$ we have $|\vartheta(x) - x|/x \le 1.151 \cdot 10^{-8}$, establishing our result in this range. For the latter two ranges we argue similarly, using $|\psi(x) - x| \le 1.165 \cdot 10^{-9}$ when $x \ge e^{50}$ and $|\psi(x) - x| \le 2.885 \cdot 10^{-10}$ for $x \ge e^{55}$, both of these inequalities coming from [4]. ∎

We remark that there are improved inequalities at higher values of x, found in [4] and [7], which one would want to use if estimating $\Psi(x, y)$ for larger values of y than we have done here.

3. The main argument

As in the Introduction, for complex s, define

$$\zeta(s, y) = \sum_{\substack{n \ge 1 \\ P(n) \le y}} n^{-s} = \prod_{p \le y} (1 - p^{-s})^{-1},$$

which is the Riemann zeta function restricted to y-smooth numbers, and for $j \ge 0$, let

$$\phi_j(s, y) = \frac{\partial^j}{\partial s^j} \log \zeta(s, y).$$

We have the explicit formulae,

$$\phi_1(s,y) = -\sum_{p\le y}\frac{\log p}{p^s-1},$$
$$\phi_2(s,y) = \sum_{p\le y}\frac{p^s\log^2 p}{(p^s-1)^2},$$
$$\phi_3(s,y) = -\sum_{p\le y}\frac{(p^{2s}+p^s)\log^3 p}{(p^s-1)^3},$$
$$\phi_4(s,y) = \sum_{p\le y}\frac{(p^{3s}+4p^{2s}+p^s)\log^4 p}{(p^s-1)^4},$$
$$\phi_5(s,y) = -\sum_{p\le y}\frac{(p^{4s}+11p^{3s}+11p^{2s}+p^s)\log^5 p}{(p^s-1)^5}.$$

Note that for $y\ge 2$, $\sigma>0$, $\phi_1(\sigma,y)$ is strictly increasing from 0, so there is a unique solution $\alpha=\alpha(x,y)>0$ to the equation

$$\log x+\phi_1(\alpha,y)=0.$$

Since we cannot exactly solve this equation for α, we must take into account an upper bound for the difference between our value and the exact value. We denote

$$\phi_j=\phi_j(\alpha,y),\quad \sigma_j=|\phi_j|=(-1)^j\phi_j,\quad B_j=B_j(t)=\sigma_j t^j/j!$$

so that the Taylor series of $\phi(s,y)=\log\zeta(s,y)$ about $s=\alpha$ is

$$\phi(\alpha+it,y)=\sum_{j\ge 0}\frac{\sigma_j}{j!}(-it)^j=\sum_{j\ge 0}(-i)^jB_j.$$

Our first result, which is analogous to Lemma 10 in [11], sets the stage for our estimates.

Lemma 3.1. *Let $0<d<1$ and $T>1$. We have that*

$$\left|\Psi(x,y)-\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\zeta(s,y)\frac{x^s}{s}\,ds\right|$$
$$\le\frac{x^\alpha\zeta(\alpha,y)}{\pi T^d}+e^{\alpha T^{d-1}}\Big[\Psi(xe^{T^{d-1}},y)-\Psi(xe^{-T^{d-1}},y)\Big].$$

Proof. We have

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\zeta(s,y)\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\sum_{P(n)\le y}\left(\frac{x}{n}\right)^s\frac{ds}{s}$$
$$=\sum_{P(n)\le y}\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s},$$

where the interchange of sum and integral is justified since $\zeta(s,y)$ is a finite product, hence uniformly convergent as a sum.

By Perron's formula (see [12, Theorem G] and its proof), we have

$$\Big|\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\Big(\frac{x}{n}\Big)^s\frac{ds}{s}\Big| \le \frac{(x/n)^\alpha}{\max\Big(1,\pi T|\log(x/n)|\Big)} \qquad \text{if } n>x,$$

$$\Big|1-\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\Big(\frac{x}{n}\Big)^s\frac{ds}{s}\Big| \le \frac{(x/n)^\alpha}{\max\Big(1,\pi T|\log(x/n)|\Big)} \qquad \text{if } n\le x.$$

Together these imply

$$\begin{aligned}\Big|\Psi(x,y)-\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\zeta(s,y)\frac{x^s}{s}\,ds\Big| &\le x^\alpha\sum_{P(n)\le y}\frac{n^{-\alpha}}{\max\Big(1,\pi T|\log(x/n)|\Big)}\\ &\le x^\alpha \sum_{\substack{P(n)\le y\\ |\log(x/n)|>T^{d-1}}}\frac{1}{n^\alpha}\frac{1}{\pi T|\log(x/n)|}+x^\alpha\sum_{\substack{P(n)\le y\\ |\log(x/n)|\le T^{d-1}}}\frac{1}{n^\alpha}\\ &\le \frac{x^\alpha\zeta(\alpha,y)}{\pi T^d}+e^{\alpha T^{d-1}}\Big[\Psi(xe^{T^{d-1}},y)-\Psi(xe^{-T^{d-1}},y)\Big].\end{aligned}$$

This completes the proof. □

In using this result we have the problems of performing the integration from $\alpha-iT$ to $\alpha+iT$ and estimating the number of y-smooth integers in the interval $\big(xe^{-T^{d-1}},xe^{T^{1-d}}\big]$. We turn first to the integral evaluation.

Recall that $B_j=B_j(t)=\sigma_j(x,y)t^j/j!$ and let $B_1^*=B_1^*(t)=t\log x-B_1(t)$. Note that $B_1^*=0$ if α is chosen perfectly.

Lemma 3.2. *For $s=\alpha+it$, we have*

$$\mathfrak{Re}\Big\{\zeta(s,y)\frac{x^s}{s}\Big\}=$$
$$\frac{x^\alpha\zeta(\alpha,y)}{\alpha^2+t^2}\big(\alpha\cos(B_3+B_1^*+b_5)+t\sin(B_3+B_1^*+b_5)\big)\exp\big\{-B_2+B_4+a_5\big\},$$

where a_5,b_5 are real numbers, depending on the choice of t, with $|a_5+ib_5|\le B_5(t)$.

Proof. We expand $\phi(\alpha+it,y)=\log\zeta(\alpha+it,y)$ in a Taylor series around $t=0$. There exists some real ξ between 0 and t such that

$$\begin{aligned}\phi(\alpha+it,y)&=\phi(\alpha,y)+it\phi_1-\frac{t^2}{2}\phi_2-\frac{it^3}{3!}\phi_3+\frac{t^4}{4!}\phi_4-i\frac{t^5}{5!}(\alpha+i\xi,y)\\ &=B_0-iB_1-B_2+iB_3+B_4-i\frac{t^5}{5!}\phi_5(\alpha+i\xi,y).\end{aligned}$$

Since $\zeta(s,y)=\exp(\phi(s,y))$, we obtain

$$\begin{aligned}\zeta(s,y)\frac{x^s}{s}&=\frac{\zeta(\alpha,y)x^\alpha}{\alpha+it}\exp\Big\{it\log x-iB_1-B_2+iB_3+B_4+i\frac{t^5}{5!}\phi_5(\alpha+i\xi,y)\Big\}\\ &=\frac{x^\alpha\zeta(\alpha,y)}{\alpha+it}\exp\Big\{-B_2+B_4+i(B_1^*+B_3)+i\frac{t^5}{5!}\phi_5(\alpha+i\xi,y)\Big\}.\end{aligned}$$

Letting $i\phi_5(\alpha+i\xi)t^5/5!=a_5+b_5i$, we have

$$\zeta(s,y)\frac{x^s}{s}=$$
$$\frac{x^\alpha\zeta(\alpha,y)}{\alpha^2+t^2}(\alpha-it)\big(\cos(B_1^*+B_3+b_5)+i\sin(B_1^*+B_3+b_5)\big)\exp\big\{-B_2+B_4+a_5\big\},$$

and taking the real part gives the result. □

The main contribution to the integral in Lemma 3.1 turns out to come from the interval $[-T_0, T_0]$, where T_0 is fairly small. We have

$$\frac{1}{2\pi i}\int_{\alpha-iT_0}^{\alpha+iT_0}\zeta(s,y)\frac{x^s}{s}\,ds = \frac{1}{2\pi}\int_{-T_0}^{T_0}\zeta(\alpha+it,y)\frac{x^{\alpha+it}}{\alpha+it}\,dt.$$

Note that the integrand, written as a Taylor series around $s=\alpha$, has real coefficients, so the real part is an even function of t and the imaginary part is an odd function. Thus, the integral is real, and its value is double the value of the integral on $[0, T_0]$.

Consider the cosine, sine combination in Lemma 3.2:

$$f(t,v) := \alpha\cos(B_3(t)+v) + t\sin(B_3(t)+v),$$

and let

$$v_0(t) = |B_1^*(t)| + B_5(t).$$

We have, for each value of t, the constraint that $|v| \le v_0(t)$. The partial derivative of $f(t,v)$ with respect to v is zero when $\arctan(t/\alpha) - B_3(t) \equiv 0 \pmod{\pi}$. Let

$$u(t) = \arctan(t/\alpha) - B_3(t).$$

If $u(t) \notin [-v_0(t), v_0(t)]$, then $f(t,v)$ is monotone in v on that interval; otherwise it has a min or max at $u(t)$. Let T_3, T_2, T_1, T_0 be defined, respectively, as the least positive solutions of the equations

$$u(t) = v_0(t), \quad u(t) = -v_0(t), \quad u(t)+\pi = v_0(t), \quad u(t)+\pi = -v_0(t).$$

Then $0 < T_3 < T_2 < T_1 < T_0$. We have the following properties for $f(t,v)$:

(1) For t in the interval $[0, T_3]$ we have $f(t,v)$ increasing for $v \in [-v_0(t), v_0(t)]$, so that
$$f(t,-v_0(t)) \le f(t,v) \le f(t,v_0(t)).$$

(2) For t in the interval $[T_3, T_2]$, we have $f(t,v)$ increasing for $-v_0(t) \le v \le u(t)$ and then decreasing for $u(t) \le v \le v_0(t)$. Thus,
$$\min\{f(t,-v_0(t)), f(t,v_0(t))\} \le f(t,v) \le f(t,u(t)).$$

(3) For $t \in [T_2, T_1]$, $f(t,v)$ is decreasing for $v \in [-v_0(t), v_0(t)]$, so that
$$f(t,v_0(t)) \le f(t,v) \le f(t,-v_0(t)).$$

(4) For $t \in [T_1, T_0]$, we have $f(t,v)$ decreasing for $v \in [-v_0(t), u(t)+\pi]$ and increasing for $v \in [u(t)+\pi, v_0(t)]$; that is,
$$f(t,u(t)+\pi) \le f(t,v) \le \max\{f(t,-v_0(t)), f(t,v_0(t))\}.$$

Note too that $f(t,v)$ has a sign change from positive to negative in the interval $[T_2, T_1]$. Let Z^-, Z^+ be, respectively, the least positive roots of $f(t,v(t)) = 0$, $f(t,-v(t)) = 0$.

Let I_0^+ be an upper bound for the function appearing in Lemma 3.2 on $[0, T_0]$ using $|a_5|, |b_5| \le B_5$ and the above facts about $f(t,v)$, and let I_0^- be the corresponding lower bound. We choose $a_5 = B_5$ in I_0^+ when the cos, sin combination is positive, and $a_5 = -B_5$ when it is negative. For I_0^-, we choose a_5 in the reverse way.

Let

$$J_0^+ = \int_0^{T_0} I_0^+(t)\, dt, \quad J_0^- = \int_0^{T_0} I_0^-(t)\, dt. \tag{3.1}$$

We thus have the following result, which is our analogue of Lemma 11 in [11].

Lemma 3.3. *We have*

$$\frac{x^\alpha \zeta(\alpha, y)}{\pi} J_0^- \le \frac{1}{2\pi i} \int_{\alpha - iT_0}^{\alpha + iT_0} \zeta(s,y) \frac{x^s}{s}\, ds \le \frac{x^\alpha \zeta(\alpha, y)}{\pi} J_0^+.$$

In order to estimate the integral in Lemma 3.1 when $|t| > T_0$ we must know something about prime sums to y.

Lemma 3.4. *We have*

$$\Big| \int_{\alpha + iT_0}^{\alpha + iT} \zeta(s,y) \frac{x^s}{s}\, ds \Big| \le x^\alpha \zeta(\alpha, y) J_1,$$

where

$$J_1 := \int_{T_0}^{T} \exp\big(- W(y,1,t)\big)\, \frac{dt}{\sqrt{\alpha^2 + t^2}}$$

and

$$W(v,w,t) := \sum_{w < p \le v} \frac{1 - \cos(t \log p)}{p^\alpha}. \tag{3.2}$$

Proof. For $0 \le v \le 1 < t$, equation (3.14) in [11] states that

$$(1 + 4vt/(t-1)^2)^{-1} \le \exp\{-4v/t\}.$$

Applied to (3.17) in [11] with $v = (1 - \cos(t \log p))/2$, we have that

$$\begin{aligned} \Big|\frac{\zeta(s,y)}{\zeta(\alpha,y)}\Big| &= \prod_{p \le y} \Big| \frac{1 - p^{-\alpha}}{1 - p^{-s}} \Big| = \prod_{p \le y} \Big(1 + \frac{2(1 - \cos(t \log p))}{p^\alpha (1 - p^{-\alpha})^2}\Big)^{-1/2} \\ &\le \exp\Big\{ - \sum_{p \le y} \frac{1 - \cos(t \log p)}{p^\alpha} \Big\}. \end{aligned} \tag{3.3}$$

This completes the proof. □

Our goal now is to find a way to estimate $W(v,w,t)$. The following result is analogous to Lemma 6 in [11].

Lemma 3.5. *Let s be a complex number, let $1 < w < v$, and define*

$$F_s(v,w) := \sum_{w < p \le v} \frac{\log p}{p^s} - \frac{v^{1-s} - w^{1-s}}{1-s}.$$

(i) If $v \le 10^{19}$ we have

$$|F_s(v,w)| \le 2(v^{1/2 - \alpha} + w^{1/2 - \alpha}) + 2|s| \frac{w^{1/2-\alpha} - v^{1/2-\alpha}}{\alpha - 1/2}.$$

(ii) If $10^{19} \le w \le v$ we have

$$|F_s(v,w)| \le \varepsilon_w \Big(v^\beta + w^\beta + |s| \frac{v^\beta - w^\beta}{\beta} \Big),$$

where $\beta = 1-\alpha$ *and*

$$\varepsilon_w = \begin{cases} 2.3\cdot 10^{-8}, & w\in(10^{19}, e^{45}],\\ 1.2\cdot 10^{-8}, & w\in(e^{50}, e^{55}],\\ 1.2\cdot 10^{-9}, & w\in(e^{50}, e^{55}],\\ 2.9\cdot 10^{-10}, & w> e^{55}.\end{cases}$$

Proof. (i) By partial summation,

$$\begin{aligned}\sum_{w<p\le v}\frac{\log p}{p^s} &= \frac{\vartheta(v)}{v^s}-\frac{\vartheta(w)}{w^s}+\int_w^v s\frac{\vartheta(t)}{t^{s+1}}\,dt\\ &= \frac{v^{1-s}-w^{1-s}}{1-s}-\frac{E(v)}{v^s}+\frac{E(w)}{w^s}-\int_w^v s\frac{E(t)}{t^{s+1}}\,dt,\end{aligned}$$

so by the first part of Proposition 2.1,

$$\begin{aligned}|F_s(v,w)| &\le \frac{|E(v)|}{v^\alpha}+\frac{|E(w)|}{w^\alpha}+|s|\int_w^v\frac{E(t)}{t^{1+\alpha}}\,dt\\ &\le 2v^{1/2-\alpha}+2w^{1/2-\alpha}+2|s|\frac{v^{1/2-\alpha}-w^{1/2-\alpha}}{1/2-\alpha}.\end{aligned}$$

(ii) Similarly, by the second part of Proposition 2.1,

$$\begin{aligned}|F_s(v,w)| &\le \frac{|E(v)|}{v^\alpha}+\frac{|E(w)|}{w^\alpha}+|s|\int_w^v\frac{E(t)}{t^{1+\alpha}}\,dt \le \varepsilon_w\Big(v^{1-\alpha}+w^{1-\alpha}+|s|\int_w^v\frac{dt}{t^\alpha}\Big)\\ &= \varepsilon_w\Big(v^{1-\alpha}+w^{1-\alpha}+|s|\frac{v^{1-\alpha}-w^{1-\alpha}}{1-\alpha}\Big).\end{aligned}$$

□

The following result plays the role of Corollary 6.1 in [11].

Lemma 3.6. *For* $t\in\mathbb{R}$, $z>1$, *and* $\beta=1-\alpha$, *let*

$$\delta_z := t\log z-\arctan(t/\beta).$$

(i) For $1427\le w<v\le 10^{19}$ *we have that* $W(v,w,t)\ge W_0(v,w,t)$, *where*

$$\begin{aligned}W_0(v,w,t)\log v &= \frac{v^\beta-w^\beta}{\beta}-\frac{v^\beta\cos\delta_v-w^\beta\cos\delta_w}{\sqrt{\beta^2+t^2}}\\ &\quad -4(v^{1/2-\alpha}+w^{1/2-\alpha})-2(\alpha+|s|)\frac{w^{1/2-\alpha}-v^{1/2-\alpha}}{\alpha-1/2}.\end{aligned}$$

(ii) For $10^{19}\le w<v$ *we have that* $W(v,w,t)\ge W_0(v,w,t)$, *where*

$$\begin{aligned}W_0(v,w,t)\log v &= \frac{v^\beta-w^\beta}{\beta}-\frac{v^\beta\cos\delta_v-w^\beta\cos\delta_w}{\sqrt{\beta^2+t^2}}\\ &\quad -2\varepsilon_w\big(v^\beta+w^\beta\big)-\varepsilon_w(\alpha+|s|)\Big(\frac{v^\beta-w^\beta}{\beta}\Big).\end{aligned}$$

Proof. We apply Lemma 3.5 with $s = 1-\beta$ and $s = 1-\beta+it$, and take the real part of the difference. Letting the difference of the sums be S, we have that

$$S := \sum_{w<p\le v}\Big(\frac{\log p}{p^{1-\beta}} - \frac{\log p}{p^{1-\beta+it}}\Big) = \sum_{w<p\le v}\frac{\log p}{p^{1-\beta}}(1-p^{-it}), \text{ so}$$
$$\mathfrak{Re}(S) = \sum_{w<p\le v}\frac{\log p}{p^{1-\beta}}(1-\cos(t\log p)),$$

which is the sum we wish to bound.

For a positive real number z, let $S_z := \frac{z^\beta}{\beta} - \frac{z^{\beta-it}}{\beta-it}$. We have that

$$S_z = \frac{z^\beta}{\beta}\Big(1 - \frac{\beta}{\beta-it}z^{-it}\Big) = \frac{z^\beta}{\beta}\Big(1-\beta\frac{\beta+it}{\beta^2+t^2}e^{-it\log z}\Big)$$
$$= \frac{z^\beta}{\beta}\Big(1-\beta\frac{\beta+it}{\beta^2+t^2}[\cos(t\log z) - i\sin(t\log z)]\Big),$$

so

$$\mathfrak{Re}(S_z) = \frac{z^\beta}{\beta}\Big(1-\frac{\beta}{\beta^2+t^2}[\beta\cos(t\log z)+t\sin(t\log z)]\Big)$$
$$= \frac{z^\beta}{\beta}\Big(1-\frac{\beta}{\sqrt{\beta^2+t^2}}\Big[\frac{\beta\cos(t\log z)}{\sqrt{\beta^2+t^2}} + \frac{t\sin(t\log z)}{\sqrt{\beta^2+t^2}}\Big]\Big)$$
$$= \frac{z^\beta}{\beta}\Big(1-\frac{\beta}{\sqrt{\beta^2+t^2}}\cos(t\log z+\arctan(\beta/t))\Big) = \frac{z^\beta}{\beta}\Big(1-\frac{\beta\cos\delta_z}{\sqrt{\beta^2+t^2}}\Big).$$

Thus,

$$\mathfrak{Re}(S_v - S_w) = \frac{v^\beta - w^\beta}{\beta} - \frac{v^\beta\cos\delta_v - w^\beta\cos\delta_w}{\sqrt{\beta^2+t^2}}. \tag{3.4}$$

Recalling the definition of $F_s(v,w)$, we have

$$\mathfrak{Re}(S) = \mathfrak{Re}(S_v - S_w + F_\alpha(v,w) - F_s(v,w))$$
$$\ge \mathfrak{Re}(S_v - S_w) - |F_\alpha(v,w)| - |F_s(v,w)|$$

which gives the desired result by (3.4) and Lemma 3.5. ☐

From Lemma 3.4, we see that a goal is to bound $W(y,1,t)$ from below, and pieces of this sum are bounded by Lemma 3.6. Ideally, if y were sufficiently small W could be computed directly and the problem settled. In practice W might only be computed up to some convenient number L, suitable for numerical integration, after which the analytic bound $W_0(y,w,t)$ may be used. Still, there are further refinements to be made. Just as $x/\log x$ loses out to $\mathrm{li}(x)$, W_0 on a long interval is smaller than W_0 summed on a partition of the interval into shorter parts. This plan is reflected in the following lemma.

Lemma 3.7. *If v,w satisfy the hypotheses of Lemma 3.5, let*

$$W_*(v,w,t) := W_0(v/e^{\lfloor\log(y/w)\rfloor}, w, t) + \sum_{j=0}^{\lfloor\log(v/w)\rfloor-1} W_0(v/e^j, v/e^{j+1}, t).$$

Suppose that w, L satisfy $1427, L\le w$. If $y\le 10^{19}$, then

$$J_1 \le \int_{T_0}^{T}\exp\big(-W_*(y,w,t) - W(L,1,t)\big)\,\frac{dt}{\sqrt{\alpha^2+t^2}}.$$

If $y > e^{55}$ *and* $1427, L \le w \le 10^{19}$*, let*

$$W_1 = W_*(10^{19}, w, t),\ W_2 = W_*(e^{45}, 10^{19}, t),\ W_3 = W_*(e^{50}, e^{45}, t),$$
$$W_4 = W_*(e^{55}, e^{50}, t),\ W_5 = W_*(y, e^{55}, t).$$

Then

$$J_1 \le \int_{T_0}^{T} \exp\big(-W_1 - W_2 - W_3 - W_4 - W_5 - W(L, 1, t)\big)\,\frac{dt}{\sqrt{\alpha^2 + t^2}}.$$

We remark that if $10^{19} < y \le e^{55}$, then there is an appropriate inequality for J_1 involving fewer W_j's. If y is much larger than our largest example of $y = 10^{35}$, one might wish to use better approximations to $\vartheta(y)$ than were used in Proposition 2.1.

Proof. If $1427 \le w < v$ and $[w, v]$ satisfy the hypotheses of Lemma 3.5, we have

$$\begin{aligned} W(v, w, t) &= W(v/e^{\lfloor \log(v/w)\rfloor}, w, t) + \sum_{j=0}^{\lfloor \log(v/w)\rfloor - 1} W(v/e^j, v/e^{j+1}, t) \\ &\ge W_0(v/e^{\lfloor \log(v/w)\rfloor}, w, t) + \sum_{j=0}^{\lfloor \log(v/w)\rfloor - 1} W_0(v/e^j, v/e^{j+1}, t). \end{aligned}$$

The result then follows from Lemma 3.4. □

Remark 3.8. We implement Lemma 3.7 by choosing L as large as possible so as not to interfere overly with numerical integration. We have found that $L = 10^6$ works well. The ratio e in the definition of W_* is convenient, but might be tweaked for slightly better results. The individual terms in the sum $W(L, 1, t)$ are as in (3.2), except for the first 30 primes, where instead we forgo using the inequality in (3.3), using instead the slightly larger expression

$$\frac{1}{2}\log\Big(1 + \frac{2(1 - \cos(t\log p))}{p^\alpha(1 - p^{-\alpha})^2}\Big).$$

We choose w as a function $w(t)$ in such a way that the bound in Lemma 3.6 is minimized. For simplicity, we ignore the oscillating terms, i.e., we set

$$\begin{aligned} &\frac{\partial}{\partial w}\Big[-w^\beta/\beta - 4w^{1/2-\alpha} + 2(\alpha + |s|)w^{1/2-\alpha}/(1/2 - \alpha)\Big] \\ &\quad = -w^{\beta-1} - 4w^{-1/2-\alpha}/(1/2 - \alpha) + 2(\alpha + |s|)w^{-1/2-\alpha} \end{aligned}$$

equal to 0. Multiplying by $w^{1/2+\alpha}$ and solving for w gives

$$w(\alpha, t) := \Big(\frac{4}{\alpha - 1/2} + 2\alpha + 2\sqrt{\alpha^2 + t^2}\Big)^2.$$

We let

$$w(t) := \max\{L, w(\alpha, t)\}.$$

Our next result, based on [11, Lemma 9], gives a bound on the number of y-smooth integers in a short interval.

Lemma 3.9. *Let* $0 < d < 1$*,* $T > 1$ *be such that* $z := (e^{2T^{d-1}} - 1)^{-1} > 1$*. We have*

$$\Psi(xe^{T^{1-d}}, y) - \Psi(xe^{-T^{1-d}}, y) \le e^{\alpha^2/2z^2 - \alpha T^{d-1}} x^\alpha \zeta(\alpha, y)\sqrt{\frac{2e}{\pi}}\frac{J_2}{z}.$$

where, with $W(y,w,t)$ as in Lemma 3.6,

$$J_2 := \int_0^\infty \exp\Big\{-\frac{t^2}{2z^2} - W(y,1,t)\Big\}\, dt.$$

Proof. Let $\xi = xe^{-T^{d-1}}$, so that

$$\Psi(xe^{T^{d-1}}, y) - \Psi(xe^{-T^{d-1}}, y) = \Psi(\xi + \xi/z, y) - \Psi(\xi, y). \tag{3.5}$$

For $\xi < n \le \xi + \frac{\xi}{z}$, we have that

$$1 > \frac{\xi}{n} \ge \Big(1 + \frac{1}{z}\Big)^{-1},$$

so $0 > \log(\xi/n) \ge -\log(1+1/z) \ge -\frac{1}{z}$, which implies that $0 < [z\log(\xi/n)]^2 \le 1$. Thus,

$$\Psi(\xi+\xi/z,y) - \Psi(\xi,y) = \sum_{\substack{P(n)\le y\\ \xi<n\le\xi+\xi/z}} 1 \le \sqrt{e} \sum_{\substack{P(n)\le y\\ \xi<n\le\xi+\xi/z}} \exp\Big\{-\frac{1}{2}[z\log(\xi/n)]^2\Big\}.$$

For $\sigma, v \in \mathbb{R}$, we have the formula

$$e^{-v^2/2} = \frac{1}{\sqrt{2\pi}} e^{\sigma^2/2 - \sigma v} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{1}{2}t^2 + it(\sigma - v)\Big\}\, dt$$

Letting $\sigma = \alpha/z$, $v = -z\log(\xi/n)$, we obtain

$$\begin{aligned}\Psi(\xi+\xi/z,y) &- \Psi(\xi,y)\\ &\le \sqrt{\frac{e}{2\pi}} \sum_{\substack{P(n)\le y\\ \xi<n\le\xi+\xi/z}} e^{\sigma^2/2-\sigma v} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{1}{2}t^2 + it(\sigma-v)\Big\}\, dt\\ &= e^{\alpha^2/2z^2}\sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{1}{2}t^2 + it\alpha/z\Big\} \sum_{\substack{P(n)\le y\\ \xi<n\le\xi+\xi/z}} \Big(\frac{\xi}{n}\Big)^{\alpha+itz}\, dt.\end{aligned}$$

Since $\alpha \le 1 \le z$, changing variables $t \mapsto t/z$ and taking the modulus gives

$$\begin{aligned}\Psi(\xi+\xi/z,y) &- \Psi(\xi,y)\\ &\le z^{-1} e^{\alpha^2/2z^2}\sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} \exp\Big\{-\frac{t^2}{2z^2} + it\alpha/z^2\Big\} \sum_{\substack{P(n)\le y\\ \xi<n\le\xi+\xi/z}} \Big(\frac{\xi}{n}\Big)^{\alpha+it}\, dt\\ &\le \frac{\xi^\alpha}{z} e^{\alpha^2/2z^2}\sqrt{\frac{e}{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2z^2} |\zeta(\alpha+it,y)|\, dt\\ &= \frac{\xi^\alpha}{z} e^{\alpha^2/2z^2}\sqrt{\frac{2e}{\pi}} \int_0^\infty e^{-t^2/2z^2} |\zeta(\alpha+it,y)|\, dt.\end{aligned}$$

This last integral may be estimated by the method of Lemma 3.4, giving

$$\int_0^\infty e^{-t^2/2z^2}|\zeta(\alpha+it,y)|\, dt \le \zeta(\alpha,y)\int_0^\infty \exp\Big(-\frac{t^2}{2z^2} - W(y,1,t)\Big)\, dt = \zeta(\alpha,y)J_2.$$

Thus, we have

$$\Psi(\xi+\xi/z,y) - \Psi(\xi,y) \le \xi^\alpha \zeta(\alpha,y) e^{\alpha^2/2z^2}\sqrt{\frac{2e}{\pi}}\frac{J_2}{z},$$

and the lemma now follows from (3.5) and the definition of ξ. □

Remark 3.10. For t large, say $t > 2z \log z$, we can ignore the term $W(y,1,t)$ in J_2, getting a suitably tiny numerical estimate for the tail of this rapidly converging integral. The part for t small may be integrated numerically with $w(t), L$ as in Remark 3.8.

With these lemmas, we now have our principal result.

Theorem 3.11. *Let d, T, z be as in Lemma 3.9, let $J_0^{\pm}$ be as in* (3.1)*, J_1 as in Lemma 3.4, and J_2 as in Lemma 3.9. We have*

$$\Psi(x,y) \geq \frac{x^\alpha \zeta(\alpha,y)}{\pi}\Big(J_0^- - J_1 - T^{-d} - e^{\alpha^2/2z^2}\sqrt{2\pi e}\frac{J_2}{z}\Big)$$

and

$$\Psi(x,y) \leq \frac{x^\alpha \zeta(\alpha,y)}{\pi}\Big(J_0^+ + J_1 + T^{-d} + e^{\alpha^2/2z^2}\sqrt{2\pi e}\frac{J_2}{z}\Big).$$

4. Computations

In this section we give some guidance on how, for a given pair x, y, the numbers α, $\zeta(\alpha, y)$, and σ_j for $j \leq 5$ may be numerically approximated. Further, we discuss how these data may be used to numerically approximate $\Psi(x, y)$ via Theorem 3.11.

4.1. **Computing α.** Given a number $a \in (0, 1)$ and a large number y we may obtain upper and lower bounds for the sum

$$\sigma_1(a,y) = \sum_{p\leq y} \frac{\log p}{p^a - 1}.$$

First, we choose a moderate bound $w_0 \leq y$ where we can compute the sum $\sigma_1(a, w_0)$ relatively easily, such as $w_0 = 179{,}424{,}673$, the ten-millionth prime. The sum

$$\sum_{w_0<p\leq y} \frac{\log p}{p^a} \tag{4.1}$$

may be approximated easily with Proposition 2.1 and partial summation. Let $l^-(a, w_0, y)$ be a lower bound for this sum and let $l^+(a, w_0, y)$ be an upper bound. Then

$$l^-(a,w_0,y) + \sigma_1(a,w_0) \leq \sigma_1(a,y) \leq \frac{w_0^a}{w_0^a - 1} l^+(a,w_0,y) + \sigma_1(a,w_0).$$

We choose α as a number a where $\log x$ lies between these two bounds. If a given trial for a is too small, this is detected by our lower bound for $\sigma_1(a, y)$ lying above $\log x$, and if a is too large, we see this if our upper bound for $\sigma_1(a, y)$ lies below $\log x$. It does not take long via linear interpolation to find a reasonable choice for α. While narrowing in, one might use a less ambitious choice for w_0.

The partial summation used to estimate (4.1) and similar sums may be summarized in the following result.

Lemma 4.1. *Suppose $f(t)$ is positive and $f'(t)$ is negative on $[w_0, w_1]$. Suppose too that $t - 2\sqrt{t} < \vartheta(t) \le t$ on $[w_0, w_1]$. Then*

$$\int_{w_0}^{w_1} (1 - 1/\sqrt{t}) f(t)\, dt + (w_0 - \vartheta(w_0) - 2\sqrt{w_0}) f(w_0)$$
$$\le \sum_{w_0 < p \le w_1} f(p) \log p \le \int_{w_0}^{w_1} f(t)\, dt + (w_0 - \vartheta(w_0)) f(w_0).$$

Because of Proposition 2.1, the condition on ϑ holds if $[w_0, w_1] \subset [1427, 10^{19}]$. For intervals beyond 10^{19}, it is easy to fashion an analogue of Lemma 4.1 using the other estimates of Proposition 2.1.

4.2. **Computing $\sigma_0 = \log \zeta(\alpha, y)$ and the other σ_j's.** Once a choice for α is computed it is straightforward to compute σ_0 and the other σ_j's.

We have

$$\sigma_0(\alpha, y) = \sum_{p \le y} -\log(1 - p^{-\alpha}).$$

We may compute this sum up to some moderate w_0 as with the α computation. For the range $w_0 < p \le y$ we may approximate the summand by $p^{-\alpha}$ and sum this over $(w_0, y]$ using partial summation (Lemma 4.1) and Proposition 2.1, yielding, say, a lower bound l_0^- and an upper bound l_0^+. Then

$$l_0^- + \sigma_0(\alpha, w_0) \le \sigma_0(\alpha, y) \le \frac{-\log(1 - w_0^{-\alpha})}{w_0^{-\alpha}} l_0^+ + \sigma_0(\alpha, w_0).$$

The other σ_j's are computed in a similar manner.

4.3. **Data.** In Fig. 2, we record our calculations of α and the numbers σ_j for two examples. Note that we obtain bounds for ζ via $\sigma_0 = \log \zeta$.

FIGURE 2. Data.

x	10^{100}	10^{500}
y	10^{15}	10^{35}
α	.9111581	.94932677
ζ	$352{,}189 \pm 16$	$2.09222 \cdot 10^{10} \pm 5 \cdot 10^5$
σ_1^*	$4.3 \cdot 10^{-4}$	$5.6 \cdot 10^{-4}$
σ_2	$5{,}763.47 \pm 0.03$	$71{,}689.2 \pm 0.02$
σ_3	$159{,}066.8 \pm 0.5$	$4{,}779{,}948.5 \pm 0.5$
σ_4	$4{,}604{,}079 \pm 8$	$330{,}260{,}722 \pm 21$
σ_5^+	$1.3725 \cdot 10^8$	$2.3353 \cdot 10^{10}$

Note that σ_1^* is an upper bound for $|\sigma_1 - \log x|$, and σ_5^+ is an upper bound for σ_5.

The functions $\alpha(x, y)$ and $\sigma_j(x, y)$ are of interest in their own right. A simple observation from their definitions allows for more general bounds on α and σ_j using the data in Figure 2, as described in the following remark.

Remark 4.2. For pairs x, y and x', y', if $x \ge x'$ and $y \le y'$ then $\alpha(x, y) \le \alpha(x', y')$. Similarly, if $\alpha(x, y) \ge \alpha(x', y')$ and $y \le y'$ then $\sigma_j(x, y) \le \sigma_j(x', y')$.

4.4. **A word on numerical integration.** The numerical integration needed to estimate J_1, J_2 is difficult, especially when we choose a large value of L, like $L = 10^6$. We performed these integrals independently on both Mathematica and Sage platforms. It helps to segment the range of integration, but even so, the software can report an error bound in addition to the main estimate. In such cases we have always added on this error bound and then rounded up, since we seek upper bounds for these integrals. In a case where one wants to be assured of a rigorous estimate, there are several options, each carrying some costs. One can use a Simpson or midpoint quadrature with a mesh say of 0.1 together with a careful estimation of the higher derivatives needed to estimate the error. An alternative is to do a Riemann sum with mesh 0.1, where on each interval and for each separate cosine term appearing, the maximum contribution is calculated. If this is done with $T = 4 \cdot 10^5$ and $L = 10^6$, there would be magnitude 10^{11} of these calculations. The extreme value of the cosine contribution would either be at an endpoint of an interval or -1 if the argument straddles a number that is $\pi \bmod 2\pi$. We have done a mild form of this method in our estimation of the integrals $J_0^{\pm}$, see the discussion leading up to Lemma 3.3.

4.5. **Example estimates.** We list some example values of x, y and the corresponding estimates in Fig. 3.

Figure 3. Results.

x	10^{100}	10^{500}
y	10^{15}	10^{35}
T_3	.00642708	.00114940
T_2	.00644109	.00115038
Z^-	.0385260	.0124202
Z^+	.0403125	.0127461
T_1	.0478624	.0155272
T_0	.0514483	.0161799
T	$4 \cdot 10^5$	10^9
d	0.57	0.58
J_0^-	$1.78554 \cdot 10^{-2}$	$4.90043 \cdot 10^{-3}$
J_0^+	$1.80312 \cdot 10^{-2}$	$4.92738 \cdot 10^{-3}$
J_1	$7.236 \cdot 10^{-4}$	$1.717 \cdot 10^{-6}$
J_2	$1.758 \cdot 10^{-2}$	$4.745 \cdot 10^{-3}$
Ψ^-	$2.3302 \cdot 10^{94}$	$1.4989 \cdot 10^{482}$
Ψ^+	$2.9227 \cdot 10^{94}$	$1.5118 \cdot 10^{482}$

5. Appendix

We prove the following theorem.

Theorem 5.1 (Granville and Soundararajan)**.** *If* $3 \le y \le x$ *and* $1/\log y \le \sigma \le 1$, *then*

$$\Psi(x, y) \le 1.39 \frac{y^{1-\sigma}}{\log x} x^{\sigma} \zeta(\sigma, y).$$

Proof. By the identity $\log n = \sum_{d|n} \Lambda(d)$, we have

$$\sum_{\substack{n\le x\\ P(n)\le y}} \log n = \sum_{\substack{m\le x\\ P(m)\le y}} \sum_{\substack{d\le x/m\\ P(d)\le y}} \Lambda(d) = \sum_{\substack{m\le x\\ P(m)\le y}} \sum_{p\le \min\{y,x/m\}} \log p \Big\lfloor \frac{\log(x/m)}{\log p} \Big\rfloor$$
$$\le \sum_{\substack{m\le x\\ P(m)\le y}} \pi\big(\min\{y,x/m\}\big)\log(x/m).$$

Thus,

$$\Psi(x,y)\log x = \sum_{\substack{n\le x\\ P(n)\le y}} (\log n + \log(x/n)) \le \sum_{\substack{n\le x\\ P(n)\le y}} \big(1+\pi(\min\{y,x/n\})\big)\log(x/n).$$

Using the estimates in [18] we see that the maximum of $(1+\pi(t))/(t/\log t)$ occurs at $t=7$, so that

$$1+\pi(t) < 1.39t/\log t$$

for all $t>1$. The above estimate then gives

$$\Psi(x,y)\log x < 1.39 \sum_{\substack{x/y<n\le x\\ P(n)\le y}} x/n + 1.39 \sum_{\substack{n\le x/y\\ P(n)\le y}} y\log(x/n)/\log y.$$

We now note that if $1/\log y \le \sigma \le 1$, then

$$y^{1-\sigma}(x/n)^\sigma \ge \begin{cases} x/n, & \text{if } x/y<n\le x,\\ y\log(x/n)/\log y, & \text{if } n\le x/y.\end{cases}$$

Indeed, in the first case, since $t^{1-\sigma}$ is non-decreasing in t, we have $(x/n)^{1-\sigma} \le y^{1-\sigma}$. And in the second case, since $t^{-\sigma}\log t$ is decreasing in t for $t\ge y$, we have $(x/n)^{-\sigma}\log(x/n) \le y^{-\sigma}\log y$.

We thus have

$$\Psi(x,y)\log x < 1.39 \sum_{\substack{n\le x\\ P(n)\le y}} y^{1-\sigma}(x/n)^\sigma < 1.39y^{1-\sigma}x^\sigma\zeta(\sigma,y).$$

This completes the proof. □

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