Sociable numbers: new developments on an ancient problem

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Sum of proper divisors

Let $s(n)$ be the sum of the *proper* divisors of n:

For example:

$s(10) = 1+2+5 = 8,\ \ s(11) = 1,\ \ s(12) = 1+2+3+4+6 = 16.$

Thus, $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.

Pythagoras, ca. 2500 years ago

Notice that $s(6) = 1 + 2 + 3 = 6$.
If $s(n) = n$, we say n is *perfect*.

Notice that

$$s(220) = 284, \quad s(284) = 220.$$

If $s(n) = m$, $s(m) = n$, and $m \neq n$, we say n, m are an *amicable pair* and that they are amicable numbers.

In the bible?

St. Augustine, ca. 1600 years ago in "City of God":

" Six is a perfect number in itself, and not because God created all things in six days; rather the converse is true — God created all things in six days because the number is perfect."

It was also noted that 28, the second perfect number, is the number of days in a lunar month. A coincidence? Numerologists thought not.

In Genesis it is related that Jacob gave his brother Esau a lavish gift so as to win his friendship. The gift included 220 goats and 220 sheep.

Abraham Azulai, ca. 500 years ago:

"Our ancestor Jacob prepared his present in a wise way. This number 220 is a hidden secret, being one of a pair of numbers such that the parts of it are equal to the other one 284, and conversely. And Jacob had this in mind; this has been tried by the ancients in securing the love of kings and dignitaries."

Ibn Khaldun, ca. 600 years ago in "Muqaddimah":

"Persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals."

Al-Majriti, ca. 1050 years ago reports in "Aim of the Wise" that he had put to the test the erotic effect of

"giving any one the smaller number 220 to eat, and himself eating the larger number 284."

(This was a very early application of number theory, far predating public-key cryptography . . .)

Euclid, ca. 2300 years ago:

"If as many numbers as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes a prime, and if the sum multiplied into the last make some number, the product will be perfect."

For example: $1 + 2 + 4 = 7$ is prime, so $7 \times 4 = 28$ is perfect.

That is, if $1 + 2 + \cdots + 2^k = 2^{k+1} - 1$ is prime, then $2^k(2^{k+1} - 1)$ is perfect.

For example, take $k = 43{,}112{,}608$.

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29. The 46th Mersenne Prime

A Mersenne number is a positive number that can be expressed in the form 2n-1. A Mersenne prime is a Mersenne number that is, well, prime. Searching for higher and higher Mersenne primes is the unofficial national sport of mathematicians. The 45th and 46th (right) Mersenne primes were found this year, the latter by a team at UCLA. It has almost 13 million digits.

Nicomachus, ca. 1900 years ago:

A natural number n is *abundant* if $s(n) > n$ and is *deficient* if $s(n) < n$. These he defined in "Introductio Arithmetica" and went on to give what I call his 'Goldilocks Theory':

" In the case of too much, is produced excess, superfluity, exaggerations and abuse; in the case of too little, is produced wanting, defaults, privations and insufficiencies. And in the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect."

Abundant numbers are like an animal with *"ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands..."* while with deficient numbers, *"a single eye,..., or if he does not have a tongue."*

Actually, Nicomachus only defined deficient and abundant for even numbers, since he likely thought all odd numbers are deficient. However, 945 is abundant; it is the smallest odd abundant number.

Nicomachus conjectured that there are infinitely many perfect numbers and that they are all given by the Euclid formula. Euler, ca. 250 years ago, showed that all even perfect numbers are given by the formula. We still don't know if there are infinitely many, or if there are any odd perfect numbers.

In 1888, Catalan suggested that we iterate the function s, and conjectured that one would always end at 0 or a perfect number. For example:

$$s(12) = 16, \;\; s(16) = 15, \;\; s(15) = 9, \;\; s(9) = 4, \;\; s(4) = 3, \;\; s(3) = 1,$$

and $s(1) = 0$. Perrott in 1889 pointed out that one might also land at an amicable number, in 1907, Meissner said there may well be cycles of length > 2, and in 1913, Dickson amended the conjecture to say that the sequence of s-iterates is always bounded.

Now known as the Catalan–Dickson conjecture, the least number n for which it is in doubt is 276. Guy and Selfridge have the counter-conjecture that in fact there are a positive proportion of numbers for which the sequence is unbounded.

Suppose that

$$s(n_1) = n_2, \quad s(n_2) = n_3, \quad \ldots \ , \quad s(n_k) = n_1,$$

where $n_1, n_2, \ldots, n_k$ are distinct. We say these numbers form a *sociable* cycle of length k, and that they are *sociable* numbers of order k.

Thus, sociable numbers of order 1 are perfect and sociable numbers of order 2 are amicable.

Though Meissner first posited in 1907 that there may be sociable numbers of order > 2, Poulet found the first ones in 1918: one cycle of length 5 and another of length 28. The smallest of order 5 is 12,496, while the smallest of order 28 is 14,316.

Today we know of 175 sociable cycles of order > 2, all but 10 of which have order 4. (The smallest sociable number of order 4 was found by Cohen in 1970; it is 1,264,460.)

We know 46 perfect numbers and over 12 million amicable pairs.

A modern perspective on these problems: what can we say about their distribution in the natural numbers, in particular, do they have density 0?

From work of Descartes and Euler, it is not hard to see that perfect numbers are sparsely distributed within the natural numbers; that is, they have density 0. It is instructive though to look at a result of Davenport from 1933 that implies the same.

For each real number $u > 0$, let $\mathcal{D}_s(u)$ denote the set of natural numbers n with $s(n)/n \le u$. Davenport proved that $\mathcal{D}_s(u)$ has a positive density $D_s(u)$ within the natural numbers; properties for the function $D_s(u)$ include

$$\text{continuous}, \quad \text{strictly increasing}, \quad D_s(0+) = 0, \quad D_s(+\infty) = 1.$$

Note that the first item implies that the perfect numbers have density 0.

Davenport was preceded by Schoenberg in 1928 who had analogous results for Euler's function. Later, Erdős and Wintner considered general multiplicative functions. This (and the Turán proof of the Hardy–Ramanujan theorem) was the dawn of the field of *probabilistic number theory*.

The Davenport distribution result also implies that the deficient numbers ($s(n)/n < 1$) and the abundant numbers ($s(n)/n > 1$) have positive densities. From the very start, people were interested in computing these densities, especially since it seemed that the even numbers are about equally split between abundant and deficient. After work of Behrend in the 1930's, Wall et al. in the 1970's, Deléglise in the 1990's, and now Kobayashi, we know that the density of the abundant numbers is ≈ 0.2476.

But what of the density of amicable numbers or more generally, sociable numbers?

In a *Monthly* problem from 1975, Lenstra proposed: for each k there are infinitely many integers n with the s-sequence starting from n (namely, $n, s(n), s(s(n)), \ldots$) being strictly increasing for the first k steps. Call such a number n a *k-climber*.

This inspired Erdős to prove a remarkable and at first counter-intuitive theorem: *For each fixed k, the set of abundant numbers which are not k-climbers has density* 0. That is, if $n < s(n)$, then almost surely, $s(n) < s(s(n)) < \ldots$ for $k - 1$ more steps.

Now, if you have a sociable k-cycle with $k \geq 2$, then it contains an abundant number that is *not* a k-climber. Thus, for each fixed k, the sociable numbers of order at most k have density 0.

Any given natural number is either sociable or it is not sociable.

Does the set of sociable numbers have density 0?
We conjecture yes.

One thing that makes this a possibly tough question is that we don't have a simple algorithm that can test membership in the set of sociable numbers. For example, is 276 sociable? This is an unsolved problem, despite *much* computation.

Our (Kobayashi, Pollack, P) principal result: *But for a set of density 0, all sociable numbers are contained within the odd abundant numbers.*

Further, the density of all odd abundant numbers is $\approx 1/500$.

Call a sociable number n *special* if

- n is odd abundant,
- the number preceding n in its cycle exceeds

$$n\ \exp\left(\tfrac{1}{2}\sqrt{\log\log\log n\ \log\log\log\log n}\right).$$

We prove that if the special sociable numbers have density 0, then so too do all sociable numbers have density 0. Further, we prove that the special sociable numbers have upper density at most $\approx 1/6000$.

M. Deléglise, Bounds for the density of abundant integers, Experimental Math. **7** (1998), 137–143.

P. Erdős, On asymptotic properties of aliquot sequences, Math. Comp. **30** (1976), 641–645,

P. Erdős, A. Granville, C. Pomerance, and C. Spiro, On the normal behavior of the iterates of some arithmetic functions, pp. 165–204 in Analytic number theory, Progr. Math. vol. 85, Birkhäuser, Boston, 1990.

M. Kobayashi, P. Pollack, C. Pomerance, On the distribution of sociable numbers, J. Number Theory, to appear.

(The last two papers and these slides are available at `www.dartmouth.edu/~carlp` .)