The first dynamical system?

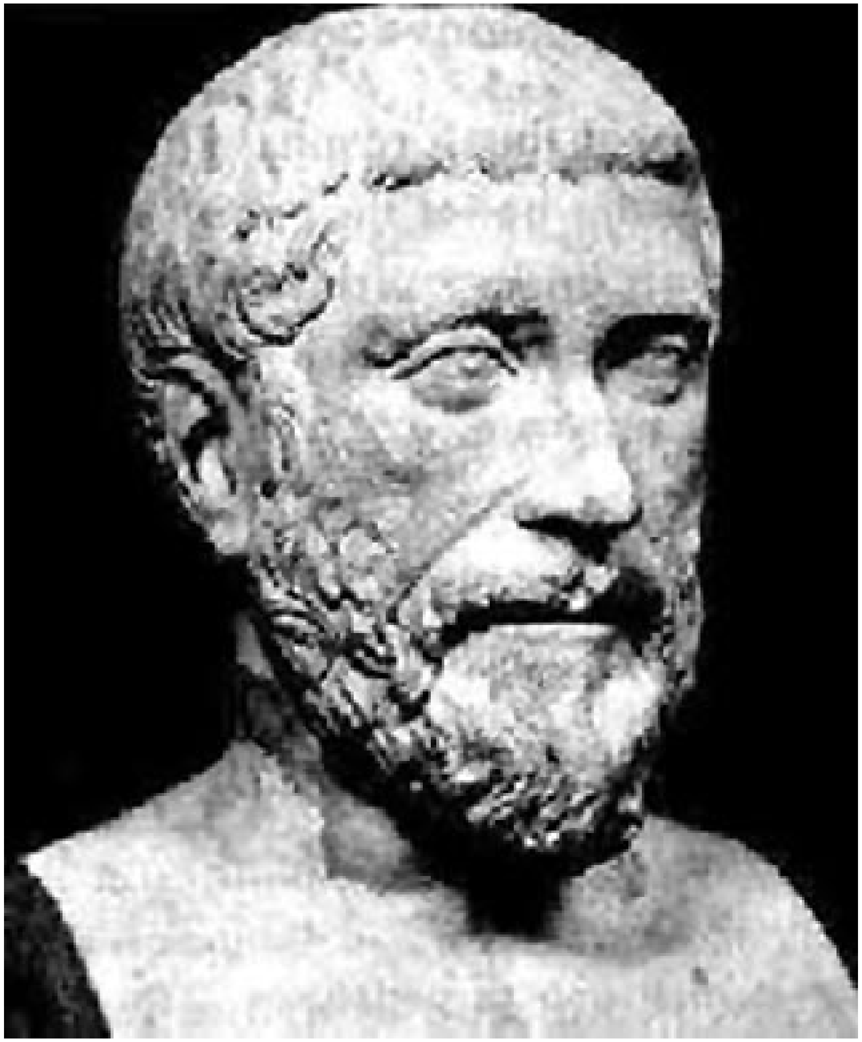
Carl Pomerance, **Dartmouth College**

with

Mitsuo Kobayashi, **Cal Poly Pomona**

Paul Pollack, **University of Georgia**

In the beginning ...



Pythagoras

Sum of proper divisors

Let $s(n)$ be the sum of the *proper* divisors of n:

Thus, $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.

The function $s(n)$ was considered by Pythagoras, about 2500 years ago.

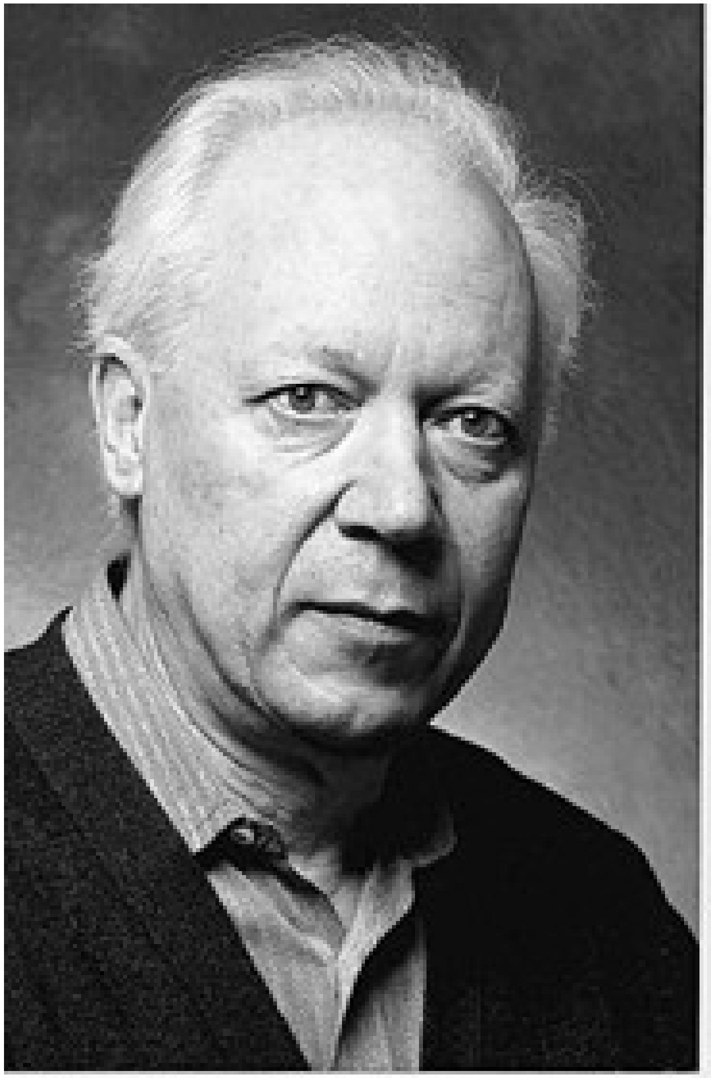
Pythagoras:

noticed that $s(6) = 1 + 2 + 3 = 6$
(If $s(n) = n$, we say n is *perfect*.)

and noticed that

$$s(220) = 284, \quad s(284) = 220.$$

(If $s(n) = m$, $s(m) = n$, and $m \neq n$, we say n, m are an *amicable pair* and that they are *amicable* numbers.)



Enrico Bombieri

In 1976, Bombieri wrote:

"There are very many old problems in arithmetic whose interest is practically nil, e.g. the existence of odd perfect numbers, problems about the iteration of numerical functions, the existence of infinitely many Fermat primes $2^{2^n}+1$, etc."

Sir Fred Hoyle

Hoyle wrote in 1962 that there were two difficult astronomical problems faced by the ancients. One was a good problem, the other was not so good.

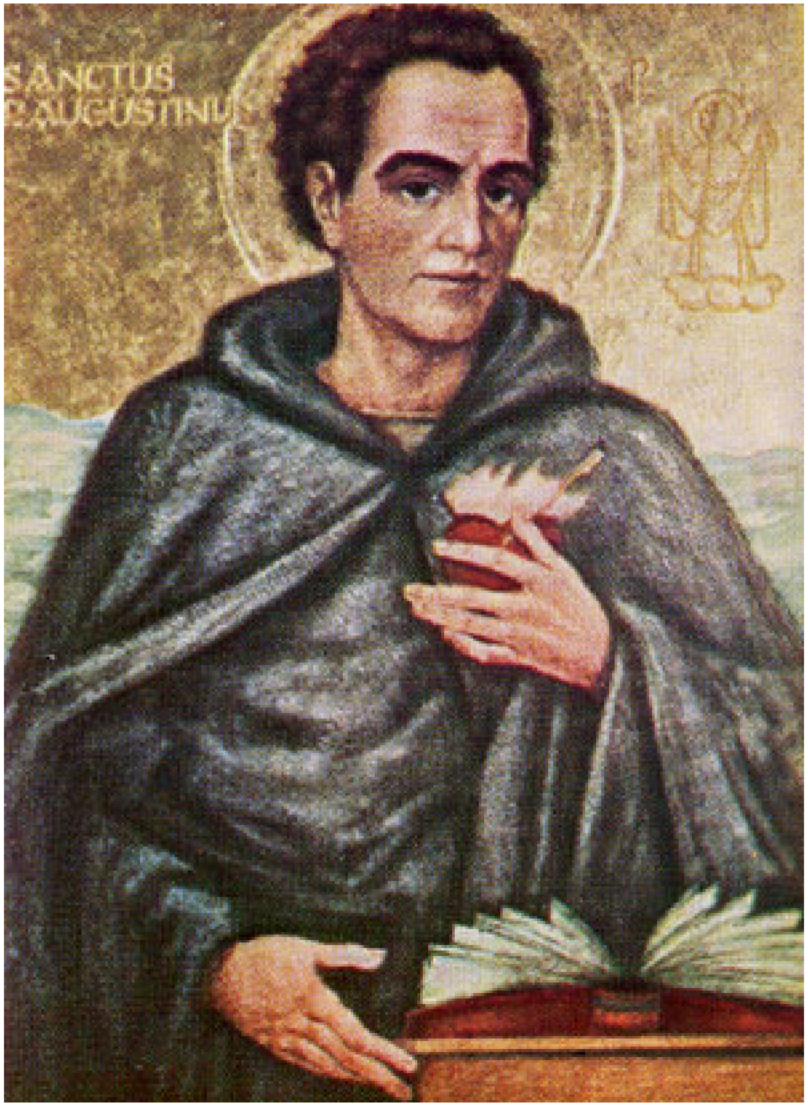
The good problem: Why do the planets wander through the constellations in the night sky?

The not-so-good problem: Why is it that the sun and the moon are the same apparent size?

Historically, perfect numbers, amicable numbers, and topics like these (figurate numbers, e.g.) were important to the development of elementary number theory. So, perhaps it could be argued that they were "good" problems, in the sense of Hoyle.

Further, the search for large numerical examples (say of Mersenne primes) has greatly spurred number-theoretic computing. There are also the twin problems of primality testing and factoring that have been historically linked to these topics.

In addition, they are fascinating to more than just number theorists . . .



St. Augustine

In the bible?

St. Augustine, ca. 1600 years ago in "City of God":

" Six is a perfect number in itself, and not because God created all things in six days; rather the converse is true — God created all things in six days because the number is perfect."

It was also noted that 28, the second perfect number, is the number of days in a lunar month. A coincidence? Numerologists thought not.

In Genesis it is related that Jacob gave his brother Esau a lavish gift so as to win his friendship. The gift included 220 goats and 220 sheep.

Abraham Azulai, ca. 500 years ago:

"Our ancestor Jacob prepared his present in a wise way. This number 220 is a hidden secret, being one of a pair of numbers such that the parts of it are equal to the other one 284, and conversely. And Jacob had this in mind; this has been tried by the ancients in securing the love of kings and dignitaries."

Ibn Khaldun, ca. 600 years ago in "Muqaddimah":

"Persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals."

Ibn Khaldun

Al-Majriti, ca. 1050 years ago reports in “Aim of the Wise” that he had put to the test the erotic effect of

“giving any one the smaller number 220 to eat, and himself eating the larger number 284.”

(This was a very early application of number theory, far predating public-key cryptography . . .)

Nicomachus, ca. 1900 years ago:

A natural number n is *abundant* if $s(n) > n$ and is *deficient* if $s(n) < n$. These he defined in "Introductio Arithmetica".

Abundant numbers are like an animal with *"ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands..."* while with deficient numbers, *"a single eye,..., or if he does not have a tongue."*

It was not known till 1933 that the abundant, deficient, and perfect numbers all possess an asymptotic density. This was shown by Harold Davenport, and more precisely: *For each real number $u \geq 0$, the density $D(u)$ of integers n with $s(n) \geq un$ exists, and $D(u)$ is continuous and strictly increasing.*

Mits Kobayashi: $D(1) = 0.2476\ldots$.

Eugène Catalan

Leonard Dickson

In 1888, Catalan suggested that we iterate the function s and conjectured that one would always end at 0 or a perfect number. For example:

$$s(12) = 16, \;\; s(16) = 15, \;\; s(15) = 9, \;\; s(9) = 4, \;\; s(4) = 3, \;\; s(3) = 1,$$

and $s(1) = 0$. Perrott in 1889 pointed out that one might also land at an amicable number. In 1907, Meissner said there may well be cycles of length > 2. And in 1913, Dickson amended the conjecture to say that the sequence of s-iterates is always bounded.

Now known as the Catalan–Dickson conjecture, the least number n for which it is in doubt is 276. Guy and Selfridge have the counter-conjecture that in fact there are a positive proportion of numbers for which the sequence is unbounded.

Suppose that

$$s(n_1) = n_2, \quad s(n_2) = n_3, \quad \dots \ , \quad s(n_k) = n_1,$$

where $n_1, n_2, \dots, n_k$ are distinct. We say these numbers form a *sociable* cycle of length k, and that they are *sociable* numbers of order k.

Thus, sociable numbers of order 1 are perfect and sociable numbers of order 2 are amicable.

Though Meissner first posited in 1907 that there may be sociable numbers of order > 2, Poulet found the first ones in 1918: one cycle of length 5 and another of length 28. The smallest of order 5 is 12,496, while the smallest of order 28 is 14,316.

Today we know of 217 sociable cycles of order > 2, all but 11 of which have order 4. (The smallest sociable number of order 4 was found by Henri Cohen in 1970; it is 1,264,460.)

We know 48 perfect numbers and about 12 million amicable pairs.

A modern perspective on these problems: what can we say about their distribution in the natural numbers, in particular, does the set of sociable numbers have density 0?

We know after Hornfeck & Wirsing: The number of perfect numbers in $[1, x]$ is $O(x^{c/\log\log x})$.

And P: The number of amicable numbers in $[1, x]$ is less than $x/\exp((\log x)^{1/3})$ for all large x.

In particular the sum of reciprocals of these two sets is bounded.

This is not known for numbers in longer cycles, even if you fix the length. Lets try and count all of the sociables.

Up to 100 the only sociable numbers are the perfect numbers 6 and 28, so $N(100) = 2$ and $N(100)/100 = 0.02$.

Up to 1000 we pick up the perfect number 496 and the Pythagorean amicables 220 and 284. So $N(1000) = 5$ and $N(1000)/1000 = 0.005$.

Up to 10,000 we pick up the perfect number 8128 and the amicable pairs

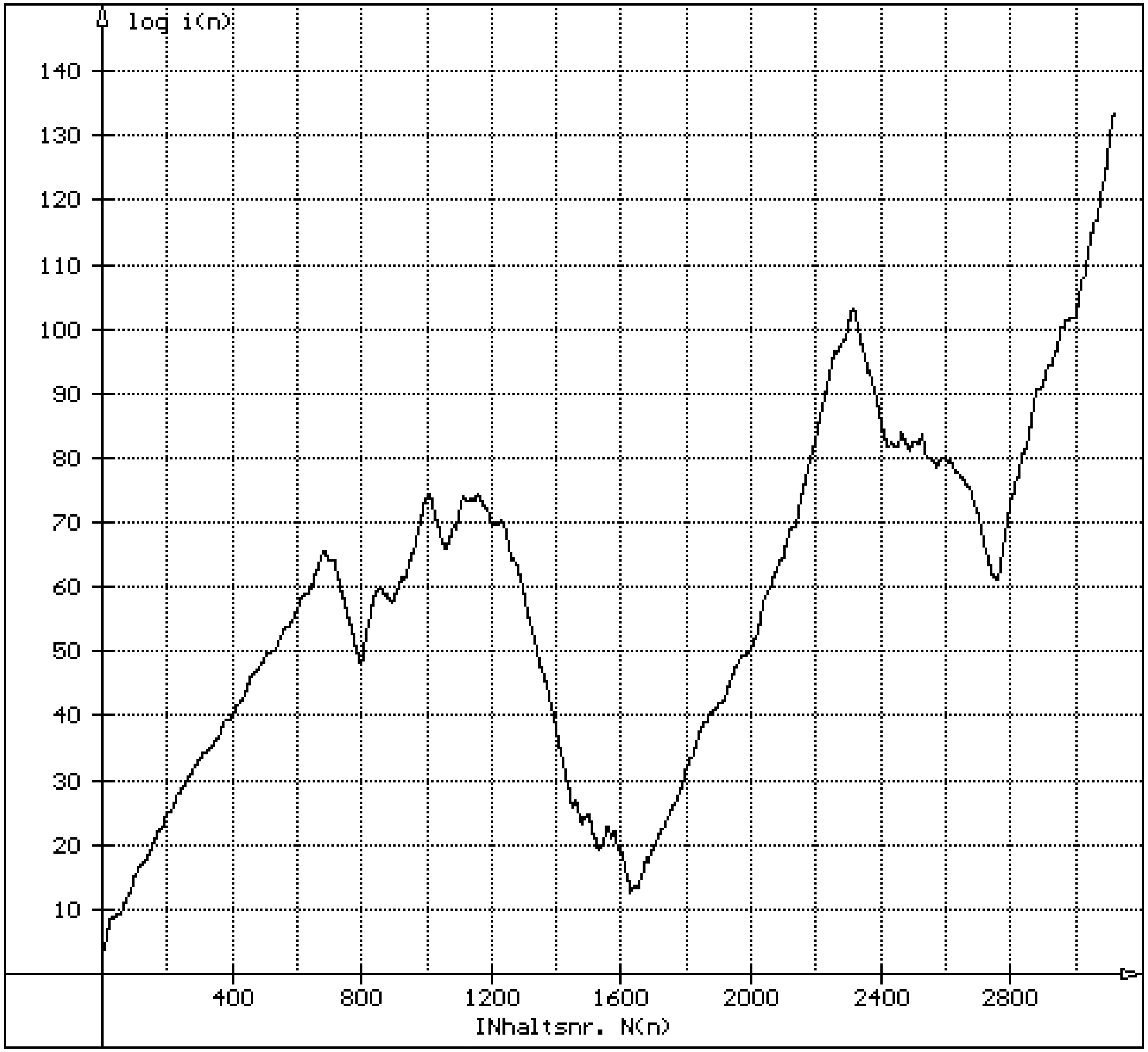
$$1184,\ 1210;\quad 2620,\ 2924;\quad 5020,\ 5564;\quad 6232,\ 6368.$$

(The first was found by Paganini in 1860, the others by Euler.) So $N(10{,}000) = 14$ and $N(10{,}000)/10{,}000 = 0.0014$.

Are we sure we have the counts right? These are the correct counts for perfects and amicables, and also for sociables of order at most 600.

In fact, there are 81 starting numbers below 10,000 where we have iterated $s(n)$ over 600 times, and it is not yet clear what is happening. Some of these are known *not* to be sociable, for example the least number in doubt, 276. (It is not sociable because it is not in the range of the function s.) But some of them might end up being sociable after travelling a very long distance through its s-chain. The least such possibility is 564. (Creyaufmueller)

So, we are having trouble even computing $N(1000)$ much less showing the sociable numbers have density 0.



564 iteration

In January 1973, te Riele published an example of a number n such that the sequence $n, s(n), s(s(n)), \ldots$ is strictly increasing for more than 5092 steps. At the end he remarks that Hendrik Lenstra communicated a proof to him that for every k there is some number n, with the sequence $n, s(n), s(s(n)), \ldots$ strictly increasing for at least k steps. I believe this is the earliest citation of a result of Lenstra.

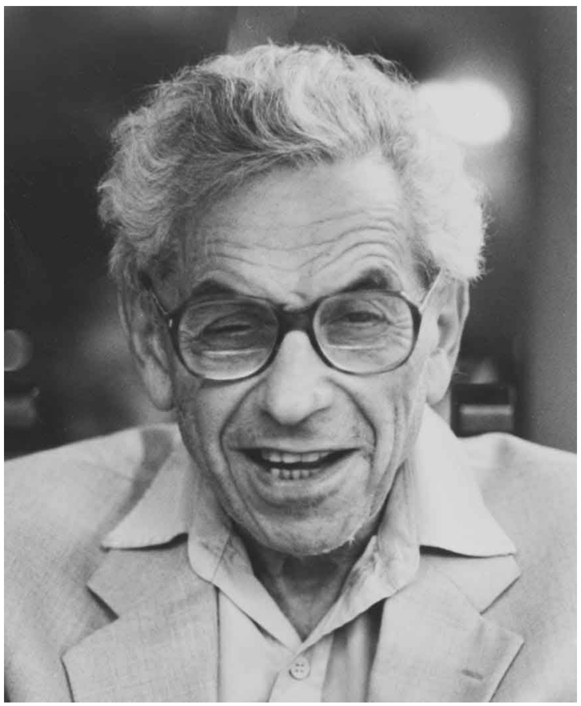
Lenstra published this assertion as a *Monthly* problem in 1975, and his solution in 1977.

Let $s_1(n) = s(n)$, and for all positive integers k, n, if $s_k(n)$ is defined and not 0, let $s_{k+1}(n) = s(s_k(n))$.

From Lenstra's proof we have for each k, the set of numbers n where $n < s_1(n) < \cdots < s_k(n)$ contains a set of positive asymptotic density.

The Lenstra problem and solution inspired Erdős to prove a remarkable and at first counter-intuitive theorem: *Let C_k be the set of integers n such that $n < s_1(n) < \cdots < s_k(n)$. Then each set C_k has the same asymptotic density as the set C_1, the set of abundant numbers.* That is, if $n < s(n)$, then almost surely, $s(n) < s(s(n)) < \ldots$ for $k - 1$ more steps.

Now, if you have a sociable k-cycle with $k \geq 2$, then it contains an abundant number n not in C_k. Thus, for each fixed k, the sociable numbers of order at most k have density 0.


Paul Erdős

Any given natural number is either sociable or it is not sociable.

I ask again: Does the set of sociable numbers have density 0? We think the answer is yes, but we have no heuristic for this. (Despite the fact that Cohen and Lenstra have worked in this subject, I don't think the Cohen–Lenstra heuristics apply.)

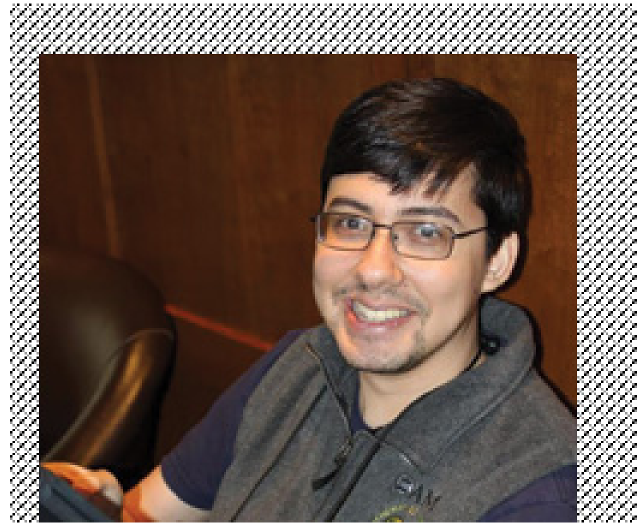
Kobayashi, Pollack, P:
But for a set of density 0*, all sociable numbers are contained within the odd abundant numbers.*

Further, the density of all odd abundant numbers is $\approx 1/500$.

One helpful tool in the proof was to essentially prove the reverse of the Erdős abundant-perpetuation theorem for deficient sociable numbers. So, in particular, if one has a deficient sociable number n, the sequence $n, s_1(n), s_2(n), \ldots$ usually decays exponentially for a long way, and so n is associated to a very small sociable number in its cycle.

We'd like to do the same for abundant numbers, but by backing up to earlier members of the cycle. For even abundants, this idea works fine, but not for odd abundants.

Mitsuo Kobayashi



Paul Pollack

THANK YOU