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There Are No Odd Super Perfect Numbers Less Than $7 \cdot 10^{24}$

J. L. Hunsucker and Carl Pomerance

§1. Introduction. D. Suryanarayana [7] called a natural number n super perfect if $\sigma(\sigma(n)) = 2n$ where σ is the sum of the divisors function. Other papers on this subject are Hunsucker and Pomerance [2], Kanold [3], Lord [4], Niederreiter [5], and Suryanarayana [8]. No one knows if any odd super perfect numbers exist, nor do we know a proof of their non-existence. Our result here, that the smallest such number must be $> 7 \cdot 10^{24}$, certainly puts them beyond reach of a casual search.

The main technique of our proof is to do case studies using prime factorizations of various $\sigma(p^a)$ where p is a prime. Most of the prime factorizations used in this paper are found in a computer print-out at the end of Tuckerman [9]. Several other factorizations we have established ourselves, namely $\sigma(3^{40})$, $\sigma(3^{42})$, $\sigma(3^{46})$, $\sigma(5^{28})$, $\sigma(7^{22})$, $\sigma(11^{18})$, $\sigma(13^{18})$, $\sigma(61^{12})$, $\sigma(71^{10})$, and $\sigma(1093^{6})$. We used the CDC 6400 at the University of Georgia. We wish to thank Dr. D. E. Penney for his expert assistance in finding the factorizations.

We should say a word about our bound $7 \cdot 10^{24}$. If n is an odd super perfect number for which $(n, \sigma(n)) = 1$, then it is easy to show that $n\sigma(n)$ is an odd perfect number. However, recently Hagis [1] showed that every odd perfect number is $> 10^{50}$. Hence if $(n,\sigma(n)) = 1$, and if n is an odd super perfect number, then $n\sigma(n) > 10^{50}$. Now $\sigma(n) < \sigma(\sigma(n)) = 2n$, so $n\sigma(n) < 2n^2$. Hence $n > (\frac{1}{2} \cdot 10^{50})^{\frac{1}{2}} > 7 \cdot 10^{24}$. So we might assume throughout that $(n,\sigma(n)) \neq 1$. However, we found this condition only slightly useful, so we decided to make this paper independent of the Hagis result and not use the condition $(n,\sigma(n)) \neq 1$. The only remnant of our original approach is our unusual bound $7 \cdot 10^{24} \approx (\frac{1}{2} \cdot 10^{50})^{\frac{1}{2}}$.

§2. Preliminaries. By a Fermat prime, we mean a prime of the form $2^{2^k} + 1$. By $m \| n$, we mean $m | n$ and $(m, \frac{n}{m}) = 1$. Note that $\sigma(p^a) = 1 + p + \ldots + p^a = (p^{a+1}-1)/(p-1)$. Hence if $a | b$ then $\sigma(p^{a-1}) | \sigma(p^{b-1})$. Our first lemma comes from Pomerance [6, p. 269.].

Lemma 1. Let p be an arbitrary prime, q a Fermat prime, and b, x positive integers, where $x \not\equiv 3 \ (4)$. Then $q^b \| \sigma(p^x)$ if and only if either

(i) $p \equiv 1 (q)$ and $q^b \| x+1$; or

(ii) $x \equiv 1(4)$, $q^a \| p+1$ for some $a > 0$, and $q^{b-a} \| x+1$.

Our second lemma is a catalogue of some known facts on odd super perfect numbers. Suryanarayana [7] and Kanold [3] noted that (i), (ii), and (iii) hold. Suryanarayana [8] and

Hunsucker and Pomerance [2] are responsible for (iv) and (v).

Lemma 2. Let n be an odd super perfect number. then

(i) $\sigma(n)$ is odd;

(ii) n is a square;

(iii) $\sigma(n)$ is an Euler number, that is $\sigma(n) = p_1^{a_1} \cdot p_2^{2a_2} \cdot \ldots \cdot p_k^{2a_k}$ where $p_1,\ldots,p_k$ are distinct odd primes and $p_1 \equiv a_1 \equiv 1(4)$ (we will call p_1 the special prime);

(iv) n is not a prime power;

(v) $\sigma(n)$ is not a prime power.

If x is a positive integer, we define $h(x) = \sigma(x)/x$. If $x_1,\ldots,x_k$ are positive integers, we define $h(x_1,\ldots,x_k) = h(x_1) \cdot \ldots \cdot h(x_k)$. Hence if $y_i | x_i$ for $i = 1,\ldots,k$, then $h(y_1,\ldots,y_k) \le h(x_1,\ldots,x_k)$ where equality holds if and only if $y_i = x_i$ for $i = 1,\ldots,k$. We clearly have that n is super perfect if and only if $h(n,\sigma(n)) = 2$. Note also that if $3|(n,\sigma(n))$, then lemma 2 implies $9|(n,\sigma(n))$. But $h(9,9) > 2$. Hence we have

Lemma 3. If $a|n$, $b|\sigma(n)$, and $h(a,b) > 2$, then n is not super perfect. In particular n is not super perfect if $3|(n,\sigma(n))$.

Lemma 4. For n an odd super perfect number either of the following conditions implies $n > 7\cdot 10^{24}$:

(i) $\sigma(n) > 14\cdot 10^{24}$

(ii) n or $\sigma(n)$ has a square divisor $> 7\cdot 10^{24}$.

Proof. If $\sigma(n) > 14\cdot 10^{24}$ then $2n = \sigma(\sigma(n)) \geq 1+\sigma(n) > 14\cdot 10^{24}$. If $k^2 | \sigma(n)$ where $k^2 > 7\cdot 10^{24}$ then lemma 2 implies $\sigma(n) \geq 5k^2 > 14\cdot 10^{24}$.

We use the notation $v(q,n)$ to denote the exponent of the prime q in the prime factorization of n. Hence $v(q,n) = a$ if and only if $q^a \| n$.

§3. The smallest prime factor of $n\sigma(n)$ is not 7. In this section we will prove the title and more:

Theorem 1. If n is an odd super perfect number and either

(A) 7 is the smallest prime in $n\sigma(n)$; or

(B) $3 | 1 + v(3,n)$ and 13 is the special prime;

then $n > 7\cdot 10^{24}$.

Note that if either (A) or (B) holds for n odd and super perfect, then we also have

(C) x is one of n and $\sigma(n)$, y is the other,
$7|x$, $3 \nmid y$, $5 \nmid xy$, the special prime is $\not\equiv 2(3)$,
and if $x = \sigma(n)$ then $3 \nmid x$.

Indeed, note that if $p \equiv 2(3)$ and $a \equiv 1(4)$, then

$3 \mid p+1 \mid \sigma(p^a)$, so (A) clearly implies (C). If (B) holds, then $7 \mid \sigma(13) \mid 2n$. Hence we take $x = n$ so that (C) holds provided $3 \nmid y$ and $5 \nmid xy$. But $h(3^2, 5, 7^2) > 2$, so lemma 3 implies $5 \nmid xy$. Lemma 3 also implies $3 \nmid y$.

Hence to prove theorem 1 it will be sufficient to prove that for an odd super perfect number n for which (C) holds we have $n > 7 \cdot 10^{24}$.

We will denote the special prime power in $\sigma(n)$ by p^a. From condition (C) we have that $p \equiv 1(3)$. Also since $5 \nmid n\sigma(n)$ we have that $p \not\equiv 4(5)$ for otherwise $5 \mid \sigma(p^a) \mid 2n$. Also for $q \equiv 1(5)$ we have $5 \nmid 1 + v(q,z)$ where $z = n$ or $\sigma(n)$. Since $3 \mid \sigma(7^2)$ and $3 \nmid y$, we have $3 \nmid 1 + v(7,x)$. Since $7^{30} > 14 \cdot 10^{24}$ we will prove theorem 1 by showing in propositions 1.1 through 1.4 that there is no allowable value for $v(7,x) \leq 28$ for which $n < 7 \cdot 10^{24}$.

Proposition 1.1. $5 \nmid 1 + v(7,x)$.

Proof. Suppose $5 \mid 1 + v(7,x)$. Then $2801 = \sigma(7^4) \mid y$. Since $2801 \equiv 2(3) \equiv 1(5)$, it is not special, and its exponent is not 4. Since $2801^{10} > 14 \cdot 10^{24}$, it will be sufficient to show that $v(2801,y) \neq 2, 6$, or 8. Note that $\sigma(2801^6) = 7 \cdot 71m$ where $(7 \cdot 71, m) = 1$ and $m \equiv 3(4)$. Now $7^4 \cdot 71^2 \cdot m > 7 \cdot 10^{24}$ so for $v(2801,y) = 6$, we have $7^4 \cdot 71^2 m = \sigma(n)$. Lemma 2 implies then that $m \equiv 1(4)$, a contradiction. Hence, it will be sufficient to show that $3 \nmid v(2801,y)$.

Suppose $3 \mid 1 + v(2801,y)$, so that $37 \cdot 43 \cdot 4933 = \sigma(2801^2) \mid x$. Since $43 \equiv 1(3) \not\equiv 1(4)$, we have $v(43,x) \geq 4$. Suppose 4933 is special. Then $2467 = \frac{1}{2}\sigma(4933) \mid y$. Since $2467 \equiv 1(3)$, its exponent is not 2 for otherwise $3 \mid x = \sigma(n)$. Now $v(2467,y) \nmid 4$ since $(\sigma(2467^4), 7 \cdot 37 \cdot 43 \cdot 4933) = 1$ and $7^4 \cdot 37^2 \cdot 43^4 \cdot 4933 \cdot \sigma(2467^4) > 14 \cdot 10^{24}$. Hence $v(2467,y) \geq 6$, so that $y \geq 2801^2 \cdot 2467^6 > 14 \cdot 10^{24}$, a contradiction. Hence 4933 is not special. But $v(4933,x) \nmid 2$ since otherwise $3 \mid \sigma(4933^2) \mid y$. Finally if $v(4933,x) \geq 4$, we have $x \geq 7^4 \cdot 37 \cdot 43^4 \cdot 4933^4 > 14 \cdot 10^{24}$.

Proposition 1.2. $v(7,x) \nmid 6$

Proof. If $v(7,x) = 6$, then $29 \cdot 4733 = \sigma(7^6) \mid y$. Since $29 \equiv 4733 \equiv 2(3)$ neither prime is special. Since $29^2 \cdot 4733^6 > 7 \cdot 10^{24}$ it will suffice to show that $v(4733,x) \nmid 2$ or 4. If $v(4733,x) = 2$, then $22406023 = \sigma(4733^2) \mid x$. Since $22406023 \equiv 1(3) \not\equiv 1(4)$, its exponent is ≥ 4. But then $x \geq 7^6 \cdot 22406023^4 > 14 \cdot 10^{24}$. Hence $v(4733,x) \nmid 2$. If $v(4733,x) = 4$, then $11 \cdot 41 \cdot 101 \cdot 11018941331 = \sigma(4733^4) \mid x$. But $\sigma(4733^4)^2 > 14 \cdot 10^{24}$, so that $v(4733,x) \nmid 4$.

Proposition 1.3. $v(7,x) \nmid 10$ or 12.

Proof. If $v(7,x) = 10$, then $1123 \cdot 293459 = \sigma(7^{10}) \mid y$. Since $1123^2 \cdot 293459^4 > 14 \cdot 10^{24}$, we may assume $v(293459,y) = 2$. Then $277 \cdot 310897033 = \sigma(293459^2) \mid x$. But both of these primes are $\equiv 1(3)$ so their exponents are $\nmid 2$ or else $3 \mid y$. Then

$x \geq 7^{10} \cdot 277^{4} \cdot 310897033 > 14 \cdot 10^{24}$.

If $v(7,x) = 12$, then $16148168401 = \sigma(7^{12}) \mid y$. If 16148168401 is special, then $103 \cdot m = \frac{1}{2}\sigma(16148168401) \mid x$ where $(7 \cdot 103, m) = 1$. Since $103 \equiv 1(3) \not\equiv 1(4)$, we have $v(103,x) \geq 4$. Then $x \geq 7^{12} \cdot 103^{4} \cdot m > 7 \cdot 10^{24}$. But clearly $v(16148168401,y) < 4$, so we must have $3 \cdot m' = \sigma(16148168401^{2}) \mid x$ where $(3 \cdot 7, m') = 1$. Then $x \geq 3^{2} \cdot 7^{12} \cdot m' > 14 \cdot 10^{24}$.

Proposition 1.4. $v(7,x) \neq 16, 18, 22$ or 28.

Proof. If $v(7,x) = 16$, then $14009 \cdot 2767631689 = \sigma(7^{16}) \mid y$. Both of these primes are $\equiv 4(5)$ so neither is special. Then $y \geq \sigma(7^{16})^{2} > 14 \cdot 10^{24}$.

If $v(7,x) = 18$, then $419 \cdot 4534166740403 = \sigma(7^{18}) \mid y$, and again neither prime is special, so $y > 14 \cdot 10^{24}$.

If $v(7,x) = 22$, then $47 \cdot 3083 \cdot 31479823396757 = \sigma(7^{22}) \mid y$, and since these primes are all $\equiv 2(3)$, none is special.

Finally if $v(7,x)$ 28, we note that $59 \| \sigma(7^{28}) \mid y$. Then $y \geq 59 \cdot \sigma(7^{28}) > 14 \cdot 10^{24}$, since $v(59,y) \geq 2$.

§4. If $3 \mid n\sigma(n)$ then $3 \nmid 1 + v(3, n\sigma(n))$. Suppose $3 \mid n\sigma(n)$. Since $3 \nmid (n, \sigma(n))$, let x be the one of $n, \sigma(n)$ divisible by 3, and let y be the other. Then $v(3, n\sigma(n)) = v(3,x)$.

Theorem 2. If n is an odd super perfect number and $3 \mid 1 + v(3, n\sigma(n))$, then $n > 7 \cdot 10^{24}$.

First we note that if $3 \mid 1 + v(3,x)$, then $13 = \sigma(3^2) \mid y$. Theorem 1 implies $v(13,y)$ is even. If $v(13,y) \geq 22$, then lemma 2 implies $y \geq 5 \cdot 13^{22} > 14 \cdot 10^{24}$, so we may assume $v(13,y) < 22$. In propositions 2.1 to 2.5 we show there is no allowable value for $v(13,y) < 22$ for which $n < 7 \cdot 10^{24}$.

Proposition 2.1. If $3 \mid 1 + v(13,y)$, then 61 is not the special prime in x.

Proof. Assume $3 \mid 1 + v(13,y)$ and 61 is the special prime in x. Then $x = \sigma(n)$, $y = n$ and $31 = \frac{1}{2}\sigma(61) \mid n$. Suppose $3 \mid 1 + v(31,n)$. Then $331 \mid \sigma(31^2) \mid \sigma(n)$. Now $3 \mid \sigma(331^2)$, so since $3 \nmid (n, \sigma(n))$, we have $3 \nmid 1 + v(331, \sigma(n))$. Since $5 \mid \sigma(331^4)$ and $h(5^2 \cdot 13^2 \cdot 31^2, 3^2) > 2$, lemma 3 implies $v(331, \sigma(n)) \neq 4$. If $v(331, \sigma(n)) = 6$, then $2180921 \cdot 604842197 = \sigma(331^6) \mid n$, and $n > \sigma(331^6)^2 > 7 \cdot 10^{24}$. Note that $331^{10} > 14 \cdot 10^{24}$. We conclude that $3 \nmid 1 + v(31,n)$.

Now $5 \cdot 11 \mid \sigma(31^4)$ and $h(13^2 \cdot 31^4, 3^2 \cdot 5^2 \cdot 11^2) > 2$, so $v(31,n) \neq 4$. If $v(31,n) = 6$, then $917087137 = \sigma(31^6) \mid \sigma(n)$. Since $3 \mid \sigma(917087137^2)$, we have $\sigma(n) \geq 61 \cdot \sigma(31^6)^4 > 14 \cdot 10^{24}$, so $v(31,n) \neq 6$. If $v(31,n) = 10$, then $23 \cdot 397 \cdot 617 \cdot 150332843 \mid \sigma(n)$, so $\sigma(n) \geq 61 \cdot \sigma(31^{10})^2 > 14 \cdot 10^{24}$. If $v(31,n) = 12$, then $42407 \cdot 2426789 \cdot 7908811 = \sigma(31^{12}) \mid \sigma(n)$, so $\sigma(n) \geq 61 \cdot \sigma(31^{12})^2 > 14 \cdot 10^{24}$. Finally, if $v(31,n) \geq 16$, then $n \geq 13^2 \cdot 31^{16} > 7 \cdot 10^{24}$.

Proposition 2.2. $3 \nmid 1 + v(13,y)$.

Proof. Assume $3 \mid 1 + v(13,y)$. Then $61 \mid \sigma(13^2) \mid x$. We have seen that $v(61,x)$ is even. Since $3|\sigma(61^2)$, we have $3\nmid 1+v(61,x)$. Since $61^{16} > 14\cdot 10^{24}$, we need only consider $v(61,x) = 4$, 6, 10 and 12.

Suppose $v(61,x) = 4$. Then $5\cdot 131\cdot 21491|y$. Since 131 and $21491 \not\equiv 1(4)$ they are not special. If $v(21491,y) = 2$, then $421\cdot 1097113 = \sigma(21491^2)|x$. If 1097113 is special, then $548557 = \frac{1}{2}\sigma(1097113)|y$ and $n \geq 5^2\cdot 13^2\cdot 131^2\cdot 21491^2\cdot 548557^2 > 7\cdot 10^{24}$. Now $3|\sigma(1097113^2)$ and $3^2\cdot 61^4\cdot 1097113^4 > 14\cdot 10^{24}$, so $v(21491,y) \neq 2$. Since $5|\sigma(21491^4)$ and $h(5\cdot 13^2, 3^2\cdot 5^2) > 2$, we have $v(21491,y) \geq 6$. But $21491^6 > 14\cdot 10^{24}$, so $v(61,x) \neq 4$.

If $v(61,x) = 6$, then $52379047267 = \sigma(61^6)|y$. Now $\sigma(52379047267^2) = 3\cdot m$ where $(3\cdot 61,m) = 1$. Then since $3^2\cdot 61^6\cdot m > 14\cdot 10^{24}$, we have $v(52379047267,y) \geq 4$. But then $y > 14\cdot 10^{24}$, so $v(61,x) \neq 6$.

If $v(61,x) = 10$, then $199\cdot 859\cdot 4242586390571 = \sigma(61^{10})|y$, so that $y \geq \sigma(61^{10})^2 > 14\cdot 10^{24}$. Finally, if $v(61,x) = 12$, then $187123\|\sigma(61^{12})|y$. But $187123\cdot\sigma(61^{12}) > 14\cdot 10^{24}$, so $v(61,x) \neq 12$.

Proposition 2.3. $v(13,y) \neq 4$.

Proof. If $v(13,y) = 4$, then $30941 = \sigma(13^4)|x$. Now 30941 is not special since otherwise $3|\sigma(30941)|y$. If $v(30941,x) = 2$ then $157\cdot 433\cdot 14083 = \sigma(30941^2)|y$. Since $13^4\cdot 157^2\cdot 433\cdot 14083^4 > 14\cdot 10^{24}$, we need only examine the case $v(14083,y) = 2$. In this case we have $3\cdot 4591\cdot 14401 = \sigma(14083^2)|x$. Now if

$v(14401,x) = 1$, then $19\cdot 379 = \frac{1}{2}\sigma(14401)|y$ and $y \geq 13^4\cdot 19^2\cdot 157^2\cdot 379^2\cdot 433^2\cdot 14083^2 > 14\cdot 10^{24}$. But $v(14401,x) \nmid 2$ since otherwise $3|y$. Then $v(14401,x) \geq 4$ so $x \geq 3^2\cdot 4591^2\cdot 14401^4\cdot 30941^2 > 14\cdot 10^{24}$. Hence $v(30941,x) \nmid 2$.

If $v(30941,x) = 4$, then $5\cdot 11|\sigma(30941^4)|y$. But $h(3^2, 5\cdot 11^2\cdot 13^4) > 2$. Hence $v(30941,x) \geq 6$, so $x \geq 3^2\cdot 30941^6 > 14\cdot 10^{24}$.

Proposition 2.4. $v(13,y) \nmid 6$, 10, 12, 16 or 18.

Proof. If $v(13,y) = 6$, then $5229043 = \sigma(13^6)|x$. But $5229043 \equiv 3(4) \equiv 1(3)$ so $v(5229043,x) \geq 4$. Then $x > 14\cdot 10^{24}$.

If $v(13,y) = 10$, then $23\cdot 419\cdot 859\cdot 18041 = \sigma(13^{10})|x$. None of these primes are special since the first 3 are $\equiv 3(4)$ and $18041 \equiv 2(3)$. Also $v(859,x) \nmid 2$ since otherwise $3|\sigma(859^2)|y$. Hence $x \geq 3^2\cdot 23^2\cdot 419^2\cdot 859^4\cdot 18041^2 > 14\cdot 10^{24}$.

If $v(13,y) = 12$, then $53\cdot 264031\cdot 1803647|x$. Hence $x \geq 3^2\cdot 53\cdot 264031^2\cdot 1803647^2 > 14\cdot 10^{24}$.

If $v(13,y) = 16$, then $103\cdot 443\cdot 157984613575O9 = \sigma(13^{16})|x$. Since $157984613575O9^2 > 14\cdot 10^{24}$, we have $5\cdot 13\cdot 73\cdot 21487\cdot 77477 = \frac{1}{2}\sigma(157984613575O9)|y$. Then $y \geq 5^2\cdot 13^{16}\cdot 73^2\cdot 21487^2\cdot 77477^2 > 14\cdot 10^{24}$.

Finally, if $v(13,y) = 18$, then $12865927\|\sigma(13^{18})|x$, so that $x \geq 3^2\cdot 12865927\cdot \sigma(13^{18}) > 14\cdot 10^{24}$.

§5. If $3|n\sigma(n)$ then $5\nmid 1 + v(3,n\sigma(n))$ and $7\nmid 1 + v(3,n\sigma(n))$. As in §4, we assume $3|n\sigma(n)$ and we let $x = n$ or $\sigma(n)$ depending on which is divisible by 3. Then $v(3,n\sigma(n)) = v(3,x)$. Also we let y be the other of $n,\sigma(n)$ so that $3\nmid y$.

Theorem 3. If n is an odd super perfect number and either

$5 \mid 1 + v(3,n\sigma(n))$ or $7 \mid 1 + v(3,n\sigma(n))$, then $n > 7\cdot 10^{24}$.

Note that $\sigma(3^4) = 11^2$ and $\sigma(3^6) = 1093$. Hence if $5 \mid 1 + v(3,x)$ then $11 \mid y$ and if $7 \mid 1 + v(3,x)$, then $1093 \mid y$. Since $1093^9 > 11^{26} > 14\cdot 10^{24}$, we need only consider $v(11,y) < 26$ and $v(1093,y) < 9$.

Proposition 3.1. If $5 \mid 1 + v(3,x)$ then $1 + v(11,y)$ is not divisible by 3 or 5.

Proof. If $3 \mid 1 + v(11,y)$, then $7\cdot 19 = \sigma(11^2) \mid x$. Since 7 and 19 are non-special, we have $h(n,\sigma(n)) \geq h(3^4\cdot 7^2\cdot 19^2, 11^2) > 2$. So $3 \nmid 1 + v(11,y)$.

If $5 \mid 1 + v(11,y)$, then $5 \mid \sigma(11^4) \mid x$. But $v(5,x) \nmid 1$ or else $3 \mid y$. Also $h(3^4\cdot 5^2, 11^4) > 2$. Hence $5 \nmid 1 + v(11,y)$.

We state the next proposition in a more general setting so that we may use it in proposition 7.1. Note that we do not assume that $3 \mid n\sigma(n)$.

Proposition 3.2. If n is an odd super perfect number, $x = n$ or $\sigma(n)$, y is the other, $3 \nmid y$, and $v(11,y) = 6$, 10, 12, 16, 18 or 22, then $n > 7\cdot 10^{24}$.

Proof. Suppose $v(11,y) = 6$. Then $43\cdot 45319 = \sigma(11^6) \mid x$. Since these primes are $\equiv 1(3)$ and $\equiv 3(4)$, we have that their exponents are ≥ 4. But $\sigma(11^6)^4 > 14\cdot 10^{24}$.

If $v(11,y) = 10$, then $15797\cdot 1806113 = \sigma(11^{10}) \mid x$. Since these primes are $\equiv 2(3)$, they are not special. Since $15797^4\cdot 1806113^2 > 14\cdot 10^{24}$, we may assume that $\sigma(11^{10})^2 \| x$.

But $11 \nmid \sigma(\sigma(11^{10})^2)$ so $y \geq 11^{10}\cdot\sigma(\sigma(11^{10})^2) > 14\cdot 10^{24}$.

Now assume $v(11,y) = 12$ so that $1093\cdot 3158528101 = \sigma(11^{12})\,|\,x$. Now $1093^4\cdot 315828101^2 > 14\cdot 10^{24}$ so $1093^a\cdot 3158528101^b \| x$ where $\{a,b\} \subset \{1,2\}$ and $ab \neq 1$. Then $11 \nmid \sigma(1093^2\cdot 3158528101^b)$ so $y \geq 11^{12}\cdot\frac{1}{2}\sigma(1093^2\cdot 3158528101) > 14\cdot 10^{24}$.

If $v(11,y) = 16$, then $\sigma(11^{16})\,|\,x$. But $\sigma(11^{16})$ is prime. Now $\sigma(11^{16})^2 > 14\cdot 10^{24}$ so $\sigma(11^{16}) \| x$. Now $11 \nmid \sigma(\sigma(11^{16}))$ so $y \geq 11^{16}\cdot\frac{1}{2}\sigma(\sigma(11^{16})) > 14\cdot 10^{24}$.

If $v(11,y) = 18$, then $\sigma(11^{18})\,|\,x$. But $\sigma(11^{18})$ is prime. As before $\sigma(11^{18}) \| x$ and $11 \nmid \sigma(\sigma 11^{18}))$, so $y \geq 11^{18}\cdot\frac{1}{2}\sigma(\sigma(11^{18})) > 14\cdot 10^{24}$.

Finally, suppose $v(11,y) = 22$. Then $829 \| \sigma(11^{22})\,|\,x$. Now $829\cdot\sigma(11^{22}) > 14\cdot 10^{24}$ so $829 \| x$. But $11 \nmid \sigma(829)$, so $y \geq 11^{22}\cdot\frac{1}{2}\sigma(829) > 14\cdot 10^{24}$.

Proposition 3.3. If $7\,|\,1 + v(3,x)$, then $2 \nmid 1 + v(1093,y)$.

Proof. Assume $7\,|\,1 + v(3,x)$ and 1093 is the special prime in y. Then $547 = \frac{1}{2}\sigma(1093)\,|\,x$. Now $3\,|\,\sigma(547^2)$ so $3 \nmid 1 + v(547,x)$. Suppose $v(547,x) = 4$. Then $431\cdot 208097431 = \sigma(547^4)\,|\,y$. Now $1093\cdot\sigma(5744^4)^2 > 7\cdot 10^{24}$, so we have $\sigma(547^4)^2 \| y$. But $547 \nmid \sigma(\sigma(547^4)^2)$ so $x \geq 547^4\cdot\sigma(\sigma(547^4)^2) > 14\cdot 10^{24}$. Hence $v(547,x) \neq 4$.

If $v(547,x) = 6$, then $7\cdot 29\cdot 13219730563 5599 = \sigma(547^6)\,|\,y$. Then $y > \sigma(547^6)^2 > 14\cdot 10^{24}$. Finally, if $v(547,x) \geq 10$, then $x \geq 3^6\cdot 547^{10} > 14\cdot 10^{24}$.

Proposition 3.4. If $7 \mid 1 + v(3,x)$ then $3 \nmid 1 + v(1093,y)$.

Proof. Assume $7 \mid 1 + v(3,x)$ and $3 \mid 1 + v(1093,y)$. Then $398581 \mid \sigma(1093^2) \mid x$. Since $3^6 \cdot 398581^4 > 14 \cdot 10^{24}$ and $3 \mid \sigma(398581^2)$, we may assume $398581 \| x$. Then $17 \cdot 19 \cdot 617 = \frac{1}{2}\sigma(398581) \mid y$. If $v(617,y) = 2$, then $97 \cdot 3931 = \sigma(617^2) \mid x$. Since these primes are $\equiv 1(3)$, their exponents are ≥ 4 and $x \geq 3^6 \cdot 97^4 \cdot 3931^4 \cdot 398581 > 14 \cdot 10^{24}$. If $v(617,y) = 4$, then $145159381141 = \sigma(617^4) \mid x$, so that $x \geq 3^6 \cdot \sigma(617^4)^2 \cdot 398581 > 14 \cdot 10^{24}$. Hence $v(617,y) \geq 6$. But then $y \geq 17^2 \cdot 19^2 \cdot 617^6 \cdot 1093^2 > 14 \cdot 10^{24}$.

Proposition 3.5. If $7 \mid 1 + v(3,x)$, then $v(1093,y) \neq 4$ or 6.

Proof. Assume $7 \mid 1 + v(3,x)$. If $v(1093,y) = 4$, then $11 \cdot 31 \cdot 4189129561 = \sigma(1093^4) \mid x$. Since $3^6 \cdot \sigma(1093^4)^2 > 14 \cdot 10^{24}$, we may assume $4189129561 \| x$. Then $17^2 \cdot 7247629 = \frac{1}{2}\sigma(4189129561) \mid y$, so that $y \geq 17^2 \cdot 1093^4 \cdot 7247629^2 > 14 \cdot 10^{24}$.

If $v(1093,y) = 6$, then $7 \cdot 29 \cdot 14939 \cdot 562731116179 = \sigma(1093^6) \mid x$. Then $x \geq 3^6 \cdot 7^2 \cdot 29 \cdot 14939^2 \cdot 562731116179^2 > 14 \cdot 10^{24}$.

§6. $3 \nmid n\sigma(n)$. We first prove

Theorem 4. If n is an odd super perfect number and $3 \mid \sigma(n)$, then $n > 7 \cdot 10^{24}$.

Proof. Suppose $3 \mid \sigma(n)$. Then theorems 2 and 3 imply that $v(3,\sigma(n)) \geq 10$. Lemma 2 allows us to write $n = p_1^{2b_1} \cdot p_2^{2b_2} \cdot \ldots \cdot p_k^{2b_k}$ where $p_1 < p_2 < \ldots < p_k$ are primes. Now lemma 1 implies that $v(3,\sigma(p_i^{2b_i})) = v(3,2b_i+1)$ or 0 depending on whether $p_i \equiv 1$ or $2(3)$. Since $3^{10} \mid \sigma(n)$, we have $n \geq 7^2 \cdot 13^2 \cdot 19^2 \cdot 31^2 \cdot 37^2 \cdot 43^2 \cdot 61^2 \cdot 67^2 \cdot 73^2 \cdot 79^2 > 7 \cdot 10^{24}$.

Theorem 5. If n is an odd super perfect number and $3 | n\sigma(n)$, then $n > 7\cdot 10^{24}$.

Assume $3 | n\sigma(n)$. From theorem 4 we may assume $3 | n$. Since $3^{50}\cdot 5^2 > 7\cdot 10^{24}$, lemma 2 implies we may assume $v(3,n) < 50$. Also from lemma 2 and theorems 2 and 3 we may assume $1 + v(3,n)$ is not divisible by 2, 3, 5 or 7. In propositions 5.1 to 5.5 we shall show there is no allowable value for $v(3,n)$ for which $n < 7\cdot 10^{24}$.

Proposition 5.1. $v(3,n) \nmid 10$.

Proof. If $v(3,n) = 10$ then $23\cdot 3851 = \sigma(3^{10}) | \sigma(n)$. Since $3851^8 > 14\cdot 10^{24}$ and $3851 \equiv 3(4)$ all we need show is that $v(3851,\sigma(n)) \nmid 2$, 4 or 6.

If $3851^2 \| \sigma(n)$ then $13\cdot 1141081 = \sigma(3851^2) | n$. Since these primes are $\equiv 1(3)$ and $3 \nmid \sigma(n)$ we have $n \geq 3^{10}\cdot \sigma(3851^2)^4 > 7\cdot 10^{24}$. If $3851^4 \| \sigma(n)$ then $5\cdot 2289401\cdot 19218301 = \sigma(3851^4) | n$, so $n \geq 3^{10}\cdot \sigma(3851^4)^2 > 7\cdot 10^{24}$. Finally if $3851^6 \| \sigma(n)$, then since $3 \nmid \sigma(3851^6)$, we have $n \geq 3^{10}\cdot \sigma(3851^6) > 7\cdot 10^{24}$.

Proposition 5.2. The special prime is $\equiv 17(36)$.

Proof. Suppose p is the special prime and $p^b \| \sigma(n)$. Then $p \geq 5$. Since $5^{53} > 14\cdot 10^{24}$ and $b \equiv 1(4)$ we have $3^3 \nmid b+1$. Then lemma 1 implies $v(3,\sigma(p^b)) \leq 2 + v(3,p+1)$. Now $v(3,n) = v(3,\sigma(\sigma(n))) \geq 12$. Since $7^2\cdot 13^2\cdot 19^2\cdot 31^2\cdot 37^2\cdot 43^2\cdot 61^2\cdot 67^2\cdot 73^2 > 14\cdot 10^{24}$, lemma 1 implies $v(3,\sigma(p^b)) \geq 4$. Then $v(3,p+1) \geq 2$, and since $p \equiv 1(4)$ we have $p \equiv 17(36)$.

Proposition 5.3. $v(3,n) \nmid 12$ or 16.

Proof. If $v(3,n) = 12$, then $797161 = \sigma(3^{12}) | \sigma(n)$. Since $797161 \not\equiv 17(36)$, it is not special. If $v(797161, \sigma(n)) = 2$, then $3 \cdot 61 \cdot 151 \cdot 22996651 = \sigma(797161^2) | n$. Then $n \geq 3^{12} \cdot \sigma(797161^2)^2 > 7 \cdot 10^{24}$. If $797161^4 \| \sigma(n)$, then $n \geq 3^{12} \cdot \sigma(797161^4) > 7 \cdot 10^{24}$. Finally, if $v(797161, \sigma(n)) \geq 6$, then $\sigma(n) > 14 \cdot 10^{24}$.

If $v(3,n) = 16$, then $1871 \cdot 34511 = \sigma(3^{16}) | \sigma(n)$. Neither prime is $\equiv 1(4)$, so neither is special. Since the special prime is at least 17, and $17 \cdot 1871^2 \cdot 34511^4 > 14 \cdot 10^{24}$, we have $34511^2 \| \sigma(n)$. Then $13 \cdot 19 \cdot 4822039 = \sigma(34511^2) | n$. Hence $n \geq 3^{16} \cdot \sigma(34511^2)^2 > 7 \cdot 10^{24}$.

Proposition 5.4. If $v(3,n) = 2a$, if no prime factor of $\sigma(3^{2a})$ is $\equiv 17(36)$, and if $\sigma(3^{2a})$ is the product of k distinct primes, then $2a \leq \frac{1}{3}(51+k)$.

Proof. Let $\sigma(3^{2a}) = p_1 \cdot p_2 \cdot \ldots \cdot p_k$. Then proposition 5.2 and the assumption that no $p_i \equiv 17(36)$ imply $p_1^{2b_1} \cdot p_2^{2b_2} \cdot \ldots \cdot p_k^{2b_k} \| \sigma(n)$ for some $b_1, b_2, \ldots, b_k$. Lemma 1 implies $v(3, \sigma(p_i^{2b_i})) \leq v(3, 2b_i+1)$ so that if m_i is that part of $\sigma(p_i^{2b_i})$ which is prime to 3, then $m_i \geq \sigma(p_i^{2b_i})/(2b_i+1) \geq \frac{1}{3}\sigma(p_i^2)$. Hence $n \geq 3^{2a} \cdot m_1 \cdot m_2 \cdot \ldots \cdot m_k > 3^{2a-k} \cdot p_1^2 \cdot p_2^2 \cdot \ldots \cdot p_k^2 = 3^{2a-k} \cdot \sigma(3^{2a})^2 > 3^{2a-k} (\frac{4}{3} \cdot 3^{2a})^2 = 16 \cdot 3^{6a-k-2}$. Then the assumption $n < 7 \cdot 10^{24}$ implies $6a - k - 2 + \log_3 16 < \log_3(7 \cdot 10^{24})$, so that $6a - k < 52$. Then $6a - k \leq 51$, that is $2a \leq \frac{1}{3}(51+k)$.

Proposition 5.5. $v(3,n) \nmid 18, 22, 28, 30, 36, 40, 42$ or 46.

Proof. We note the following prime factorizations:

$$\sigma(3^{18}) = 1597 \cdot 363889$$

$$\sigma(3^{22}) = 47 \cdot 1001523179$$

$$\sigma(3^{28}) = 59 \cdot 28537 \cdot 20381027$$

$$\sigma(3^{30}) = 683 \cdot 102673 \cdot 4404047$$

$$\sigma(3^{36}) = 13097927 \cdot 17189128703$$

$$\sigma(3^{40}) = 83 \cdot 2526913 \cdot 86950696619$$

$$\sigma(3^{42}) = 431 \cdot 380808546861411923$$

$$\sigma(3^{46}) = 1223 \cdot 21997 \cdot 5112661 \cdot 96656723$$

In each case proposition 5.4 is applicable, but also in each case $2a > \frac{1}{3}(51+k)$.

§7. $5 \nmid n\sigma(n)$. In this section we prove

Theorem 6. If n is an odd super perfect number, and $5 | n\sigma(n)$ the $n > 7 \cdot 10^{24}$.

Suppose $5 | n\sigma(n)$. Let p be the special prime. In the previous section we showed that $3 \nmid n\sigma(n)$. Hence $p \equiv 1(3)$. Also if q is a prime factor of n or $\sigma(n)$ and if $q \equiv 1(3)$, then the exponent on q is not $\equiv 2(3)$. We shall let $x = n$ or $\sigma(n)$ and y the other assuming $5 | x$. Note that we do not exclude $5 | y$.

Since $5^{36} > 14\cdot 10^{24}$, we may assume $v(5,x) < 36$. Also we have $v(5,x)$ even. In propositions 6.1 to 6.4 we show there is no allowable value for $v(5,x)$ for which $n < 7\cdot 10^{24}$.

Proposition 6.1. $3 \nmid 1 + v(5,x)$.

Proof. If $3 \mid 1 + v(5,x)$, then $31 = \sigma(5^2) \mid y$. Since $31 \equiv 1(3)$, we have $3 \nmid 1 + v(31,y)$. If $v(31,y) = 4$, then $5\cdot 11\cdot 17351 = \sigma(31^4) \mid x$. Since $17351 \equiv 3(4)$ it is not special. If $17351^2 \| x$, then $13\cdot 1063\cdot 21787 \mid y$. Since $1063 \equiv 21787 \equiv 1(3) \equiv 3(4)$, we have $y \geq 13\cdot 31^4\cdot 1063^4\cdot 21787^4 > 14\cdot 10^{24}$. If $17351^4 \| x$ then $5\cdot 11\cdot 164801204033679l = \sigma(17351^4) \mid y$, so that $y \geq \sigma(17351^4)^2 > 14\cdot 10^{24}$. If $17351^6 \mid x$, then $x \geq 5^2\cdot 17351^6 > 14\cdot 10^{24}$. Hence $v(31,y) \neq 4$.

If $v(31,y) = 6$, then $917087137 = \sigma(31^6) \mid x$. If 917087137 is special, then $11\cdot 1451\cdot 28729 = \frac{1}{2}\sigma(917087137) \mid y$, so that $y \geq 11^2\cdot 31^6\cdot 1451^2\cdot 28729^2 > 14\cdot 10^{24}$. Since $3 \mid \sigma(917087137^2)$ we have $917087137^4 \mid x$ which gives $x > 14\cdot 10^{24}$. Hence $v(31,y) \neq 6$.

If $v(31,y) = 10$, then $23\cdot 397\cdot 617\cdot 150332843 = \sigma(31^{10}) \mid x$ and $x \geq 5^2\cdot 23^2\cdot 397^2\cdot 617\cdot 150332843^2 > 14\cdot 10^{24}$. If $v(31,y) = 12$, then $42407\cdot 2426789\cdot 7908811 = \sigma(31^{12}) \mid x$ and $x \geq 5^2\cdot 42407^2\cdot 2426789\cdot 7908811^2 > 14\cdot 10^{24}$. If $v(31,y) = 16$, then $5 \nmid \sigma(31^{16}) \mid x$, so that $x \geq 5^2\cdot \sigma(31^{16}) > 14\cdot 10^{24}$. Hence $v(31,y) \geq 18$, so that $y \geq 31^{18} > 14\cdot 10^{24}$.

Proposition 6.2. $5 \nmid 1 + v(5,x)$.

Proof. If $5 \mid 1 + v(5,x)$ then $11\cdot 71 = \sigma(5^4) \mid y$. Suppose

$3 \mid 1 + v(71,y)$. Then $5113 = \sigma(71^2) \mid x$. If 5113 is special, then $2557 = \frac{1}{2}\sigma(5113) \mid y$. Now $3 \mid \sigma(2557^2)$ and if $2557^4 \| y$, then $11 \cdot 12011 \cdot 323683781 = \sigma(2557^4) \mid x$. But then $x \geq 5^4 \cdot 5113 \cdot \sigma(2557^4)^2 > 14 \cdot 10^{24}$. Also $11^2 \cdot 71^2 \cdot 2557^6 > 14 \cdot 10^{24}$. Hence 5113 is not special. Since $5113 \equiv 1(3)$, we have $v(5113,x) \nmid 2$. If $5113^4 \| x$, then $11 \cdot 4751 \cdot 13080080081 = \sigma(5113^4) \mid y$. These primes are $\equiv 2(3)$, so they are non-special, and $y \geq \sigma(5113^4)^2 > 14 \cdot 10^{24}$. Hence $v(5113,x) \geq 6$. Then $x \geq 5^4 \cdot 5113^6 > 7 \cdot 10^{24}$ (cf. lemma 4). Hence $3 \nmid 1 + v(71,y)$.

If $71^4 \| y$ then $5 \cdot 11 \cdot 211 \cdot 2221 = \sigma(71^4) \mid x$. Now $211 \equiv 1(3) \equiv 3(4)$ so 211 is non-special and $3 \nmid 1 + v(211,x)$. If $211^4 \| x$, then $5 \cdot 1361 \cdot 292661 = \sigma(211^4) \mid y$. These primes are all $\equiv 2(3)$ so none is special. Then $y \geq 11^2 \cdot 71^4 \cdot \sigma(211^4)^2 > 14 \cdot 10^{24}$. If $211^6 \| x$ then $7 \| \sigma(211^6) \mid y$ so that $y \geq 11^2 \cdot 71^4 \cdot 7^3 \cdot \sigma(211^6) > 14 \cdot 10^{24}$, since $v(7,y) \nmid 2$. Hence $211^{10} \mid x$, so that $x \geq 5^4 \cdot 11^2 \cdot 211^{10} \cdot 2221 > 14 \cdot 10^{24}$.

If $71^6 \| y$, then $7 \cdot 883 \cdot 21020917 = \sigma(71^6) \mid x$. But $7 \equiv 883 \equiv 3(4) \equiv 1(3)$ so that $x \geq 5^4 \cdot 7^4 \cdot 883^4 \cdot 21020917 > 14 \cdot 10^{24}$. If $71^{10} \| y$, then $23q = \sigma(71^{10}) \mid x$ and q is a prime $\equiv 3(4)$. Then $x \geq 5^4 \cdot 23^2 \cdot q^2 > 14 \cdot 10^{24}$. If $71^{12} \| y$, then since $11^4 \cdot 71^{12} > 7 \cdot 10^{24}$, we have $11^2 \| y$. But $5 \nmid \sigma(11^2 \cdot 71^{12})$, so that $x \geq 5^4 \cdot \sigma(11^2 \cdot 71^{12}) > 14 \cdot 10^{24}$. Hence $71^{16} \mid y$. But then $y \geq 11^2 \cdot 71^{16} > 14 \cdot 10^{24}$.

Proposition 6.3. $x = n$ and the special prime is $\equiv 49(100)$.

Proof. Propositions 6.1 and 6.2 imply that $v(5,x) \geq 6$, so $5^6 | \sigma(y)$. Let y' be the product of the non-special primes in y with correct exponents, so that if $y = n$, then $y' = y$. If $5^5 | \sigma(y')$, then lemma 1 implies $y' \geq 11^4 \cdot 31^4 \cdot 41^4 \cdot 61^4 \cdot 71^4 > 14 \cdot 10^{24}$. Hence $y = \sigma(n)$, $x = n$ and if p^b is the special prime power in y, then $5^2 | \sigma(p^b)$. Lemma 1 and the above estimate show that $p \not\equiv 1(5)$, so that $p \equiv 4(5)$. Suppose $p \not\equiv 49(100)$. Then $5 \| p+1$. Now $p \geq 109$, and since $109^{49} > 14 \cdot 10^{24}$, we have $v(5,b+1) \leq 1$. But $5^2 | \sigma(p^b)$, so $v(5,b+1) = 1$ and $5^2 \| \sigma(p^b)$. Then $y \geq 109^9 \cdot 11^4 \cdot 31^4 \cdot 41^4 \cdot 61^4 > 14 \cdot 10^{24}$. Hence $p \equiv 49(100)$.

Proposition 6.4. $v(5,n) \not\equiv 6, 10, 12, 16, 18, 22, 28$, or 30.

Proof. If $v(5,n) = 6$, then $19531 = \sigma(5^6) | \sigma(n)$. Since $19531 \equiv 1(3) \equiv 3(4)$, we have $v(19531, \sigma(n)) \geq 4$ and even. If $v(19531, \sigma(n)) = 4$, then $5 \cdot 191 \cdot 4760281 \cdot 32009891 = \sigma(19531^4) | n$. Then $n \geq 5^4 \cdot \sigma(19531^4)^2 > 7 \cdot 10^{24}$. Hence $19531^6 | \sigma(n)$, so $\sigma(n) \geq 14 \cdot 10^{24}$. Thus $v(5,n) \not\equiv 6$.

The primes listed in the following factorizations are all $\equiv 1(3)$ and $\not\equiv 49(100)$: $\sigma(5^{10}) = 12207031$, $\sigma(5^{12}) = 305175781$, $\sigma(5^{16}) = 409 \cdot 466344409$. Hence if $v(5,n) = 10$, 12 or 16 then $\sigma(5^{v(5,n)})^4 | \sigma(n)$, so that $\sigma(n) \geq \sigma(5^{10})^4 > 14 \cdot 10^{24}$.

The primes involved in the following factorizations on all $\not\equiv 49(100)$: $\sigma(5^{18}) = 191 \cdot 6271 \cdot 3981071$, $\sigma(5^{22}) = 8971 \cdot 332207361361$, $\sigma(5^{28}) = 59 \cdot 35671 \cdot 22125996444329$. Hence if $v(5,n) = 18$, 22, or 28, then $\sigma(5^{v(5,n)})^2 | \sigma(n)$, so that $\sigma(n) \geq \sigma(5^{18})^2 > 14 \cdot 10^{24}$.

Finally if $v(5,n) = 30$ then $1861 \| \sigma(5^{30}) | \sigma(n)$. Now $1861 \equiv 1(3) \not\equiv 49(100)$ so $\sigma(n) \geq 1861^3 \cdot \sigma(5^{30}) > 14 \cdot 10^{24}$.

§8. Conclusion. In this section we conclude our proof that there are no odd super perfect numbers $< 7 \cdot 10^{24}$. If n is an odd super perfect number, then $2 = h(n)h(\sigma(n))$. Then either $h(n) > \sqrt{2}$ or $h(\sigma(n)) > \sqrt{2}$. Let x be the one of $n, \sigma(n)$ for which $h(x) > \sqrt{2}$, and let y be the other.

Theorem 7. If n is an odd super perfect number, $x = n$ or $\sigma(n)$ where $h(x) > \sqrt{2}$, and either $11 | x$ or $13 | x$, then $n > 7 \cdot 10^{24}$.

Before we prove theorem 7 in propositions 7.1 and 7.2, we will show how it is sufficient to prove our main result. Suppose n is an odd super perfect number and $n < 7 \cdot 10^{24}$. Then $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13, x) = 1$ using theorems 1, 4, 5, 6 and 7. Now $17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37^2 \cdot 41^2 \cdot 43^2 \cdot 47^2 \cdot 53^2 \cdot 59 > 14 \cdot 10^{24}$, so x has no more than 10 distinct prime factors. If $x = \prod p_i^{a_i}$ then $h(x) = \prod h(p_i^{a_i}) = \prod (p_i^{a_i+1} - 1)/(p_i - 1)p_i^{a_i} < \prod p_i/(p_i - 1) \leq \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \frac{29}{28} \cdot \frac{31}{30} \cdot \frac{37}{36} \cdot \frac{41}{40} \cdot \frac{43}{42} \cdot \frac{47}{46} \cdot \frac{53}{52} < \sqrt{2}$. This contradiction establishes our main result.

Proposition 7.1. $11 \nmid x$.

Proof. Suppose $11 | x$. Since $7 | \sigma(11^2)$ and $5 | \sigma(11^4)$, we have $3, 5 \nmid 1 + v(11,x)$. Then proposition 3.2 implies $v(11,x) \geq 28$, so $x > 14 \cdot 10^{24}$.

Proposition 7.2. $13 \nmid x$.

Proof. Suppose $13 | x$. Since $7 | \sigma(13)$ and $3 | \sigma(13^2)$ we have $2, 3 \nmid 1 + v(13,x)$. Suppose $v(13,x) = 4$. Then $30941 = \sigma(13^4) | y$. Now $3 | \sigma(30941)$ so 30941 is not special. If $30941^2 \| y$, then $157 \cdot 433 \cdot 14083 = \sigma(30941^2) | x$. Now these primes are all $\equiv 1(3)$ and $14083 \not\equiv 1(4)$ so $x \geq 13^4 \cdot 157^4 \cdot 433 \cdot 14083^4 > 14 \cdot 10^{24}$. Since $5 | \sigma(30941^4)$ we have $v(30941,y) \geq 6$. Then $y > 14 \cdot 10^{24}$.

Hence $v(13,x) \geq 6$. Now if $19 | x$, then $v(19,x) \geq 4$ since $3 | \sigma(19^2)$. But $13^6 \cdot 17^2 \cdot 19^4 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37^2 \cdot 41 > 13^6 \cdot 17^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37^2 \cdot 41^2 \cdot 43 > 14 \cdot 10^{24}$, so x is divisible by at most 7 distinct primes. Then $h(x) < \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \frac{29}{28} \cdot \frac{31}{30} \cdot \frac{37}{36} < \sqrt{2}$, a contradiction.

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Department of Mathematics, University of Georgia, Athens, Ga. 30602