SYMMETRIC PRIMES REVISITED

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In memory of Peter Fletcher (1939–2019)

Abstract. A pair of odd primes is said to be symmetric if they are 1 modulo their difference. A theorem from 1996 by Fletcher, Lindgren, and the current author got an upper bound for the distribution of primes that belong to a symmetric pair. In this paper that theorem is improved to what is likely to be a best possible result. In addition, we show that there are infinitely many pairs of symmetric primes.

1. Introduction

A pair of odd primes $p < q$ is said to be a symmetric pair if $\gcd(p-1, q-1) = q-p$. For example, if p, q are a twin prime pair, they are also a symmetric pair. We say a prime is *symmetric* if it is a member of some symmetric pair, and otherwise, we say it is *asymmetric*. In [3] it is shown that most primes are asymmetric. In particular, the number $S(x)$ of symmetric primes $p \le x$ is $O(\pi(x)/(\log x)^{0.027})$. We conjectured that the exponent 0.027 could be improved to $\eta + o(1)$, where

$$\eta := 1 - \frac{1 + \log\log 2}{\log 2} = 0.086\ldots.$$

In this note we prove the conjecture.

Theorem 1.1. *For all large x, we have*

$$S(x) \le \frac{\pi(x)}{(\log x)^{\eta}} (\log\log x)^{O(1)}.$$

We also are able to prove another conjecture from [3] that there are infinitely many symmetric primes.

Theorem 1.2. *We have*

$$S(x) \gg \frac{\pi(x)}{(\log x)^{49}}.$$

Of the two bounds, we believe that Theorem 1.1 is closer to the truth, and in fact, it seems likely that we have equality in Theorem 1.1.

The constant η appears in a number of problems. An early appearance is in the Erdős multiplication table problem, where after work of Erdős, Tenenbaum, and Ford, we now know that the number $M(N)$ of distinct entries in the $N \times N$ multiplication table is $N^2(\log N)^{-\eta}(\log\log N)^{O(1)}$. (In fact, Ford [4] showed that the "$O(1)$" is $-3/2$ with the resulting expression of the same magnitude as $M(N)$.) Another, more recent appearance of η is in the paper [2] where the odd legs in integer-sided right triangles with prime hypotenuse are considered. This note uses some of the techniques from [2].

Our proof of Theorem 1.2 uses an old result of Heath-Brown [7] plus the framework of the new results of Zhang, Maynard, Tao, et al. on small gaps between primes.

In Sections 2 and 3 we prove our theorems. In Section 4 we present some new computations of symmetric primes. In Section 5 we close with a few problems of a somewhat different nature.

2. The proof of Theorem 1.1

Let $\omega(n)$ denote the number of distinct primes that divide n and let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity.

Let $S_1(x)$ denote the number of primes $p \le x$ with all prime factors of $p-1$ at most $x^{1/\log\log x}$. Since the number of integers up to x with all prime factors at most $x^{1/\log\log x}$ is $O(x/(\log x)^2)$ (see de Bruijn [1, Eq. (1.6)]) it follows that $S_1(x) = O(\pi(x)/\log x)$. Thus, we may assume that we are counting symmetric primes $p \le x$ where there is a prime $r \mid p-1$ with $r > x^{1/\log\log x}$.

Let $S_2(x)$ denote the number of primes $p \le x$ with $\Omega(p-1) > L$, where $L = \lfloor (1/\log 2)\log\log x \rfloor$. We show that

$$S_2(x) \le \frac{\pi(x)}{(\log x)^{\eta}}(\log\log x)^{O(1)}. \tag{2.1}$$

Write $p = ar+1$, where r is the greatest prime factor of $p-1$. Since the count for $S_1(x)$ is negligible in comparison to (2.1), we may assume that $r > x^{1/\log\log x}$. For a given choice of $a < x^{1-1/\log\log x}$, the number of primes $r \le x/a$ with $ar+1$ prime, is by Brun's method (see [5, Eq. (6.1)]), at most

$$\frac{x}{a(\log x)^2}(\log\log x)^{O(1)}.$$

We sum this expression over a, assuming that $\Omega(a) \ge L$. For $L \le \Omega(a) \le 1.9\log\log x$, we use [6, Theorem 08], finding that $\sum 1/a \le (\log x)^{1-\eta}(\log\log x)^{O(1)}$, which is consistent with our goal (2.1). For larger values of $\Omega(a)$ we use [6, Exercise 05], getting $\sum 1/a \ll (\log x)^{0.69}$. This completes the proof of (2.1).

We write $p = ar+1$ where $r > x^{1/\log\log x}$ is prime, and $\Omega(a) < L$. Since p is symmetric, there is some $d \mid a$ with at least one of $p+d$, $p-d$, $p+dr$, $p-dr$ prime. Write $a = dm$. For a given pair d, m with $dm < x^{1-1/\log\log x}$, let $R(x,d,m)$ denote the number of primes $r \le x/dm$ with $dmr+1$ prime and at least one of $dmr+d+1$, $dmr-d+1$, $dmr+dr+1$, $dmr-dr+1$ prime. Again by Brun's method we have uniformly for x large that

$$R(x,d,m) \le \frac{x}{dm(\log x)^3}(\log\log x)^{O(1)}.$$

It remains to sum this expression over pairs d, m with $dm < x^{1-1/\log\log x}$ and $\Omega(dm) < L$. Let E denote the reciprocal sum of all primes and prime powers less than x. We have

$$\sum_{\substack{dm<x^{1-1/\log\log x}\\ \Omega(dm)<L}} \frac{1}{dm} \le \sum_{i+j<L} \sum_{\substack{d<x\\ \omega(d)=i}} \frac{1}{d} \sum_{\substack{m<x\\ \omega(m)=j}} \frac{1}{m}$$

$$\le \sum_{i+j<L} \frac{1}{i!}E^i \frac{1}{j!}E^j = \sum_{k<L} \frac{1}{k!}E^k \sum_{i+j=k} \frac{k!}{i!j!}$$

$$= \sum_{k<L} \frac{1}{k!}(2E)^k \ll \frac{1}{L!}(2E)^L,$$

since $E = \log\log x + O(1)$. A short calculation then shows that this expression is $(\log x)^{2-\eta}(\log\log x)^{O(1)}$. Thus, the sum of $R(x,d,m)$ over pairs d,m is at most $\pi(x)(\log x)^{-\eta}(\log\log x)^{O(1)}$, so completing the proof.

3. Infinitely many symmetric primes

In her dissertation, Spiro showed that the equation $d(n) = d(n+5040)$ has infinitely many solutions, where $d(n)$ is the divisor function. Heath-Brown [7] was able to replace 5040 with 1 in this theorem, and a key lemma (also see [8, 9]) was the following statement. *For every k there are distinct positive integers $a_1 < \cdots < a_k$ such that for each $1 \le i < j \le k$, we have $\gcd(a_i, a_j) = a_j - a_i$.* An example of such a set when $k = 4$ is $\{6, 8, 9, 12\}$.

If we have just 2 numbers $a < b$ with $\gcd(a, b) = b - a$, then any integer n for which $p = an+1$ and $q = bn+1$ are both prime produces the symmetric prime pair p, q. And so, the prime k-tuples conjecture would immediately give us infinitely many symmetric pairs. This is just a generalization of the thought that twin prime pairs are symmetric. These statements are still conjectural, but we do have the following new theorem. Say we have k distinct linear functions $a_it + b_i$, where a_i, b_i are integers, $a_i > 0$. They are *admissible* if for each prime p there is some integer t with none of $a_it + b_i$ divisible by p.

Theorem 3.1. *For each positive integer m there is some integer $k = \exp((O(m))$ such that among k admissible distinct linear functions $a_it + b_i$, there are at least m of them which simultaneously represent primes infintiely often. In fact the number of integers $t \le x$ at which the m functions are all prime is $\gg x/(\log x)^k$.*

This is essentially Maynard [10, Theorem 3.4]. After Polymath 8b (see [11, 12]), we now know that when $m = 2$ we may take $k = 50$. We apply this to the linear polynomials $a_1t + 1, \ldots, a_{50}t + 1$ with a special Heath-Brown set $\{a_1, \ldots, a_{50}\}$ as above. Note that these polynomials form an admissible set since their common value at $t = 0$ is 1. Say $a < b$ are in this set and the number of $n \le x$ with both $p = an + 1$, $q = bn + 1$ prime is $\gg x/(\log x)^{50}$. As noted above, p, q form a symmetric pair. Our Theorem 1.2 follows immediately.

4. Computations

In [3] some values of $S(x)$ for x up to the 10^5th prime were given. The data did not strongly suggest that $S(x) = o(\pi(x))$, in fact it seemed more plausible that $S(x)/\pi(x) \approx 0.83$. Using Mathematica we have extended the calculation to the 10^8th prime and we see that $S(x)/\pi(x)$ continues to be in no hurry to get to 0, but progress towards this limit is somewhat discernible. The descent to 0 does indeed appear to be not so different than the main term in our upper bound.

Table 1. Tabulation of $S(p_n)$, the number of symmetric primes to the nth prime.

n	$S(p_n)$	$S(p_n)/n$	$1/(\log p_n)^\eta$
10	9	0.9000	0.9008
10^2	86	0.8600	0.8536
10^3	864	0.8640	0.8279
10^4	8473	0.8473	0.8101
10^5	83263	0.8326	0.7964
10^6	819848	0.8198	0.7854
10^7	8098086	0.8098	0.7761
10^8	80112625	0.8011	0.7681

5. Graph problems

Consider a graph on the odd primes where two primes are connected by an edge if they form a symmetric pair. The asymmetric primes are isolated nodes. Must every connected component be finite? At the other extreme, removing the asymmetric primes, is the graph connected? If not, what is the least symmetric prime that is not in the component containing the prime 3? Does the graph have infinitely many components? Does it contain a complete graph K_m on m vertices for every m? The answer to this last question is "yes", since we can apply Theorem 3.1 and the argument of Section 3 to see this. Clearly there cannot exist an infinite complete subgraph since if $p < q$ are a symmetric pair, then $q < 2p$. Say a prime p is m-symmetric if it is in a K_m but not a K_{m+1}. It would be interesting to investigate the distribution of m-symmetric primes; the number of them to x is $x/(\log x)^{O_m(1)}$, but what can be said about the exponent here?

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