SYMMETRIC PRIMES REVISITED

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In memory of Peter Fletcher (1939–2019)

Abstract. A pair of odd primes is said to be symmetric if they are 1 modulo their difference. A theorem from 1996 by Fletcher, Lindgren, and the current author got an upper bound for the distribution of primes that belong to a symmetric pair. In this paper that theorem is improved to what is likely to be a best possible result.

1. Introduction

A pair of odd primes \( p < q \) is said to be a symmetric pair if \( \gcd(p-1, q-1) = q-p \). For example, if \( p, q \) are a twin prime pair, they are also a symmetric pair. We say a prime is symmetric if it is a member of some symmetric pair, and otherwise, we say it is asymmetric. In [3] it is shown that most primes are asymmetric. In particular, the number \( S(x) \) of symmetric primes \( p \leq x \) is \( O(\pi(x)/(\log x)^{0.027}) \). We conjectured that the exponent 0.027 could be improved to \( \eta + o(1) \), where

\[
\eta := 1 - \frac{1 + \log \log 2}{\log 2} = 0.086\ldots.
\]

In this note we prove the conjecture.

Theorem 1.1. For all large \( x \), we have

\[
S(x) \leq \frac{\pi(x)}{(\log x)^{\eta}}(\log \log x)^{O(1)}.
\]

It seems likely that we have equality in Theorem 1.1, but we cannot even show there are infinitely many symmetric primes.

The constant \( \eta \) appears in a number of problems. An early appearance is in the Erdős multiplication table problem, where after work of Erdős, Tenenbaum, and Ford, we now know that the number \( M(N) \) of distinct entries in the \( N \times N \) multiplication table is \( N^2(\log N)^{-\eta}(\log \log N)^{O(1)} \). (In fact, Ford [4] showed that the “\( O(1) \)” is \(-3/2\) with the resulting expression of the same magnitude as \( M(N) \).) Another, more recent appearance of \( \eta \) is in the paper [2] where the odd legs in integer-sided right triangles with prime hypotenuse are considered. This note uses some of the techniques from [2].

In Section 2 we prove our theorem. In Section 3 we present some new computations of symmetric primes. In Section 4 we close with a few problems of a somewhat different nature.
2. The proof of Theorem 1.1

Let \(\omega(n)\) denote the number of distinct prime factors of \(n\) and let \(\Omega(n)\) denote the number of prime factors of \(n\) counted with multiplicity.

Let \(S_1(x)\) denote the number of primes \(p \leq x\) with all prime factors of \(p - 1\) at most \(x^{1/\log \log x}\). Since the number of integers up to \(x\) with all prime factors at most \(x^{1/\log \log x}\) is \(O(x/(\log x)^2)\) (see de Bruijn [1, Eq. (1.6)]) it follows that \(S_1(x) = O(\pi(x)/\log x)\). Thus, we may assume that we are counting symmetric primes \(p \leq x\) where there is a prime \(r \mid p - 1\) with \(r > x^{1/\log \log x}\).

Let \(S_2(x)\) denote the number of primes \(p \leq x\) with \(\Omega(p - 1) > L\), where \(L = \lfloor(1/\log 2)\log \log x\rfloor\). We show that

\[
S_2(x) \leq \frac{\pi(x)}{(\log x)^\eta} (\log \log x)^{O(1)}.
\]

Write \(p = ar + 1\), where \(r\) is the greatest prime factor of \(p - 1\). Since the count for \(S_1(x)\) is negligible in comparison to \(S_2(x)\), we may assume that \(r > x^{1/\log \log x}\). For a given choice of \(a < x^{1/\log \log x}\), the number of primes \(r \leq x/a\) with \(ar + 1\) prime, is by Brun’s method (see [5, Eq. (6.1)]), at most

\[
\frac{x}{a(\log x)^2} (\log \log x)^{O(1)}.
\]

We sum this expression over \(a\), assuming that \(\Omega(a) \geq L\). For \(L \leq \Omega(a) \leq 1.9 \log \log x\), we use [6, Theorem 08], finding that \(\sum 1/a \leq (\log x)^{1-\eta} (\log \log x)^{O(1)}\), which is consistent with our goal (2.1). For larger values of \(\Omega(a)\) we use [6, Exercise 05], getting \(\sum 1/a \ll (\log x)^{0.69}\). This completes the proof of (2.1).

We write \(p = ar + 1\) where \(r > x^{1/\log \log x}\) is prime, and \(\Omega(a) < L\). Since \(p\) is symmetric, there is some \(d \mid a\) with at least one of \(p + d, p - d, p + dr, p - dr\) prime. Write \(a = dm\). For a given pair \(d, m\) with \(dm < x^{1-1/\log \log x}\), let \(R(x, d, m)\) denote the number of primes \(r \leq x/dm\) with \(dmr + 1\) prime and at least one of \(dmr + d + 1, dmr - d + 1, dmr + dr + 1, dmr - dr + 1\) prime. Again by Brun’s method we have uniformly for \(x\) large that

\[
R(x, d, m) \leq \frac{x}{dm(\log x)^{2}} (\log \log x)^{O(1)}.
\]

It remains to sum this expression over pairs \(d, m\) with \(dm < x^{1-1/\log \log x}\) and \(\Omega(dm) < L\). Let \(E\) denote the reciprocal sum of all primes and prime powers less than \(x\). We have

\[
\sum_{dm < x^{1-1/\log \log x}} \frac{1}{dm} \leq \sum_{i+j < L} \sum_{d \leq x} \frac{1}{d} \sum_{m \leq x} \frac{1}{m} \sum_{\omega(d) = i, \omega(m) = j} \frac{1}{i!}
\]

\[
= \sum_{i+j < L} \frac{1}{i!} E^i \frac{1}{j!} E^j = \sum_{k < L} \frac{1}{k!} E^k \sum_{i+j = k} \frac{k!}{i!j!}
\]

\[
= \sum_{k < L} \frac{1}{k!} (2E)^k \ll \frac{1}{L!} (2E)^L,
\]

since \(E = \log \log x + O(1)\). A short calculation then shows that this expression is \((\log x)^{2-\eta} (\log \log x)^{O(1)}\). Thus, the sum of \(R(x, d, m)\) over pairs \(d, m\) is at most \(\pi(x)(\log x)^{-\eta} (\log \log x)^{O(1)}\), so completing the proof.
3. Computations

In [3] we gave some values of $S(x)$ for $x$ up to the $10^5$th prime. The data did not strongly suggest that $S(x) = o(\pi(x))$, in fact it seemed more plausible that $S(x)/\pi(x) \approx 0.83$. Using Mathematica we have extended the calculation to the $10^8$th prime and we see that $S(x)/\pi(x)$ continues not to be in a hurry to get to 0, but progress towards this limit is somewhat discernible. The descent to 0 does indeed appear to be not so different than the main term in our bound.

Table 1. Tabulation of $S(p_n)$, the number of symmetric primes to the $n$th prime.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S(p_n)$</th>
<th>$S(p_n)/n$</th>
<th>$1/(\log p_n)^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^1$</td>
<td>9</td>
<td>0.9000</td>
<td>0.9008</td>
</tr>
<tr>
<td>$10^2$</td>
<td>86</td>
<td>0.8600</td>
<td>0.8536</td>
</tr>
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<td>$10^3$</td>
<td>864</td>
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<td>0.8279</td>
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<td>8473</td>
<td>0.8473</td>
<td>0.8101</td>
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<td>0.7964</td>
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<td>0.7854</td>
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<td>$10^7$</td>
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<td>0.8098</td>
<td>0.7761</td>
</tr>
<tr>
<td>$10^8$</td>
<td>80112625</td>
<td>0.8011</td>
<td>0.7681</td>
</tr>
</tbody>
</table>

4. Graph problems

Consider a graph on the odd primes where two primes are connected by an edge if they form a symmetric pair. The asymmetric primes are isolated nodes. Must every connected component be finite? At the other extreme, removing the asymmetric primes, is the graph connected? Does it have infinitely many components? Does it contain a complete graph on $k$ vertices for every $k$? Probably some of these questions can be answered on assumption of the prime $k$-tuples conjecture.

References


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