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Corrigendum: 'On the average number of divisors of the Euler function'

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Sungjin Kim has brought to our attention that the proof of Lemma 3 in [2] begins with an incorrect identity. Though the stated lemma is undoubtedly correct, the proof seems elusive. The problem can be fixed by replacing that lemma with the following.

Lemma 1. *Fix real numbers* λ, C *with* $0 < \lambda < \frac{1}{10}$ *and* C *large. For* $R \le x^{\lambda}$, $2 \le M \le (\log x)^C$, *we have*

$$\sum_{R/2<r\le R} \left| \sum_{q\le x/rM} \left(\psi(x;qr,1) - \frac{x}{\varphi(qr)} \right) \right| \ll_{C,\lambda} \frac{x\log M}{M}.$$

Proof. This follows from the special case for the residue class 1 of a result of Fiorilli [1, Theorem 2.1]. In particular, under the same hypotheses, Fiorilli's theorem asserts that

$$\sum_{R/2<r\le R} \left| \sum_{q\le x/rM} \left(\psi(x;qr,1) - \frac{x}{\varphi(qr)} \right) - \frac{x}{rM}\mu(r,M) \right| \ll_{C,\epsilon,\lambda} \frac{x}{M^{743/538-\epsilon}},$$

where

$$-2\mu(r,M) = \log M + \log 2\pi + 1 + \gamma + \sum_{p} \frac{\log p}{p(p-1)} + \sum_{p\mid r} \frac{\log p}{p}.$$

Since

$$\sum_{R/2<r\le R} \frac{x}{rM} |\mu(r,M)| \ll \frac{x\log M}{M},$$

the lemma follows. $\square$

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In our paper, Lemma 3 is used in the proof of Lemma 5. Let

$$z = O(\log x/(\log\log x)^4), \quad y \in (e^{z(\log z)^2}, x],$$

let $\mathcal{D}_z(y)$ denote the set of integers $d \le y$ free of prime factors in $[1, z]$, and let $\tau_z(n)$ denote the number of divisors of n in $\mathcal{D}_z(n)$. The goal is to prove that

$$R_z(y) := \sum_{p \le y} \tau_z(p-1)$$

is equal to $c_1 y/\log z + O(y/(\log z)^2)$, where $c_1 = e^{-\gamma}$, namely (21) in [2]. It suffices to show instead that

$$R'_z(y) := \sum_{n \le y} \tau_z(n-1)\Lambda(n)$$

is equal to $c_1 y \log y/\log z + O(y \log y/(\log z)^2)$, since then (21) follows by partial summation. Indeed, $R'_z(y) = \sum_{p \le y} \tau_z(p-1)\log p + O(y^{1/2+\epsilon})$, and

$$R_z(y) = \frac{1}{\log y}\sum_{p \le y} \tau_z(p-1)\log p + \int_2^y \frac{1}{t(\log t)^2}\sum_{p \le t} \tau_z(p-1)\log p \, dt.$$

Using the trivial estimate that the sum over $p \le t$ is $O(t)$, we have

$$R_z(y) = \frac{1}{\log y} R'_z(y) + O\left(\frac{y}{\log y}\right).$$

Let P_z denote the product of the primes to z, and let $M = (\log y)^3$, so that by inclusion-exclusion,

$$\begin{aligned} R'_z(y) &= \sum_{d \in \mathcal{D}_z(y)} \psi(y; d, 1) = \sum_{r \mid P_z} \mu(r) \sum_{q \le y/r} \psi(y; qr, 1) \\ &= \sum_{r \mid P_z} \mu(r) \sum_{q \le y/rM} \psi(y; qr, 1) + \sum_{r \mid P_z} \mu(r) \sum_{y/rM < q \le y/r} \psi(y; qr, 1) \\ &= R'_1 + R'_2, \text{ say.} \end{aligned}$$

By the Brun–Titchmarsh inequality we see that

$$|R'_2| \ll \sum_{a \le M} \sum_{r \mid P_z} \pi(y; ar, 1) \log y \ll y \sum_{a \le M} \sum_{r \mid P_z} \frac{1}{\varphi(a)\varphi(r)} \ll y \log M \log z,$$

which is negligible. Let $R \le P_z$. We have by Lemma 1 that

$$\sum_{\substack{r|P_z\\ R/2<r\le R}} \mu(r) \sum_{q\le y/Mr} \psi(y;qr,1) = \sum_{\substack{r|P_z\\ R/2<r\le R}} \mu(r) \sum_{q\le y/Mr} \frac{y}{\varphi(qr)} + O\left(\frac{y\log M}{M}\right).$$

So, by summing dyadically, we have

$$R_1' = \sum_{r|P_z} \mu(r) \sum_{q\le y/Mr} \frac{y}{\varphi(qr)} + O\left(\frac{yz\log M}{M}\right).$$

The argument in [2] now suffices to obtain (21).

Lemma 3 in [2] was also used to prove part (ii) of Lemma 6 there. In that result we have an integer $u \le (\log x)^{O(1)}$ free of prime factors in $[1,z]$, where now $z = (\log x)^{1/2}/(\log\log x)^6$. The result gets an asymptotic for

$$R_{u,z}(y) := \sum_{\substack{p\le y\\ p\,\equiv\,1 \pmod u}} \tau_z(p-1),$$

where $z < \log y/(\log\log y)^4$. We deal instead with

$$R'_{u,z}(y) := \sum_{\substack{n\le y\\ n\,\equiv\,1 \pmod u}} \tau_z(n-1)\Lambda(n).$$

As with $R_z'(y)$ above, we have

$$\begin{aligned} R'_{u,z}(y) &= \sum_{d\in\mathcal{D}_z(y)} \psi(y,[u,d],1) = \sum_{r|P_z} \mu(r) \sum_{q\le y/r} \psi(y,[u,qr],1) \\ &= \sum_{r|P_z} \mu(r) \sum_{v|u} \sum_{\substack{q\le y/vr\\ (q,u)=1}} \psi(y;uqr,1) \\ &= \sum_{r|P_z} \mu(r) \sum_{v,s|u} \mu(s) \sum_{q\le y/vrs} \psi(y;usqr,1). \end{aligned}$$

The contribution when $y/Mvrs < q \le y$ is negligible (where M is as before), and since u has only $O(1)$ divisors, we can use Lemma 1 to show that replacing $\psi(y;usqr,1)$ with $y/\varphi(usqr)$ creates a negligible error. The rest of the argument is then routine.

We remark that Fiorilli [1] gives an application of his theorem to several Titchmarsh-divisor sums similar to $R_z'(y)$ and $R'_{u,z}(y)$.

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References

[1] D. Fiorilli, ‘On a theorem of Bombieri, Friedlander, and Iwaniec’, *Canad. J. Math.* **64** (2012), 1019–1035.

[2] F. Luca and C. Pomerance, “On the average number of divisors of the Euler function”, *Publ. Math. Debrecen* **70** (2007), 125–148.

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