Corrigendum: ‘On the average number of divisors of the Euler function’

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Sungjin Kim has brought to our attention that the proof of Lemma 3 in [2] begins with an incorrect identity. Though the stated lemma is undoubtedly correct, the proof seems elusive. The problem can be fixed by replacing that lemma with the following.

**Lemma 1.** Fix real numbers $\lambda, C$ with $0 < \lambda < \frac{1}{10}$ and $C$ large. For $R \leq x^\lambda$, $2 \leq M \leq (\log x)^C$, we have

$$\sum_{R/2 < r \leq R} \left| \sum_{q \leq x/rM} \left( \psi(x; qr, 1) - \frac{x}{\varphi(qr)} \right) \right| \ll_{C, \lambda} \frac{x \log M}{M}.$$  

**Proof.** This follows from the special case for the residue class 1 of a result of Fiorilli [1, Theorem 2.1]. In particular, under the same hypotheses, Fiorilli’s theorem asserts that

$$\sum_{R/2 < r \leq R} \left| \sum_{q \leq x/rM} \left( \psi(x; qr, 1) - \frac{x}{\varphi(qr)} \right) - \frac{x}{rM} \mu(r, M) \right| \ll_{C, \epsilon, \lambda} \frac{x}{M^{743/538 - \epsilon}},$$

where

$$-2\mu(r, M) = \log M + \log 2\pi + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)} + \sum_{p|r} \frac{\log p}{p}.$$  

Since

$$\sum_{R/2 < r \leq R} \frac{x}{rM} |\mu(r, M)| \ll \frac{x \log M}{M},$$

the lemma follows. □

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In our paper, Lemma 3 is used in the proof of Lemma 5. Let
\[ z = O(\log x/(\log \log x)^4), \quad y \in (e^{z(\log z)^2}, x], \]
let \( D_z(y) \) denote the set of integers \( d \leq y \) free of prime factors in \([1, z]\), and let \( \tau_z(n) \) denote the number of divisors of \( n \) in \( D_z(n) \). The goal is to prove that
\[
R_z(y) := \sum_{p \leq y} \tau_z(p - 1)
\]
is equal to \( c_1 y / \log z + O(y/(\log z)^2) \), where \( c_1 = e^{-\gamma} \), namely (21) in [2]. It suffices to show instead that
\[
R'_z(y) := \sum_{n \leq y} \tau_z(n - 1) \Lambda(n)
\]
is equal to \( c_1 y \log y / \log z + O(y \log y/(\log z)^2) \), since then (21) follows by partial summation. Indeed,
\[
R_z(y) = \frac{1}{\log y} \sum_{p \leq y} \tau_z(p - 1) \log p + \int_2^y \frac{1}{t(\log t)^2} \sum_{p \leq t} \tau_z(p - 1) \log p \, dt.
\]
Using the trivial estimate that the sum over \( p \leq t \) is \( O(t) \), we have
\[
R_z(y) = \frac{1}{\log y} R'_z(y) + O \left( \frac{y}{\log y} \right).
\]
Let \( P_z \) denote the product of the primes to \( z \), and let \( M = (\log y)^3 \), so that by inclusion-exclusion,
\[
R'_z(y) = \sum_{d \in D_z(y)} \psi(y; d, 1) = \sum_{r | P_z} \mu(r) \sum_{q \leq y/r} \psi(y; qr, 1)
= \sum_{r | P_z} \mu(r) \sum_{q \leq y/r \leq M} \psi(y; qr, 1) + \sum_{r | P_z} \mu(r) \sum_{y/r < q \leq y/r \leq M} \psi(y; qr, 1)
= R'_1 + R'_2, \text{ say.}
\]
By the Brun–Titchmarsh inequality we see that
\[
|R'_2| \ll \sum_{a \leq M} \sum_{r | P_z} \pi(y; ar, 1) \log y \ll y \sum_{a \leq M} \sum_{r | P_z} \frac{1}{\varphi(a) \varphi(r)} \ll y \log M \log z,
\]
which is negligible. Let \( R \leq P_z \). We have by Lemma 1 that
\[
\sum_{r | P_z, R/2 < r \leq R} \mu(r) \sum_{q \leq y/Mr} \psi(y; qr, 1) = \sum_{r | P_z, R/2 < r \leq R} \mu(r) \sum_{q \leq y/Mr} \frac{y}{\varphi(qr)} + O \left( \frac{y \log M}{M} \right).
\]

So, by summing dyadically, we have
\[
R'_1 = \sum_{r | P_z} \mu(r) \sum_{q \leq y/\varphi(qr)} \frac{y}{\varphi(qr)} + O \left( \frac{yz \log M}{M} \right).
\]

The argument in [2] now suffices to obtain (21).

Lemma 3 in [2] was also used to prove part (ii) of Lemma 6 there. In that result we have an integer \( u \leq (\log x)^{O(1)} \) free of prime factors in \([1, z]\), where now \( z = (\log x)^{1/2}/(\log \log x)^6\). The result gets an asymptotic for
\[
R_{u, z}(y) := \sum_{p \leq y, p \equiv 1 \pmod{u}} \tau_z(p - 1),
\]

where \( z < \log y/(\log \log y)^4\). We deal instead with
\[
R'_{u, z}(y) := \sum_{n \leq y, n \equiv 1 \pmod{u}} \tau_z(n - 1)\Lambda(n).
\]

As with \( R'_z(y) \) above, we have
\[
R'_{u, z}(y) = \sum_{d | D_{u, z}(y)} \psi(y; \lfloor u, d \rfloor, 1) = \sum_{r | P_z} \mu(r) \sum_{\substack{q \leq y/qr \leq M, \varphi(qr) \geq y/Mr}} \psi(y; qr, 1)
\]
\[
= \sum_{r | P_z} \mu(r) \sum_{\substack{q \leq y/qr \leq M, \varphi(qr) \geq y/Mr}} \psi(y; qr, 1)
\]
\[
= \sum_{r | P_z} \mu(r) \sum_{\substack{q \leq y/qr \leq M, \varphi(qr) \geq y/Mr}} \psi(y; qr, 1).
\]

The contribution when \( y/Mr < q \leq y \) is negligible (where \( M \) is as before), and since \( u \) has only \( O(1) \) divisors, we can use Lemma 1 to show that replacing \( \psi(y; usqr, 1) \) with \( y/\varphi(usqr) \) creates a negligible error. The rest of the argument is then routine.

We remark that Fiorilli [1] gives an application of his theorem to several Titchmarsh-divisor sums similar to \( R'_z(y) \) and \( R'_{u, z}(y) \).

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References


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