What we still don’t know about addition and multiplication

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You would think that all of the issues surrounding addition and multiplication were sewed up in third grade!

Well in this talk we’ll learn about some things they didn’t tell you . . .
Here’s one thing they *did* tell you:

Find $483 \times 784$. 

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\times 784 \\
\hline
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3864 \\
3381 \\
\hline
378672
\end{array}
\]
If instead you had a problem with two 23-digit numbers, well you always suspected deep down that math teachers are cruel and sadistic. Just kidding!

In principle if you really have to, you could work out 23-digits times 23-digits on paper, provided the paper is big enough, but it’s a lot of work.

So here’s the real question: How much work?
Of course the amount of work depends not only on the length of the numbers. For example, multiplying $10^{22}$ by $10^{22}$, that’s 23-digits times 23-digits, but you can do it in your head.

In general, you’ll take each digit of the lower number, and multiply it painstakingly into the top number. It’s less work if some digit in the lower number is repeated, and there are definitely repeats, since there are only 10 possible digits. But even if it’s no work at all, you still have to write it down, and that’s 23 or 24 digits. At the minimum (assuming no zeroes), you have to write down $23^2 = 529$ digits for the “parallelogram” part of the product. And then comes the final addition, where all of those 529 digits need to be processed.
So in general if you multiply two \( n \)-digit numbers, it would seem that you’d be taking \( n^2 \) steps, unless there were a lot of zeroes. This ignores extra steps, like carrying and so on, but that at worst changes \( n^2 \) to maybe \( 2n^2 \) or \( 3n^2 \). We say that the “complexity” of “school multiplication” for two \( n \)-digit numbers is of order \( n^2 \).
A. A. Karatsuba (1937–2008): Devised a faster way to multiply two $n$-digit numbers taking about $n^{1.6}$ elementary steps.
Here is Karatsuba’s idea: use high school algebra!

Say the numbers $A$ and $B$ each have $n$ digits. Let $m = n/2$ (okay, we assume that $n$ is even). Write

$$A = A_110^m + A_0, \quad B = B_110^m + B_0,$$

where $A_1, A_0, B_1, B_0$ are all smaller than $10^m$, so have at most $m$ digits. Then our product $AB$ is

$$AB = (A_1B_1)10^{2m} + (A_1B_0 + A_0B_1)10^m + A_0B_0,$$

so our problem is broken down to 4 smaller multiplication problems, each of size $m \times m$, namely

$$A_1B_1, \quad A_1B_0, \quad A_0B_1, \quad A_0B_0,$$

and each of these would seem to take $1/4$ as much work as the original problem.
So, unfortunately 4 problems each taking 1/4 as much work, is no savings!

However, we also have

\[(A_1 + A_0)(B_1 + B_0) = A_1B_1 + (A_1B_0 + A_0B_1) + A_0B_0,\]

so we can really do it in 3 multiplications, not 4 (!). Namely,

\[A_1B_1, \quad A_0B_0, \quad (A_1 + A_0)(B_1 + B_0).\]

After we do these, we have our three coefficients, where the middle one, \(A_1B_0 + A_0B_1\), is the third product minus the first two:

\[A_1B_0 + A_0B_1 = (A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0.\]

This idea can then be used on each of the three smaller multiplication problems, and so on down the fractal road, ending in about \(n^{1.6}\) elementary steps.
Karatsuba’s method was later improved by Toom, Cook, Schönhage, & Strassen. After their efforts we have the Fast Fourier Transform that allows you to multiply in about $n \cdot \ln(n)$ steps. (So $\ln(n)$ is proportional to the number of digits of the number of digits of the numbers being multiplied!)

Small improvements were made by Fürer in 2007 and by De, Kurur, Saha, & Saptharishi in 2008.

We don’t know if we have reached the limit! In particular:

**What is the fastest way to multiply?**
Let’s play **Jeopardy Multiplication!**

Here are the rules: I give you the answer to the multiplication problem, and you give me the problem phrased as a question. You must use whole numbers larger than 1.

So, if I say “15”, you say “What is $3 \times 5$?”

OK, let’s play.
Let’s do 8051.
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Thinking, thinking . . . . Hmm,
Let’s do 8051.

Thinking, thinking . . . . Hmm,

\[ 8051 = 8100 - 49 = 90^2 - 7^2 = (90 - 7)(90 + 7) = 83 \times 97. \]

Got it!

What is \( 83 \times 97 \)?
So, here’s what we don’t know:

**How many steps does it take to figure out the factors if you are given an \( n \)-digit number which can be factored?**

(A trick problem would be: 17. The only way to write it as \( a \times b \) is to use 1, and that was ruled out. So, prime numbers cannot be factored, and the thing we don’t know is how long it takes to factor the non-primes.)

The best answer we have so far is about \( 10^{n^{1/3}} \) steps, and even this is not a theorem, but our algorithm (known as the **number field sieve**) seems to work in practice.

This is all crucially important for the security of Internet commerce. Or I should say that Internet commerce relies on the premise that we **cannot** factor much more quickly than that.
A couple of words about factoring, that is, on how to win at Multiplication Jeopardy.

The trick with 8051 (due to Fermat), namely that $8051 = 8100 - 49$, is sort of generalizable as might be illustrated by 1649.
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The trick with 8051 (due to Fermat), namely that $8051 = 8100 - 49$, is sort of generalizable as might be illustrated by 1649.

We look for a square just above 1649. The first is $41^2 = 1681$. Well

$$41^2 - 1649 = 32$$

and 32 is not a square.

Try again. The next square is $42^2 = 1764$ and

$$42^2 - 1649 = 115$$

and 115 is not a square.

Trying again, the next square is $43^2 = 1849$ and

$$43^2 - 1649 = 200$$

and 200 is not a square.

Note that we can make a square out of them:

\[ 32 \times 200 = 6400 = 80^2. \]

In general, if \( N \) is a positive integer, we’ll write \( x \equiv y \pmod{N} \) if \( x, y \) leave the same remainder when divided by \( N \). For example, \( 17 \equiv 37 \pmod{10} \) and \( 43 \equiv 98 \pmod{11} \). It’s really very handy notation!
Let \( N = 1649 \), the number we’re trying to factor. Then we have

\[
41^2 \equiv 32 \pmod{N}, \quad 43^2 \equiv 200 \pmod{N},
\]

and so

\[
(41 \times 43)^2 = 41^2 \times 43^2 \equiv 32 \times 200 = 80^2 \pmod{N}.
\]

Now \( 41 \times 43 \equiv 114 \pmod{N} \), so \( 114^2 \equiv 80^2 \pmod{N} \).

It is not true that \( N = (114 - 80)(114 + 80) \), but it is true that the greatest common divisor of \( 114 - 80 = 34 \) with \( N \) is 17. (And finding the greatest common divisor of two numbers is speedy.)

Hey! That proves that \( N = 1649 \) is divisible by 17. Dividing, the other factor is 97. So, we have it: What is \( 17 \times 97 \)?
The various elements here can actually be made into a speedy algorithm, the **quadratic sieve**. The **number field sieve** is a fancier version but has the same underlying flavor of assembling squares whose difference is divisible by $N$.

Despite our success with factoring, it still is very difficult. Hard numbers with 300 decimal digits are beyond our reach at present. The really amazing thing is we can apply our ignorance to make a secure cryptographic system!
Here are three famous unsolved problems involving both addition and multiplication:

**Goldbach’s conjecture:** Every even number starting with 4 is the sum of two primes.

**The twin prime conjecture:** There are infinitely many pairs of primes that differ by 2.

**The ABC conjecture:** If $a + b = c$ where no prime divides all 3, must the product of the primes dividing $abc$ exceed $c^{1-\varepsilon}$? (Assume $\varepsilon > 0$ is fixed and $c$ is large.)
Here’s another problem that I only recently heard about.

Find 2 numbers $a, b$ where $a + b = a \times b$: 
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Are there any other integral solutions? No.

How about 3 numbers with the same sum and product?

$1 + 2 + 3 = 1 \times 2 \times 3$.

Again, there are no other solutions, and the same goes for 4 numbers ($1, 1, 2, 4$ is the unique choice).
But if I ask for 5 numbers with equal sum and product, there are 3 solutions:

1, 1, 1, 2, 5   1, 1, 1, 3, 3   1, 1, 2, 2, 2
With some work one finds that with 444 numbers there is a unique solution.

Here’s what we don’t know:

**Is 444 the largest \( n \) such that there is a unique solution to \( n \) numbers with the same sum and product?**

This has been searched for up to \( 10^{13} \) and each \( n \) to this height has at least two solutions.

One can also ask a weaker question:

**Are there infinitely many \( n \) where there is a unique solution to the sum-equals-product problem with \( n \) integers.**
Here’s a famous unsolved problem involving only simple arithmetic:

For even $n$, let $f(n) = n/2$ and for odd $n$, let $f(n) = (3n + 1)/2$.

Consider the sequence $n, f(n), f(f(n)), \ldots$.
For example: $3 \rightarrow 5 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \ldots$
Or: $7 \rightarrow 11 \rightarrow 17 \rightarrow 26 \rightarrow 13 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow \ldots \rightarrow 1$

Is it true that starting with any positive integer $n$, the sequence $n, f(n), f(f(n)), \ldots$ eventually hits the number 1?
And here’s another famous problem (in disguised form):

Consider $\ln(A(N))$, where $A(N)$ is the least common multiple of $1, 2, \ldots, N$.

For example: $A(10) = 2520$ and $\ln(A(10)) \approx 7.8$.

Another example: $\ln(A(100,000,000)) \approx 99,998,242.8$.

**Do we always have** $|\ln(A(N)) - N| < \sqrt{N}(\ln(N))^2$?

The Clay Mathematics Institute offers $1,000,000$ for a proof!
Here’s an unsolved problem concerning just addition.

We all recall the addition table:

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The 10 × 10 array of sums has all the numbers from 2 to 20 for a total of 19 different sums.

If you were to try this for the $N \times N$ addition table we’d see all of the numbers from 2 to $2N$ for a total of $2N – 1$ different sums.

Now, what if we were to be perverse and instead of having the numbers from 1 to $N$, we had some arbitrary list of $N$ different numbers.

Can you arrange it so there are fewer than $2N – 1$ different sums?
The $10 \times 10$ array of sums has all the numbers from $2$ to $20$ for a total of $19$ different sums.

If you were to try this for the $N \times N$ addition table we’d see all of the numbers from $2$ to $2N$ for a total of $2N - 1$ different sums.

Now, what if we were to be perverse and instead of having the numbers from $1$ to $N$, we had some arbitrary list of $N$ different numbers.

Can you arrange it so there are fewer than $2N - 1$ different sums?

If you answered “No, there are always at least $2N - 1$ different sums,” you’d be right.
Here’s an example where there are many different sums:

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So, sometimes there are few distinct sums and sometimes many.

**What structure is forced on the set if there are few distinct sums?**

We know the answer when there are very few distinct sums:
Here’s something with multiplication tables.

Let’s look at the $N \times N$ multiplication table using the numbers from 1 to $N$. With addition, we were able to count exactly how many distinct numbers appear in the table.

How many different numbers appear in the $N \times N$ multiplication table?
Let $M(N)$ be the number of distinct entries in the $N \times N$ multiplication table.

So $M(10) = 42$. 

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It is really amazing that though $M(N)$ is not far below $N^2$ looking “from a distance”, if we look “close up” we see that $M(N)/N^2$ tends to 0 as $N$ grows larger and larger.

It may be too difficult to expect a neat exact formula for $M(N)$.

After Erdős, Tenenbaum, and Ford, we now know the (complicated) order of magnitude for $M(N)$ as $N$ grows. (It’s something like $N^2/(\ln(N))^E(\ln(\ln(N)))^{1.5}$, where $E = 0.086…$ is an explicitly known constant.)
Find an asymptotic formula for $M(N)$ as $N$ grows?
Let me close with one unified problem about addition and multiplication tables. It’s due to Erdős & Szemerédi.

Look at both the addition and multiplication tables for $N$ carefully chosen numbers.

We’ve seen that if we take the first $N$ numbers we get close to $N^2$ distinct entries in the multiplication table, but few in the addition table.

At the other extreme, if we take for our $N$ numbers the powers of 2, namely $1, 2, 4, \ldots, 2^{N-1}$, then there are at least $\frac{1}{2}N^2$ distinct entries in the addition table and only $2N - 1$ entries in the multiplication table.
If we take $N$ random numbers, then it’s likely both tables have close to $N^2$ distinct entries.

The question is: **If we choose our numbers so that the number of distinct entries in one table is small, must the other always be large?** (More precisely, if $\epsilon > 0$ is fixed and $N$ is sufficiently large, must every choice of $N$ numbers have the number of distinct entries in the addition and multiplication tables be $> N^{2-\epsilon}$?)
The game players with the sum/product problem include: Erdős, Szemerédi, Nathanson, Chen, Elekes, Bourgain, Chang, Konyagin, Rudnev, Shkredov, Green, Tao, Solymosi, ... 

The best that’s been proved (Solymosi) is that one table must have at least $N^{4/3}$ different entries. (Improved last year by Konyagin & Rudnev to $N^{4/3+5/9813}$.)

This list of mathematicians contains two Fields Medalists, a Wolf Prize winner, an Abel Prize winner, four Salem Prize Winners, two Crafoord Prize winners, and an Aisenstadt Prize winner.

And still the problem is not solved!
My message: We could use a little help with these problems!!
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THANK YOU