Aliquot Sequences

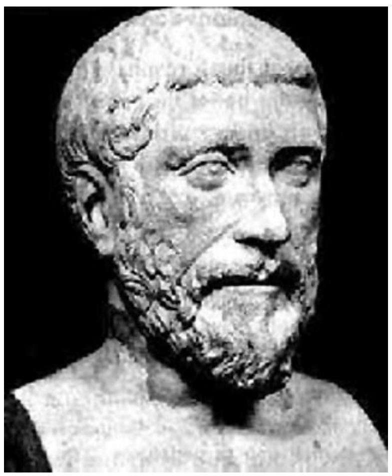
Carl Pomerance, Dartmouth College, emeritus

The Unsolved Problems Conference:

Celebrating the living legacy of the mathematics of Richard Guy

University of Calgary

October 2, 2020



As we all know, functions in mathematics are ubiquitous and indispensable.

But what was the very first function mathematicians studied?

I would submit as a candidate, the function $s(n)$ of **Pythagoras**.

Sum of proper divisors

Let $s(n)$ be the sum of the *proper* divisors of n:

For example:

$$s(10) = 1 + 2 + 5 = 8,$$
$$s(11) = 1,$$
$$s(12) = 1 + 2 + 3 + 4 + 6 = 16.$$

(In modern notation: $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.)

Pythagoras noticed that $s(6) = 1 + 2 + 3 = 6$
If $s(n) = n$, we say n is *perfect*.

And he noticed that

$$s(220) = 284, \quad s(284) = 220.$$

If $s(n) = m$, $s(m) = n$, and $m \neq n$, we say n, m are an *amicable pair* and that they are *amicable* numbers.

So 220 and 284 are amicable numbers.

An aliquot sequence continues to iterate the function s:

$1 \to 0$

any prime $p \to 1 \ldots$

$4 \to 3 \ldots$

$6 \to 6 \ldots$

$9 \to 4 \ldots$

$10 \to 8 \to 7 \ldots$

$12 \to 16 \to 15 \to 9 \ldots$

$14 \to 10 \ldots$

$18 \to 21 \to 1 \ldots$

$20 \to 22 \to 14 \ldots$

$24 \to 36 \to 55 \to 17 \ldots$

$25 \to 6 \ldots$

$30 \to 42 \to 54 \to 66 \to 78 \to 90 \to 144 \to 259 \to 45 \to 33 \to 15 \to 9 \ldots$

$220 \to 284 \to 220 \ldots$

We see that some aliquot sequences enter a fixed point or a longer cycle, and some terminate at 0.

The Catalan–Dickson conjecture: *This always happens. That is, every aliquot sequence is bounded.*

The Guy–Selfridge counter-conjecture: *Not so fast! Many aliquot sequences are unbounded.*

For starters up to 100 every aliquot sequence is bounded, the longest being the one starting at 30, having length 16 and maximum term 259.

This record is soon broken at 102, where the length is 19 and the maximum term is 759.

Here is the 120 iteration:

120, 240, 504, 1056, 1968, 3240, 7650, 14112, 32571, 27333, 12161, 1, 0

The length is just 13, but note that the maximum term is 32751.

This brings us to the 138 iteration on the next slide.

138, 150, 222, 234, 312, 528, 960, 2088, 3762, 5598, 6570, 10746, 13254, 13830, 19434, 20886, 21606, 25098, 26742, 26754, 40446, 63234, 77406, 110754, 171486, 253458, 295740, 647748, 1077612, 1467588, 1956812, 2109796, 1889486, 953914, 668966, 353578, 176792, 254128, 308832, 502104, 753216, 1240176, 2422288, 2697920, 3727264, 3655076, 2760844, 2100740, 2310856, 2455544, 3212776, 3751064, 3282196, 2723020, 3035684, 2299240, 2988440, 5297320, 8325080, 11222920, 15359480, 19199440, 28875608, 25266172, 19406148, 26552604, 40541052, 54202884, 72270540, 147793668, 228408732, 348957876, 508132204, 404465636, 303708376, 290504024, 312058216, 294959384, 290622016, 286081174, 151737434, 75868720, 108199856, 101437396, 76247552, 76099654, 42387146, 21679318, 12752594, 7278382, 3660794, 1855066, 927536, 932464, 1013592, 1546008, 2425752, 5084088, 8436192, 13709064, 20563656, 33082104, 57142536, 99483384, 245978376, 487384824, 745600776, 1118401224, 1677601896, 2538372504, 4119772776, 8030724504, 14097017496, 21148436904, 40381357656, 60572036544, 100039354704, 179931895322, 94685963278, 51399021218, 28358080762, 18046051430, 17396081338, 8698040672, 8426226964, 6319670230, 5422685354, 3217383766, 1739126474, 996366646, 636221402, 318217798, 195756362, 101900794, 54202694, 49799866, 24930374, 17971642, 11130830, 8904682, 4913018, 3126502, 1574810, 1473382, 736694, 541162, 312470, 249994, 127286, 69898, 34952, 34708, 26038, 13994, 7000, 11720, 14740, 19532, 16588, 18692, 14026, 7016, 6154, 3674, 2374, 1190, 1402, 704, 820, 944, 916, 694, 350, 394, 200, 265, 59, 1, 0

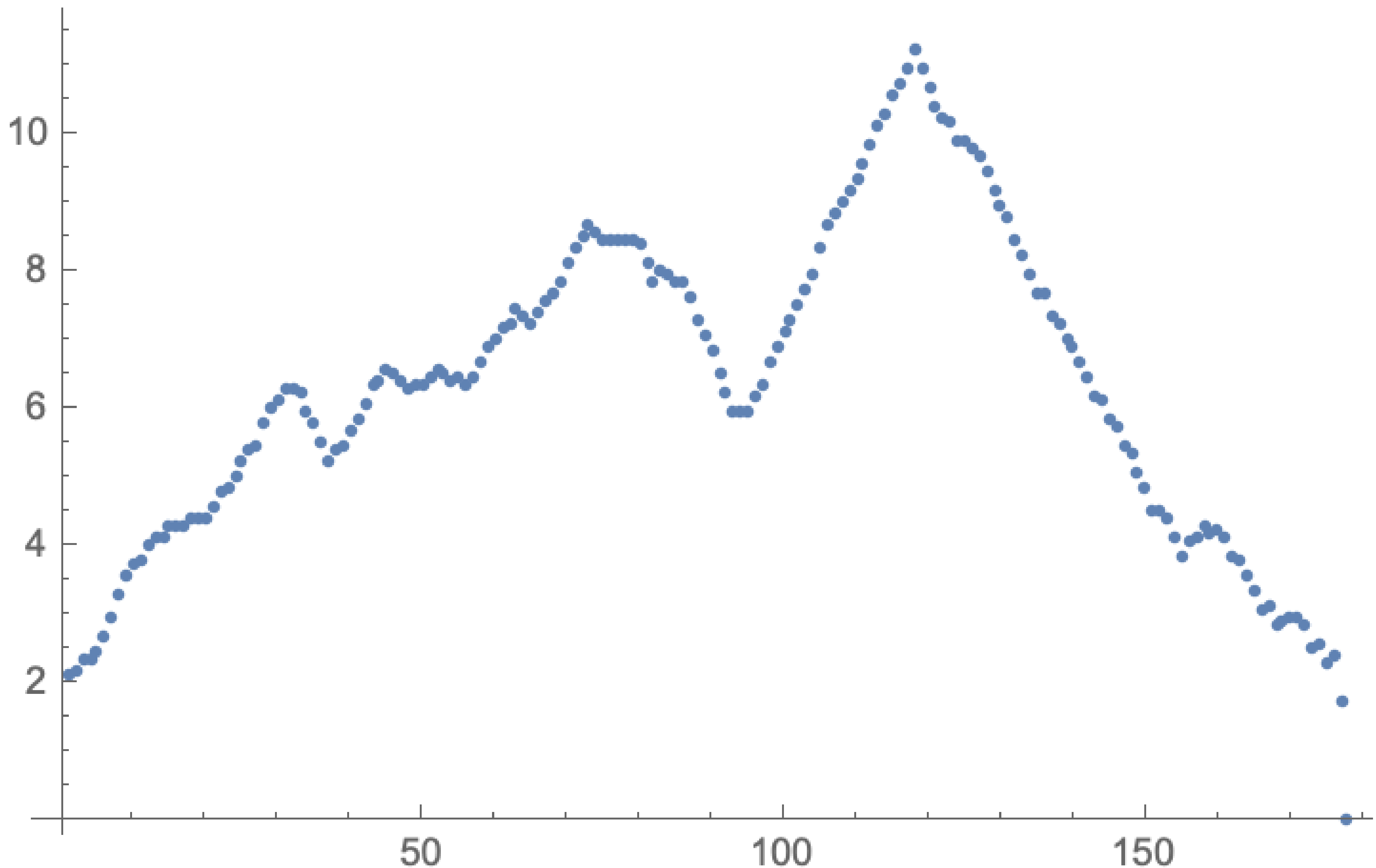
The 138 iteration has length 179 with maximum term 179,931,895,322.

We can look for patterns or clues.

One thing to note: The numbers are all even till we get to almost the end, when they turn odd for the final plunge. (Easy Fact: $s(n) \equiv n \pmod 2$ unless n is a square or twice a square.)

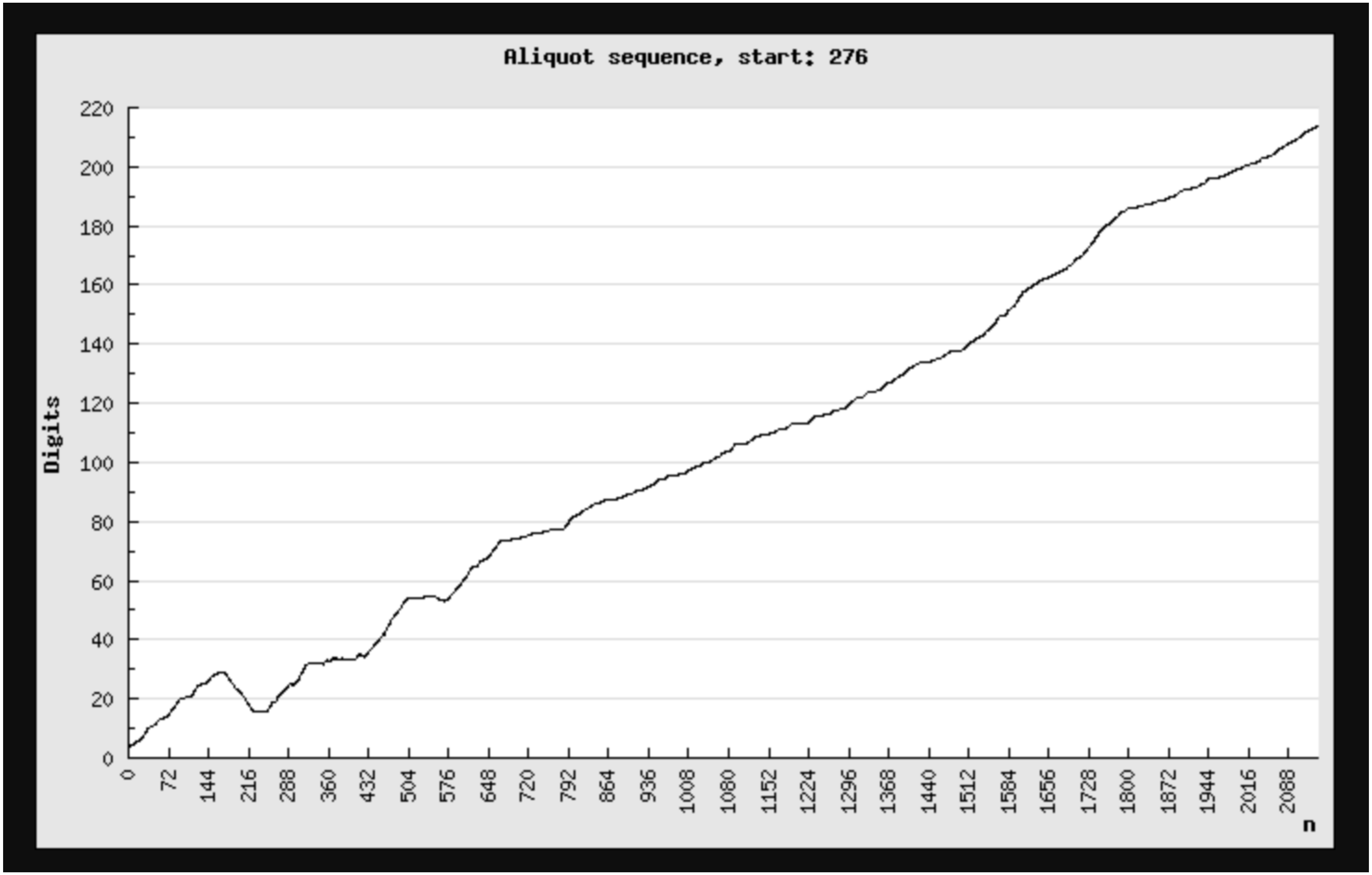
Also: The sequence tends to stay monotonic for long stretches.

Here's a plot (of the base-10 logs):


10
8
6
4
2
50
100
150

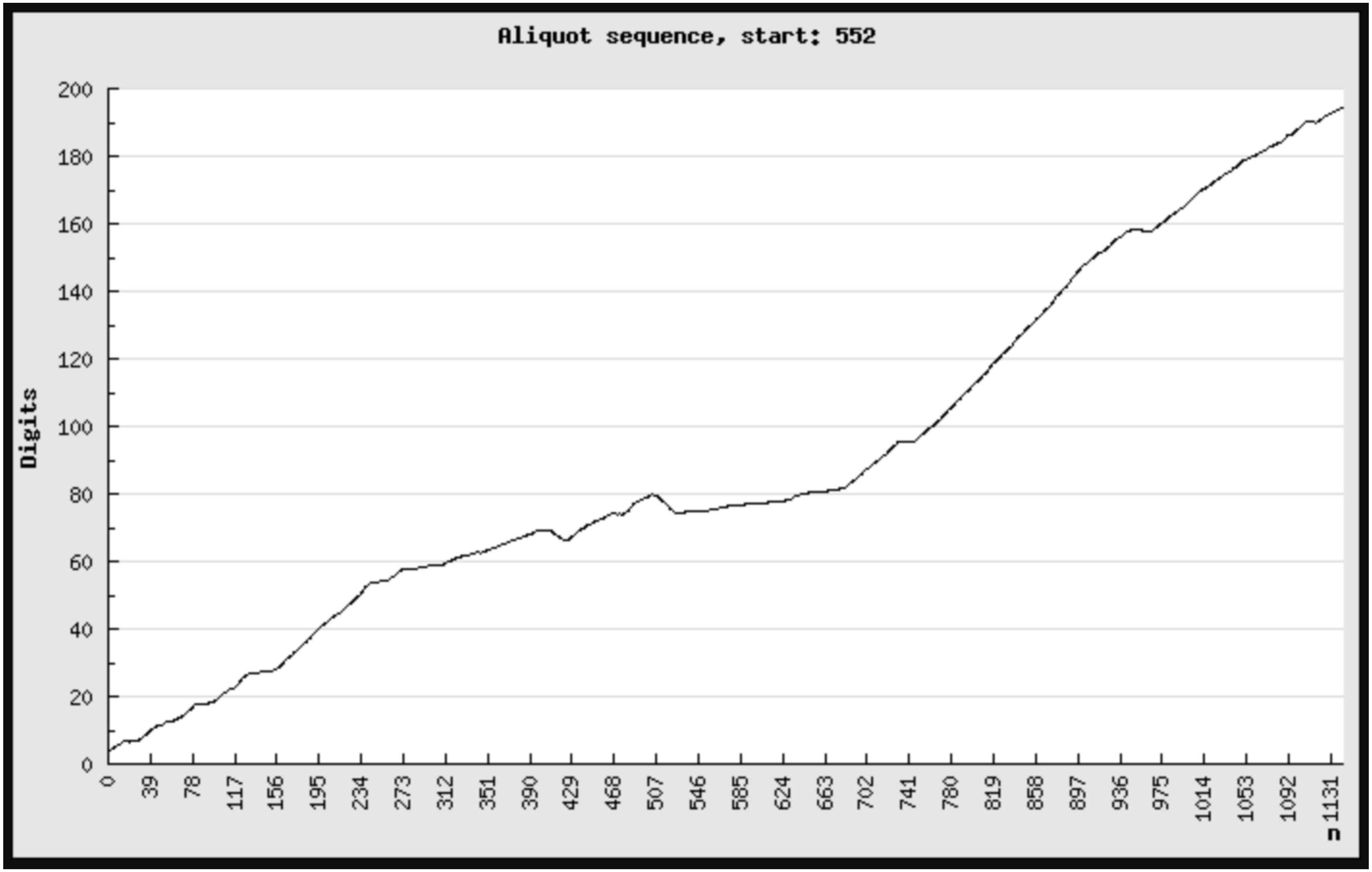
_

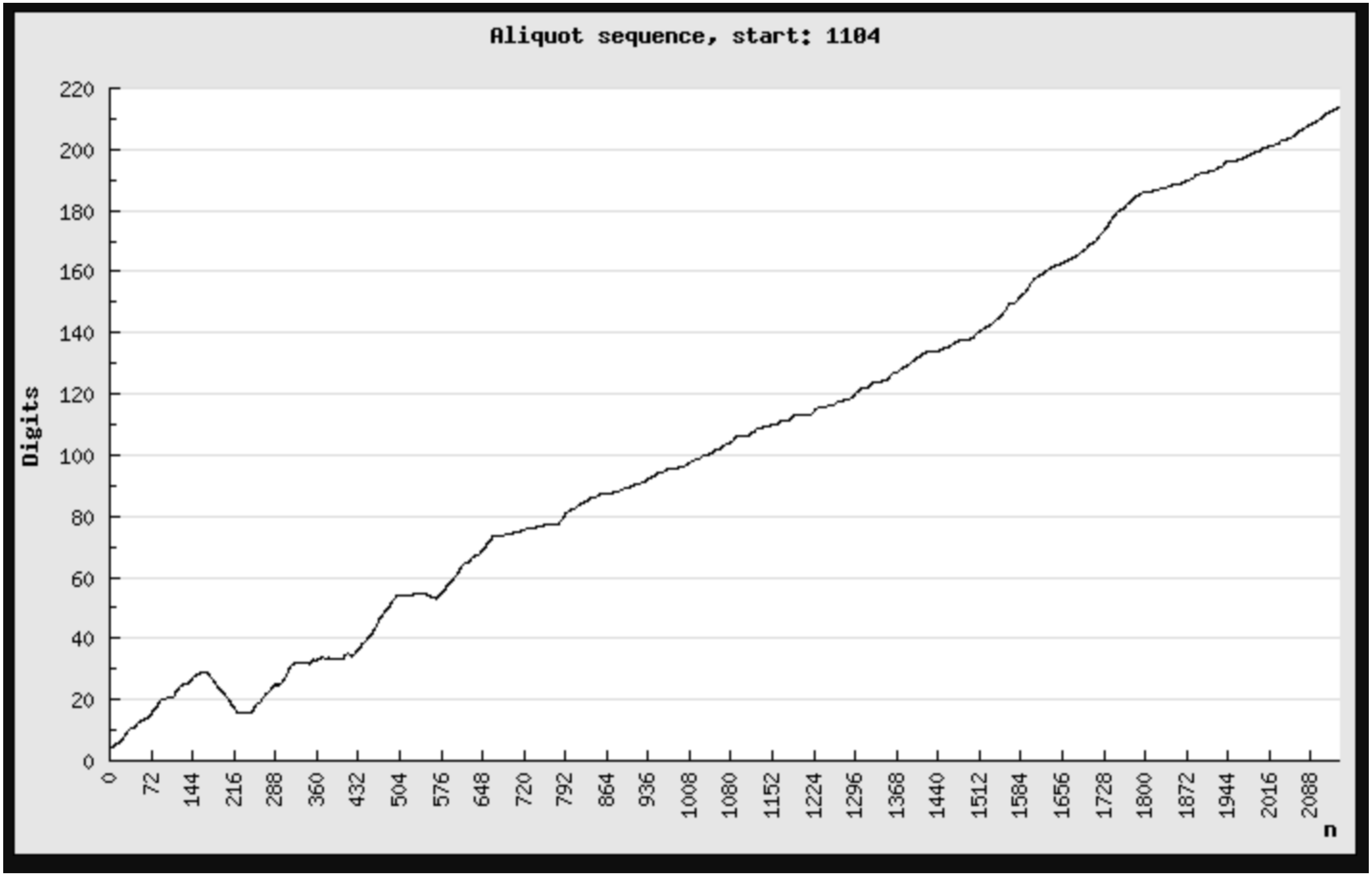
Actually, the first number in doubt for whether it terminates, enters a cycle, or diverges is 276. This has been computed for 2140 iterations, where the current value has 213 decimal digits and is of the form $6n$, where n is composite and has no small prime factors. Here is the plot: (from https://members.loria.fr/PZimmermann/records/aliquot.html)

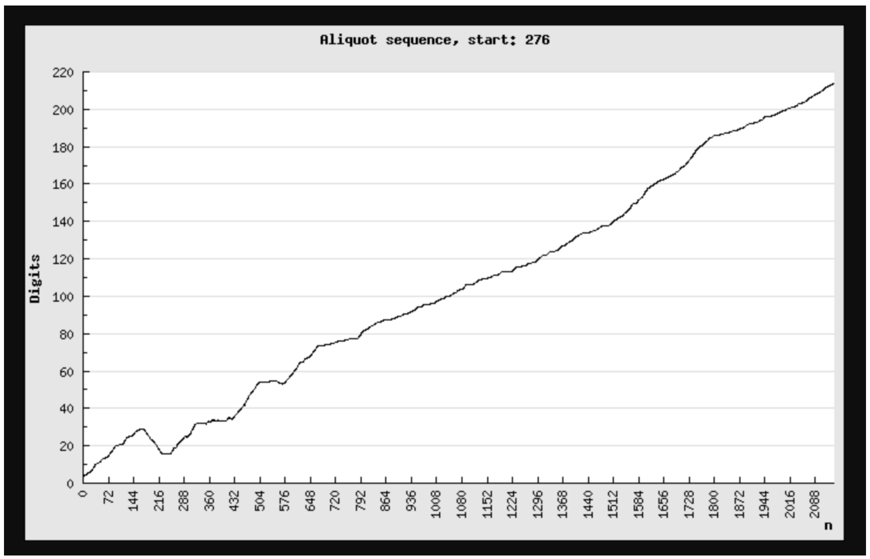

Aliquot sequence, start: 276
Digits
220
200
180
160
140
120
100
80
60
40
20
0
0 72 144 216 288 360 432 504 576 648 720 792 864 936 1008 1080 1152 1224 1296 1368 1440 1512 1584 1656 1728 1800 1872 1944 2016 2088
n

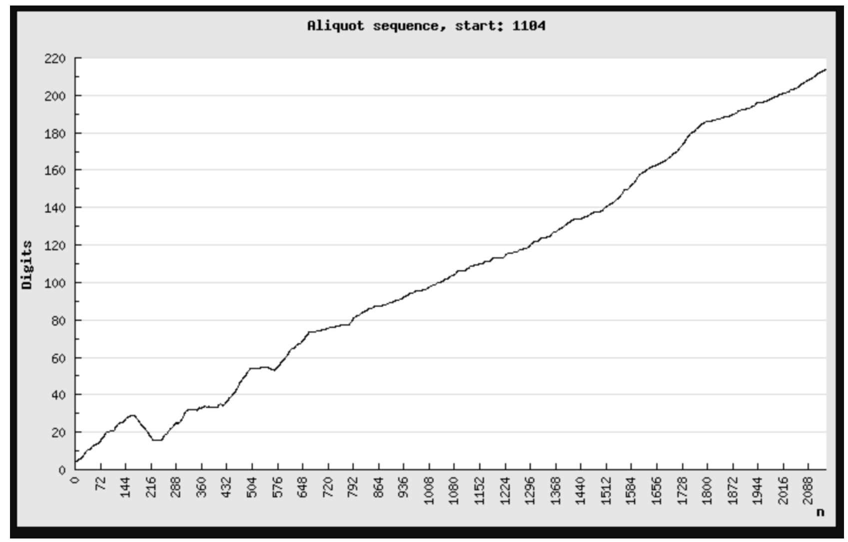
(Unlike many increasing graphs we see these days, I can assure you that aliquot sequences are totally benign!)

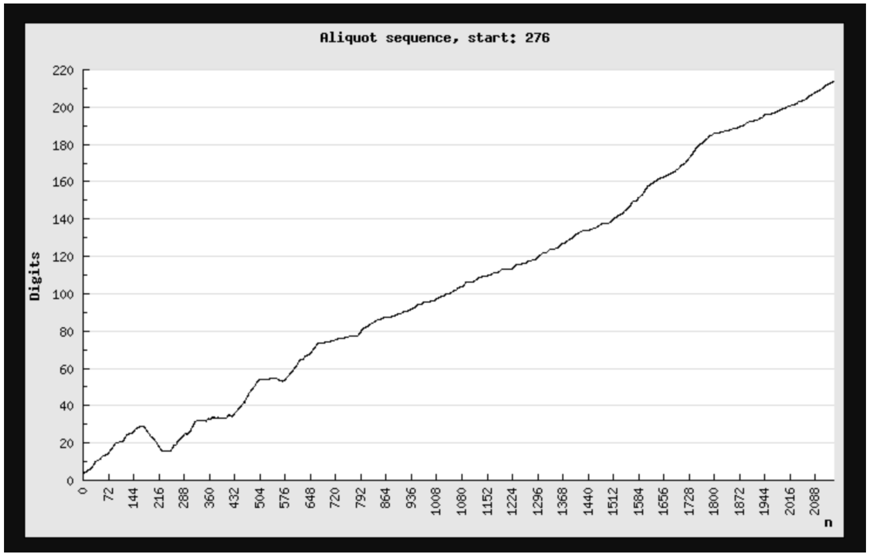
Notice that 276 is the double of 138. What if we keep doubling?

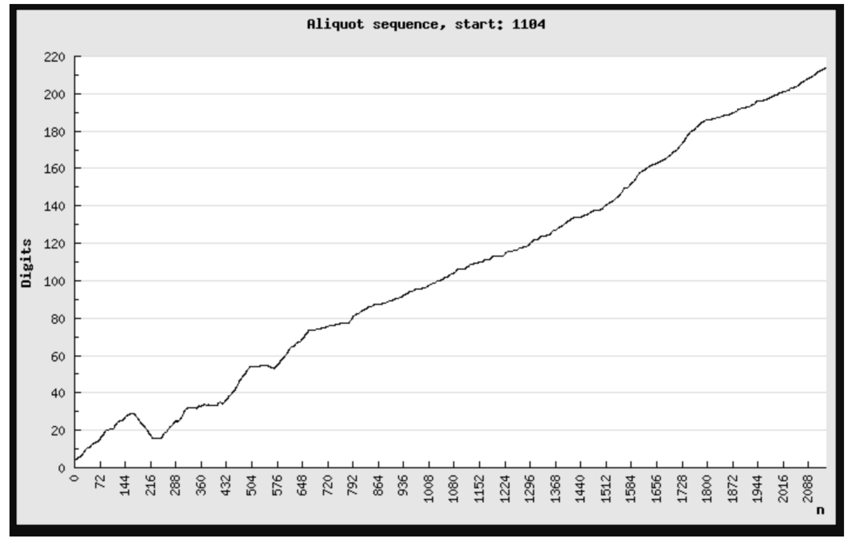

Aliquot sequence, start: 552
Digits
200
180
160
140
120
100
80
60
40
20
0
0
39
78
117
156
195
234
273
312
351
390
429
468
507
546
585
624
663
702
741
780
819
858
897
936
975
1014
1053
1092
1131
n


Aliquot sequence, start: 1104
Digits
220
200
180
160
140
120
100
80
60
40
20
0
0
72
144
216
288
360
432
504
576
648
720
792
864
936
1008
1080
1152
1224
1296
1368
1440
1512
1584
1656
1728
1800
1872
1944
2016
2088
n

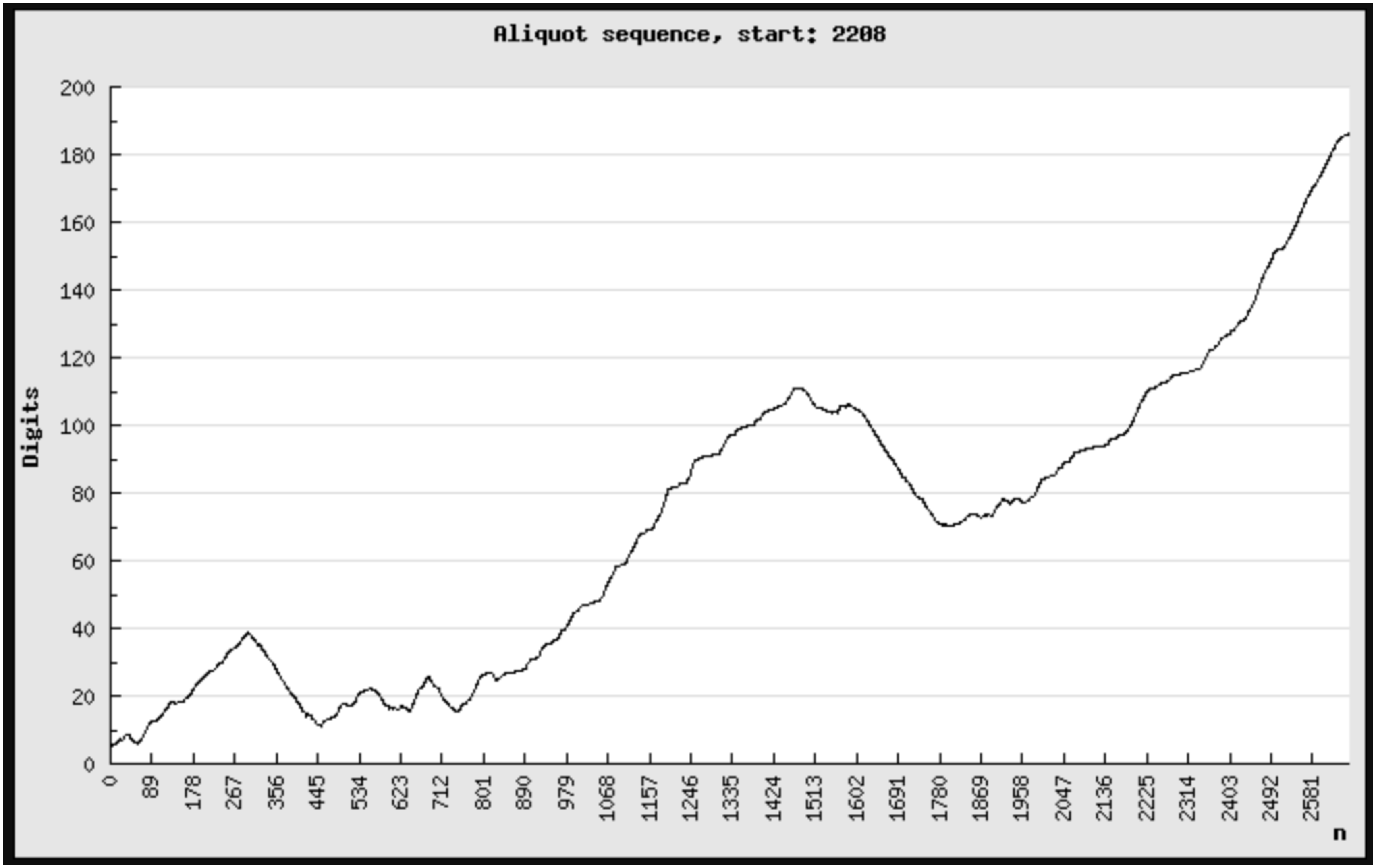

Aliquot sequence, start: 276
Digits
n

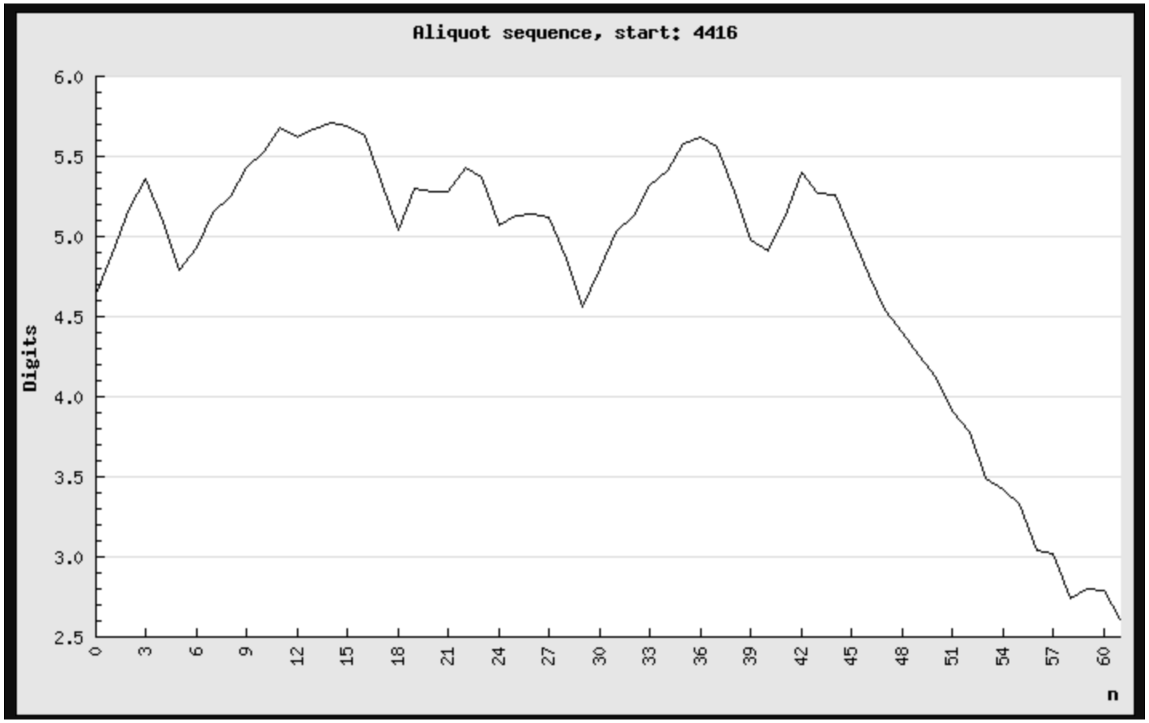

Aliquot sequence, start: 1104
Digits
n


Aliquot sequence, start: 276
Digits
n

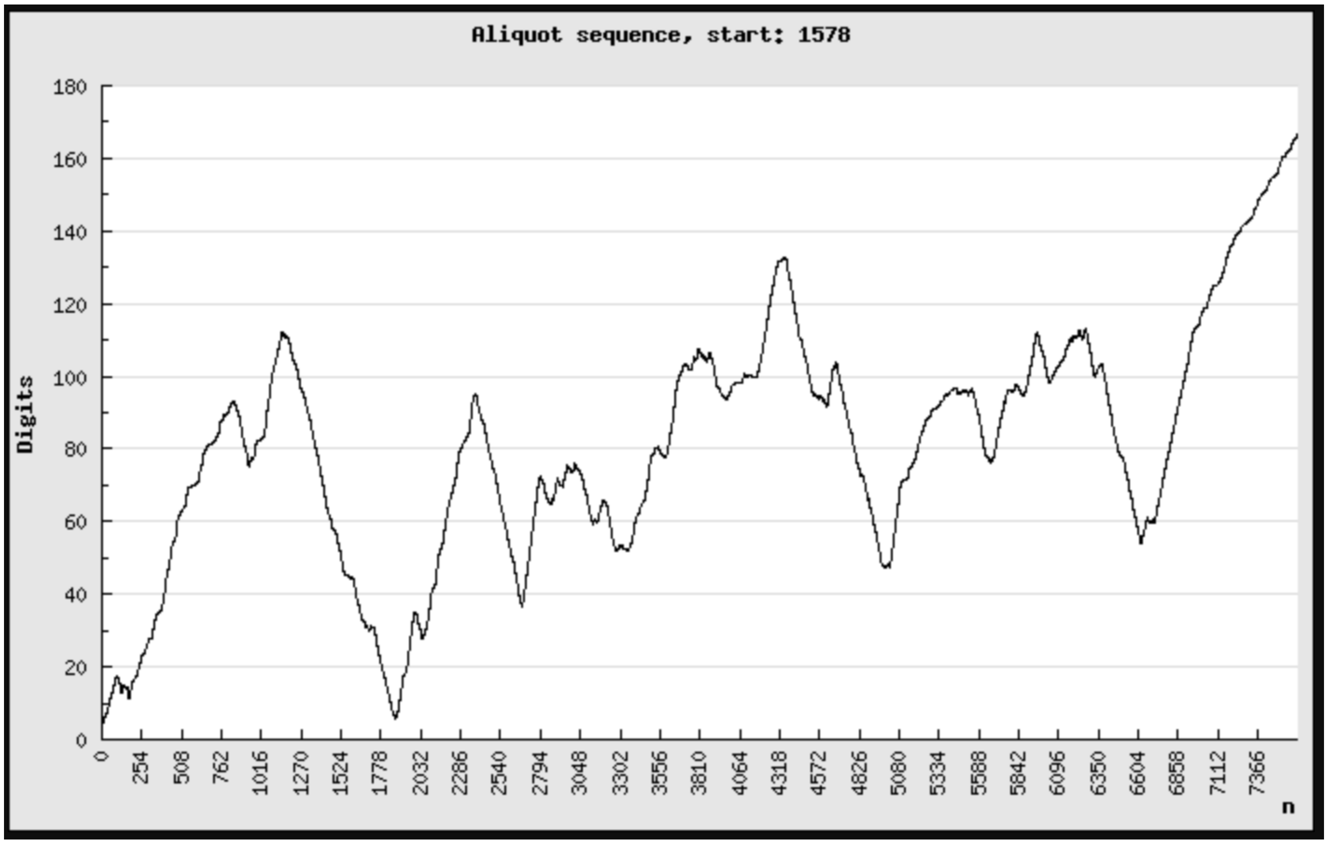

Aliquot sequence, start: 1104
Digits
n

$$s_3(276) = 1104$$

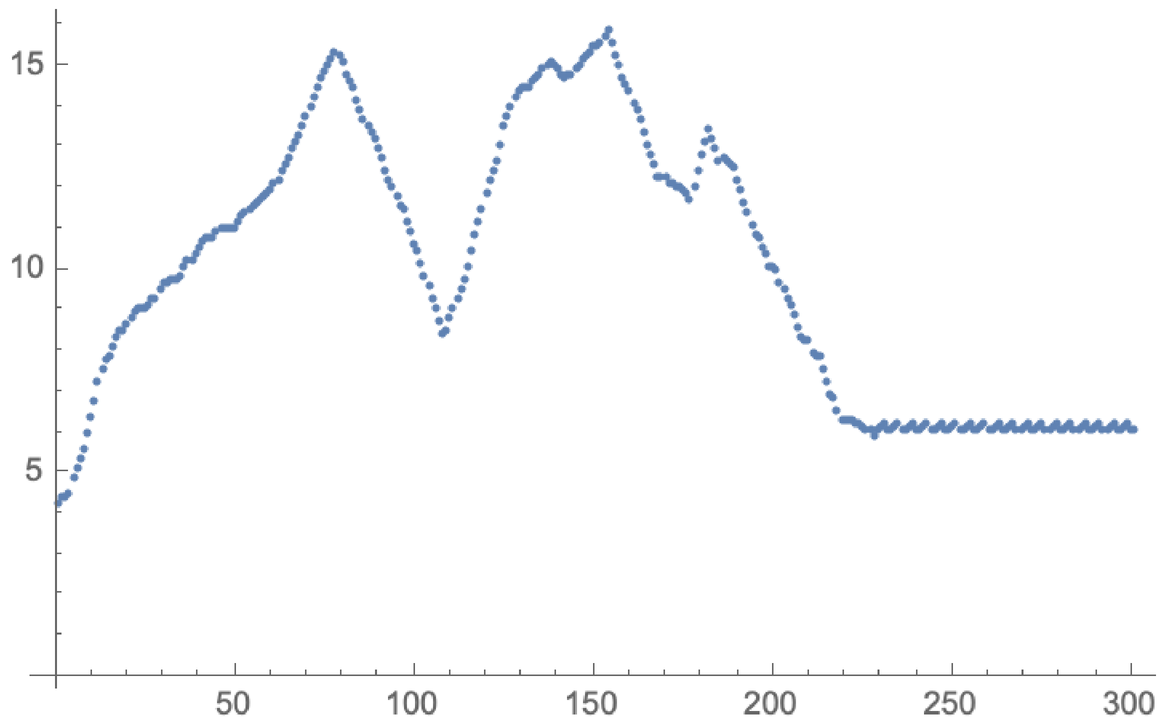

Aliquot sequence, start: 2208
Digits
n
0
20
40
60
80
100
120
140
160
180
200
0
89
178
267
356
445
534
623
712
801
890
979
1068
1157
1246
1335
1424
1513
1602
1691
1780
1869
1958
2047
2136
2225
2314
2403
2492
2581



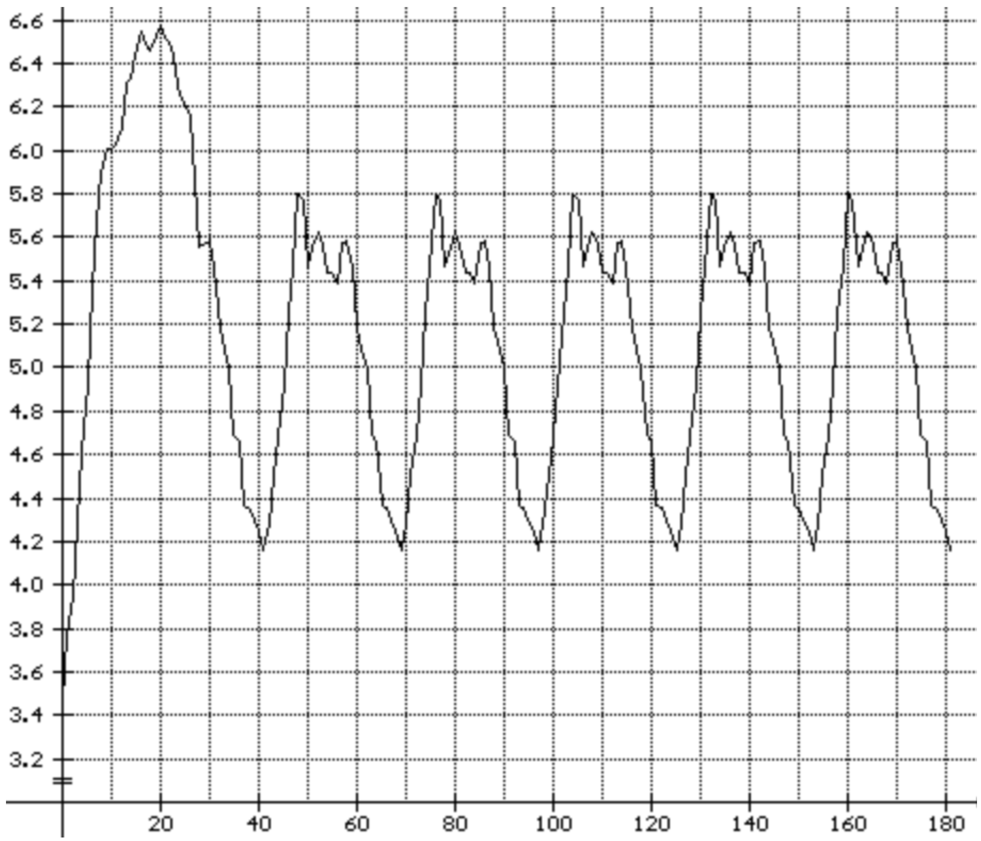
{4416, 7776, 15 156, 23 246, 12 394, 6200, 8680, 14 360, 18 040, 27 320, 34 240, 48 056, 42 064, 47 216, 51 736, 49 064, 42 946, 22 394, 11 200, 20 296, 19 304, 19 096, 26 984, 23 626, 11 816, 13 624, 14 096, 13 246, 7274, 3640, 6440, 10 840, 13 640, 20 920, 26 240, 38 020, 41 864, 36 646, 19 298, 9652, 8268, 12 900, 25 292, 18 976, 18 446, 10 498, 5882, 3514, 2534, 1834, 1334, 826, 614, 310, 266, 214, 110, 106, 56, 64, 63, 41, 1, 0}



A fairly wild one!



This one starts at 17,490 and eventually enters a 4-cycle.



The EKG graph!

It starts at 2856 and enters Poulet's 28-cycle with min value 14,316. This cycle was discovered in 1918, when Richard was two. (Graph from www.aliquot.de/aliquote.htm#programs.)

Are there any theorems? Well, here's an easy one:

$$\frac{1}{x}\sum_{n\le x}\frac{s(n)}{n}\sim\zeta(2)-1,\quad x\to\infty.$$

Since $\zeta(2)-1=\pi^2/6-1\approx 0.6449$, perhaps this lends support to **Catalan–Dickson**.

But **Guy–Selfrdige** would grant you that odd numbers would tend to give bounded aliquot sequences: what's the average over even numbers? It is

$$\frac{1}{x/2}\sum_{2n\le x}\frac{s(2n)}{2n}\sim\frac{5}{4}\zeta(2)-1,\quad x\to\infty,$$

and $5\zeta(2)/4-1\approx 1.0562$.

Well, **Catalan–Dickson** might argue that the graphs we've seen plot the log of the current iteration value and have approximately piecewise linear behavior. On such intervals we should be looking at the *geometric mean* rather than the arithmetic mean.

Bosma & Kane (2012) did this for starting values: *Over odd $n \le x$, the geometric mean is $o(1)$ as $x \to \infty$, and over even $n \le x$ it is $\sim e^{\lambda}$, where $\lambda < -0.03$.*

They call λ the "aliquot constant" and the fact that it is negative, that is, $e^{\lambda} < 1$, they say lends support to **Catalan–Dickson**.

In 2017, I computed λ to higher precision, it is $-0.03325948\ldots$. I also computed the geometric mean over multiples of 4, it is $\sim e^{\lambda_4}$, where $\lambda_4 = 0.174776$. Since this is > 1, perhaps **Guy–Selfridge** holds for many multiples of 4.

Perhaps the flaw in this thinking: Does the ratio $s(n)/n$ have any correlation with the ratio $s(s(n))/s(n)$, or if it does, what about higher iterates?

Lenstra (1975) showed that for every k there is an increasing aliquot sequence of length k, and the next year, **Erdős** showed that this commonly occurs.

Note that the inequality $n < s(n)$ is the definition of an *abundant* number, a concept that goes back to the first century in a book of **Nicomachus**. **Davenport** proved that they have a positive density Δ, and the current record, of **Kobayashi**, is that Δ to 4 decimal places is 0.2476.

Erdős (1976): *For each fixed k, but for a set of numbers of asymptotic density* 0, *if $s(n) > n$, then the aliquot sequence starting at n increases for k terms.*

In the same paper, **Erdős** claimed the same is true for $s(n) < n$; that is if the sequence decreases at n, it almost surely continues to decrease for $k-1$ additional terms. However, he later withdrew his claim of a proof, though it seems likely to be true. In 1990, **Erdős, Granville, P, & Spiro** showed it to be true if $k = 2$. That is, if $s(n) < n$, then almost surely $s(s(n)) < s(n)$.

They also conditionally proved the full statement for decreasing aliquot sequences assuming the following conjecture.

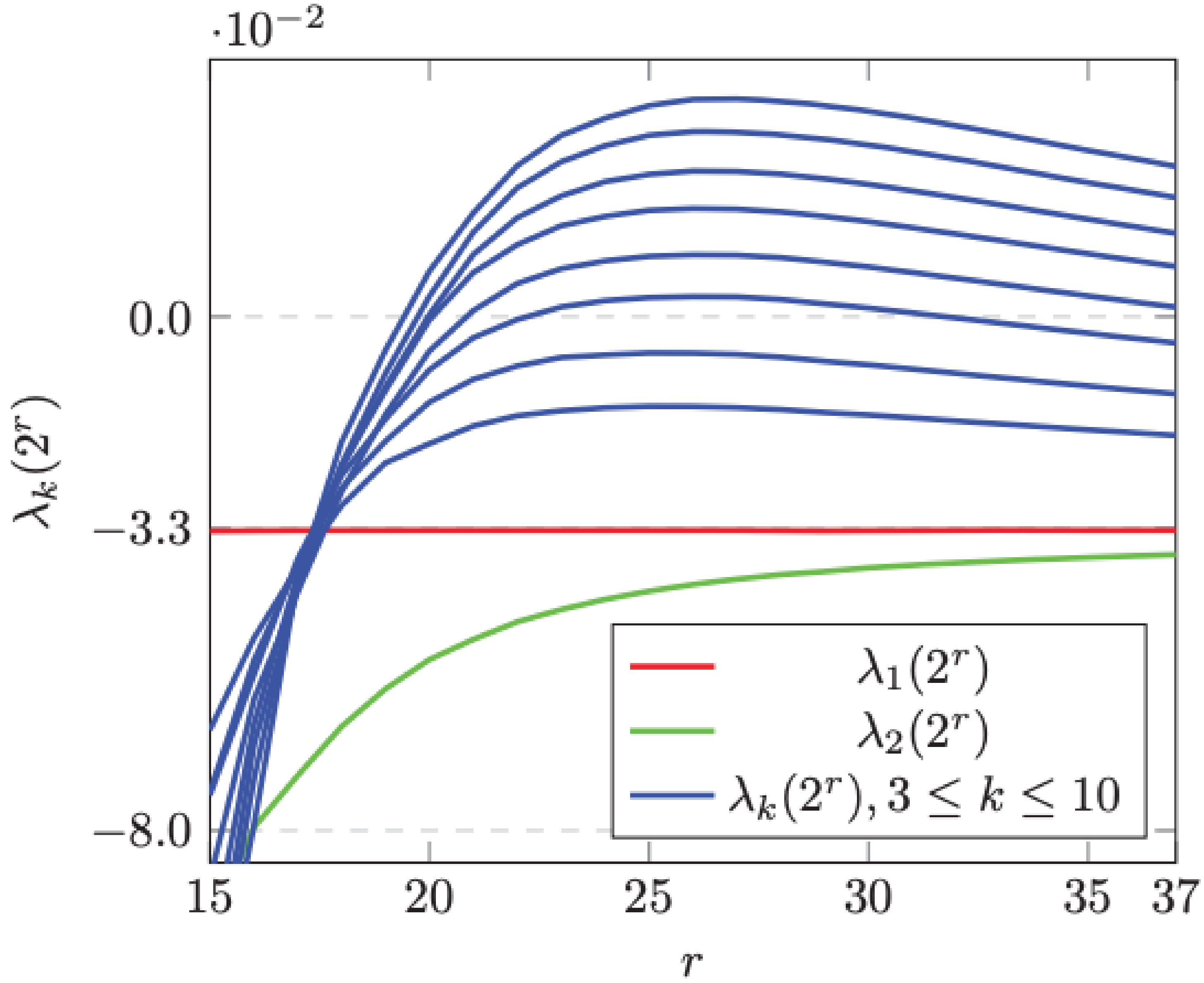
Conjecture: *If A is a set of integers of asymptotic density 0, then $s^{-1}(A)$ also has asymptotic density 0.*

Note that there is a set of asymptotic density 0 such that $s(A)$ has asymptotic density 1/2. This is the set of "almost primes" pq, where p, q are primes. The fact that the set of $s(pq)$ has density 1/2 follows from what's known about Goldbach's conjecture. (We'll revisit this situation near the end of the talk.)

Using these kinds of thoughts, it's not hard to prove that for each k there exists a decreasing aliquot sequence of length k.

Towards the issue of correlating behavior at higher iterates with what happens earlier, in 2018 I worked out the geometric mean of $s(s(n))/s(n)$ over even n: It is asymptotically the same as for $s(n)/n$.

In **Chum, Guy, Jacobson, & Mosunov** (2018) numerical experiments were performed to test the geometric mean on average for $s_k(n)/s_{k-1}(n)$ for even n with $s_k(n) > 0$ for $k \leq 10$:


$\cdot 10^{-2}$
0.0
−3.3
−8.0
$\lambda_k(2^r)$
15
20
25
30
35
37
r
$\lambda_1(2^r)$
$\lambda_2(2^r)$
$\lambda_k(2^r), 3 \leq k \leq 10$

Another extensive numerical experiment was performed by **Bosma**, 2018. For each $n < 10^6$ he computed the aliquot sequence starting at n until it either terminated at 0 or a cycle, merged with the sequence of a smaller n, or if a term exceeded 10^{99}.

Here are some of his extensive stats:

Every odd number's sequence with a term exceeding 10^{99} had already merged with a smaller, even number's sequence, and there were just 793 of these.

About $1/3$ of the even numbers $< 10^6$ had an s-iterate larger than 10^{99}. More specifically, there are 169,548 such even numbers. (After mergers, there are just 9,527 distinct such sequences.)

Another statistical angle (from an asymptotic perspective) is in a recent paper with **Pollack**, 2016. Here, among other problems, we consider "aliquot reversals". These are those exceptional numbers n where $n, s(n), s(s(n))$ is not monotone. We dedicated this paper to Richard on his 99th birthday:

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 3, Pages 1–26 (April 5, 2016)
http://dx.doi.org/10.1090/btran/10

SOME PROBLEMS OF ERDŐS ON THE SUM-OF-DIVISORS FUNCTION

PAUL POLLACK AND CARL POMERANCE

For Richard Guy on his 99th birthday. May his sequence be unbounded.

Let's assume for the sake of argument that the higher iterates of s are independent of earlier behavior. How likely is it for the sequence to maintain parity, i.e., stay even or stay odd? Well, it changes parity whenever a square or the double of a square is hit, and the chance for a random n to be of this form is proportional to $1/\sqrt{n}$. Yes, certainly a small chance, especially when n has hundreds of digits. And if the sequence is growing geometrically, then this chance decreases geometrically. However, infinity is a long way off! In the journey, once we reach high numbers, might we *never* see a parity switch?

Yes, this is a reasonable assumption, since the product of $1 - 1/\sqrt{2^n}$ (assuming the sequence is growing like 2^n) converges quickly to a positive constant, with the tail product exponentially close to 1.

OK, it may indeed be reasonable to assume constant parity for an aliquot sequence once it reaches high numbers.

What about divisibility by other small primes? For example, 3. If $6 \mid n$ and $n > 6$, then $s(n) > n$. Does 3 persist? Conversely, if $3 \nmid n$, should this persist for $s(n)$?

The answer is yes, sort of. The issue is if $3 \mid \sigma(n)$, where $\sigma(n) = s(n) + n$. If $3 \mid \sigma(n)$, then $3 \mid n$ if and only if $3 \mid s(n)$. But why should $3 \mid \sigma(n)$? A “normal” n has about $\log\log n$ prime factors, a result of **Hardy & Ramanujan**, most of which will generally be just to the first power. The number of these primes that are 2 (mod 3) normally is about $\frac{1}{2}\log\log n$. But if $p \equiv 2 \pmod 3$ is prime, then $3 \mid p + 1 = \sigma(p)$. Thus, normally, $3^k \mid \sigma(n)$, where $k \approx \frac{1}{2}\log\log n$.

Quantifying this somewhat, the probability that $3 \nmid \sigma(n)$ for a random n decays to 0 like $1/\sqrt{\log n}$. Now this is small, but again, the journey to infinity is long, and we really ought to expect to see $3 \nmid \sigma(n)$ occur infinitely often. So, when 3 occurs, it should do so for a long stretch, but not forever.

This analysis is interrupted a bit by parity again. For example, suppose that $n \equiv 2 \pmod 4$. Then we automatically have $3 \mid \sigma(n)$. But should we expect the residue class mod 4 to persevere?

A similar argument for there usually to be a high power of 3 dividing $\sigma(n)$ says that usually a high power of 2 divides $\sigma(n)$, and in fact, the probability that $\sigma(n) \not\equiv 0 \pmod 4$ (the condition that would block $2 \pmod 4$ from continuing) is proportional to $1/\log n$.

But, now here's where it gets complicated, what if both 2 and 3 appear in n to exactly the first power. Then we are guaranteed that $s(n) \equiv 2 \pmod 4$ and that $s(n) \equiv 0 \pmod 3$. What might change is that possibly $3^2 \mid s(n)$.

Again, working out all of these angles and similar scenarios convinced **Guy & Selfridge** that there will indeed be aliquot sequences that escape to infinity, and they may be right.

Say a number n is *sociable* if it is in an aliquot cycle, that is, $s_k(n) = n$ for some k. These include the perfect numbers (51 are known), amicable numbers (more than 1.2×10^9 pairs are known), length-4 cycles (5398 are known), and a few sporadic longer cycles (1 each of lengths 5, 9, and 28, 5 of length 6, and 4 of length 8).

It's known that the number of integers contained in all cycles contained in $[1, x]$ is $o(x)$ as $x \to \infty$ (with **Kobayashi & Pollack**).

Unsolved: Are there infinitely many cycles?

Unsolved: Are there any cycles of length 3?

Unsolved: Do the sociable numbers have asymptotic density 0?

I'd like to close with the problem of "untouchable" numbers, also called "nonaliquot" numbers. These are positive integers that are not values of s.

Unsolved: *The only odd untouchable number is* 5.

Conditional proof. First note that $s(2) = 1$, $s(4) = 3$, $s(8) = 7$, and there is no solution to $s(n) = 5$. Suppose the even number $2k \geq 8$ is the sum of two different primes p, q. Then $s(pq) = p + q + 1 = 2k + 1$. Thus, if every even number at least 8 is the sum of two distinct primes, then every odd number at least 9 is a value of s. $\square$

Note that from known partial results on Goldbach's conjecture, the set of even numbers that are not the sum of two distinct primes has asymptotic density 0.

What about even values of s? **Erdős**, 1973, showed that a positive proportion of even numbers are untouchable, i.e., not values of s. In 2015, **Luca** and I showed that a positive proportion of even numbers *are* values of s. Unsolved: The set of untouchable numbers has an asymptotic density.

In that 2016 paper with **Pollack** dedicated to Richard on his 99th birthday, we gave a heuristic argument that the density of untouchable numbers exists and is ≈ 0.1718. The proportion of them to 10^{10} is ≈ 0.1682. This calculation was carried much further, to 2^{40}, in **Chum, Guy, Jacobson, & Mosunov**, and the proportion of untouchables to this level is ≈ 0.1712.

Unsolved: Prove the conjecture. What can be said about even numbers of the form $s(s(n))$? (even computationally)

One might question whether the topic of aliquot sequences is "good" mathematics.

A famous number theorist once opined:

> "There are very many old problems in arithmetic whose interest is practically nil, e.g. the existence of odd perfect numbers, problems about the iteration of numerical functions, the existence of infinitely many Fermat primes $2^{2^n}+1$, etc."

But in my mind, the answer is unquestionably yes, the study of aliquot sequences has been quite worthwhile! Not only does the topic have an ancient pedigree, going back to **Pythagoras** and many other historical figures, it has helped to spur algorithms for primality testing and factoring, it has helped to spur the study of multiplicative functions, and also probabilistic number theory. It may be the first dynamical system ever studied.

The high arbiters of taste may disagree, but meanwhile, the rest of us, with Richard Guy in our midst holding the banner high, carry on.

Thank You