Some Recent Research Work and Plans

My recent work is mostly about interactions between geometric topology, contact geometry, Lorentz geometry and their applications to the theoretical aspects of general relativity.

In particular, I study relations between causality and Legendrian linking. Two points in a spacetime are said to be causally related if it is possible to get from one to the other moving at less or equal than light speed. One of the central results obtained in our joint works with Stefan Nemirovski [11, 12] is that for a large class of spacetimes two points are causally related if and only if the spheres of all light rays through these points are Legendrian linked in the contact manifold of all light rays. This means that one can tell if two events are causally related or not from the set of points in the universe where the events are visible at any given moment of time (level set of a timelike function). Below I provide the necessary background and state this and other results more formally.

1. Backrgound

Let Q^{2k-1} be an odd dimensional manifold equipped with a smooth hyperplane field $\zeta = \{\zeta_q^{2k-2} \subset T_qQ^{2k-1} \bigm| q \in Q\}$. This hyperplane field is called *a contact structure,* if it can be locally presented as the kernel of a 1-form α with $\alpha \wedge (d\alpha)^{k-1}$ that is nowhere zero.

A $(k-1)$-dimensional submanifold of a contact manifold is *Legendrian* if is is everywhere tangent to the contact hyperplanes. A standard example of the contact manifold is the *spherical cotangent bundle* ST^*M of an m-dimensional manifold M^m.

Let (X^{m+1}, g) be a Lorentz manifold. A non-zero vector $\mathbf{v} \in T_xX$ is called *non-spacelike* (respectively *null or lightlike*), if $g(\mathbf{v}, \mathbf{v})$ is non-positive (respectively zero). Such vectors correspond to velocities of particles moving at speed that is less or equal than (respectively equal to) light speed.

A *null (lightlike) curve* is a piecewise smooth curve all of whose velocity vectors are null. *Non-spacelike curves* are defined similarly. A submanifold $M \subset X$ is *spacelike* if the restriction of g to TM is a Riemann metric.

All non-spacelike vectors in T_xX form a cone consisting of two hemicones. A continuous with respect to $x \in X$ choice of one of the two hemicones is called a *time orientation* of (X, g). Vectors from the chosen hemicones are called *future pointing.* A time oriented connected Lorentz manifold is a *spacetime* and its points are *events.*

The *causal future* $J^+(x) \subset X$ of $x \in X$ is the set of all $y \in X$ that can be reached by a future pointing non-spacelike curve from x. The *causal past* $J^-(x)$ is defined similarly. Two events x, y are said to be *causally related* if $x \in J^+(y)$ or $y \in J^+(x)$.

A *Cauchy surface* in (X, g) is a subset such that every non-spacelike curve $\gamma(t)$ (defined on the maximal possible domain) intersects it at exactly one value of t. A spacetime that has a Cauchy surface is called *globally hyperbolic.* (This is equivalent to the standard definition of global hypebolicity, see [23, pp. 211–212].) Globally hyperbolic spacetimes form probably the most important and studied class of spacetimes.

Bernal and Sanchez [2, Theorem 1], [3, Theorem 1.1], [4, Theorem 1.2] strengthened the classical result of Geroch [22]. They showed that every globally hyperbolic spacetime (X, g) admits a smooth spacelike Cauchy surface M and X is diffeomorphic to the product $M \times \mathbb{R}$ with the projection to the $\mathbb{R}$-component been a timelike function.

Low [28, 29] proved that the space $\mathcal{N}_X$ of all future directed light rays (null geodeics) in a globally hyperbolic (X, g) is a contact manifold and it is contactomorphic to ST^*M, where M is any smooth spacelike Cauchy surface. (Similar results can be proved for more general spacetimes, see [28, 29, 31, 24, 7].) The redshift between two Cauchy surfaces can be interpreted [15] as the ratio of the associated contact forms on the space of light rays. The sphere of all light rays through $x \in X$ is Legendrian and it is called the *sky* S_x *of* x.

2. Legendrian Low conjecture and other recent results and research plans

In our works with Stefan Nemirovski [11, Theorem A], [12, Theorem 10.4] we proved:

Theorem 1 (Chernov, Nemirvoski)**.** *Let $X^{m+1}, m \geq 2$ be a globally hyperbolic spacetime for which the universal cover of its Cauchy surface M is noncompact. Then the statement that $x, y \in X$ are causally unrelated is equivalent to the statement that (S_x, S_y) is a trivial Legendrian link.*

The Legendrian link (S_x, S_y) can be easily reconstructed from the cooriented intersection of the cones of all light rays through x and y with a Cauchy surface M. Thus the above theorem says that causal relation between two events can be determined from the set of points in the universe where the two events are visible at a given moment of time (level set of a timelike function).

When M is homeomorphic to $\mathbb{R}^3$ the above result proves the *Legendrian Low conjecture* formulated by Natario and Tod [31]. The 3-dimensional Poincare conjecture proved by Perelman [33, 34] combined with our results says that if the universal cover of the Cauchy surface M^3 of a 4-dimensional spacetime X is **not** S^3, then causal relation of $x, y \in X$ is equivalent to the Legendrian linking of S_x and S_y. If the universal cover of M is S^3, then Elliptization conjecture [33, 34] implies that one can put a strongly refocusing Lorentz metric on $M \times \mathbb{R} = X$. (A spacetime is *strongly refocusing* if it has a pair of points u, v such that all light rays through u pass through v.) One can show that the statement of Theorem 1 is false for strongly refocusing spacetimes. Refocusing of light rays is closely related [17, Remark 7], [10] to the Y^x_ℓ Riemann manifolds that are manifolds for which all the unit speed geodesics starting from the point x return back to x after time l, see Besse [5]. So for example, the unit sphere is the $Y^x_{2\pi}$ manifold for every point x.

Theorem 1 also establishes that causality is equivalent to Legendrian linking for globally hyperbolic spacetimes $X^{m+1}, m > 3$ when the universal covering of a Cauchy surface M^m is a noncompact manifold.

This year I used the Bott-Samelson [6] type results of Frauenfelder, Labrousse and Schlenk [21] to show [9] that Legendrian linking is equivalent to causality in the case where the universal cover $\widetilde{M}$ of the Cauchy surface $M^m, m > 3$ is compact but does **not** have the integral cohomology ring of a Compact Rank One Symmetric Space (CROSS). The question about the relation between causality and linking for globally hyperbolic spacetimes for which the integral cohomology ring of $\widetilde{M}$ is the one of a CROSS remains open and I intend to study it in the future. This is closely related to the question which ones of such manifolds admit a Y^x_ℓ Riemann metric.

I also expect that causality is equivalent to linking for the non-refocusing causally simple spacetime. (A causal spacetime is *causally simple* if the causal future and past of each point is a closed set, and each globally hyperbolic spacetime is causally simple. Such spacetimes are important because they allow the so called naked singularities.) Some evidence supporting this conjecture is provided in my paper [8] and I plan to study this question in the future.

The techniques used to prove Theorem 1 also allowed us to prove [12, Corollary 9.1] that the group of contactomorphisms of ST^*M is orderable for all M. This extends the results of Eliashberg-Kim-Polterovich [19] that left some of the cases open.

Using Theorem 1 and Ding-Geiges work [18], we proved [11, Theorem B] the following result relating causality and the topological (non Legendrian) linking.

Theorem 2 (Chernov, Nemirovski)**.** *Let (X, g) be a $(2+1)$-dimensional globally hyperbolic spacetime for which the universal cover of its Cauchy surface is non-compact. Then the statement that x, y are causally unrelated is equivalent to the statement that (S_x, S_y) is a trivial topological link.*

When the Cauchy surface is noncompact, the above statement proves the *Low conjecture* [25]. Robert Low was working on the question suggested by his advisor Roger Penrose. *The problem communicated by Penrose on the Arnold's problem list* [1, Problem 1998-21] asks to study the relation between causality and linking. The above results can be viewed as the solution to the problem.

A key ingredient in our proof [11, 12] of the statement that Legendrian linking in a globally hyperbolic spacetime X is equivalent to causality, is the space $\mathcal{L}$ of all Legendrian submanifolds of $ST^*M = \mathcal{N}_X$ that are Legendrian isotopic to the fiber of ST^*M. (Here M is the Cauchy surface.) One can show that $\mathcal{L}$ is an infinite-dimensional manifold equipped with a distribution of cones. The two halves of the cones correspond to the velocity vectors of paths in $\mathcal{L}$ that are the so-called positive and negative Legendrian isotopies. Thus $\mathcal{L}$ comes with the natural causal structure. The spacetime X has a natural conformal embedding into $\mathcal{L}$ which sends $x \in X$ to the Legendrian sphere $S_x \subset \mathcal{N}_X = ST^*M$ of all the light rays through x. The fact that linking determines causality for a given spacetime can be reformulated as the statement that $\mathcal{L}$ does not have closed trajectories whose velocity vectors are always pointing inside of the half-cones. Spacetimes with such property are called *causal* and the absence of such trajectories means that time travel is not possible in a spacetime. Jointly with Nemirovski we proved that the universal cover of $\mathcal{L}$ is always causal [14] and I plan to to explore if it is in fact always strongly causal. (A spacetime is strongly causal if for every point p there exists a neighborhood U of p such that there exists no causal curve that passes through U more than once.)

The skies of events in X^{m+1} are $(m-1)$-dimensional spheres in the $(2m-1)$-dimensional manifold of all light rays. In our works [16, 17] with Yuli Rudyak we constructed the "affine linking" invariant alk that generalizes the linking number to the case where the link components are nonzero homologus and applied it to the study of causality.

Of course alk is a topological rather than contact invariant, so there is no hope that alk can always detect that the Legendrian link (S_x, S_y) is nontrivial. However the following result [17, Theorem 2, Theorem 3] says that it often completely detects causality:

Theorem 3 (Chernov, Rudyak)**.** *Let X be a globally hyperbolic spacetime, such that all of its timelike sectional curvatures are nonnegative. Assume moreover that the Cauchy surface M is* **not** *an odd-dimensional $\mathbb{Q}$-homology sphere. Then* $\operatorname{alk}(S_x, S_y)$ *is a well defined $\mathbb{Z}$-valued invariant and* $\operatorname{alk}(S_x, S_y) = 0$ *if and only if the events x, y are causally unrelated.*

Jointly with Nemirovski we showed [13] that a smooth 4-manifold homeomorphic to the product of a closed oriented 3-manifold M and $\mathbb{R}$ and admitting a globally hyperbolic Lorentz metric is in fact diffeomorphic to $M \times \mathbb{R}$. (The poof uses the Geometrization conjecture proved by Perelman [33, 34], results of Bernal and Sanchez [2, 3, 4] and results of Turaev [35].) If M is a nonorientable or a nonclosed manifold, it is not known if a similar statement is true. Globally hyperbolic spacetimes are all homeomorphic to some $M \times \mathbb{R}$, and it would be interesting to know if global hyperbolicity determines the smooth structure for such 4-manifolds. I expect this question to be very hard.

Newman and Clarke [32] showed that for a globally hyperbolic spacetime diffeomorphic to $\mathbb{R}^4$ its Cauchy surface M does not have to be homeomorhic to $\mathbb{R}^3$ and could be another contractible 3-manifold. It seems likely that for a globally hyperbolic spacetime X its contact manifold $\mathcal{N}_X = ST^*M$ of light rays should often determine M. In the language of contact topology the question is if there is a large class of manfiolds $\mathcal{M}$ such that one can conclude that $M_1, M_2 \in \mathcal{M}$ are diffeomorphic provided that ST^*M_1 and ST^*M_2 are contactomorphic. This is another problem I plan to study,

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