DETECTING THE MOMENTS OF INERTIA OF A MOLECULE VIA ITS ROTATIONAL SPECTRUM

CRAIG J. SUTTON[♯]

Abstract. The moments of inertia of a molecule can be used to help recover information about its structure. We show that for a spherical or symmetric molecule (e.g., methane, benzene and chloromethane) its moments of inertia are determined by its rotational spectrum. As we will explain, this is a corollary of the following geometric result: the left-invariant naturally reductive metrics on SO(3) can be mutually distinguished via the Laplace spectrum. We will establish this geometric result by first demonstrating that the systole of each left-invariant naturally reductive metric on SO(3) is "clean" and "audible." We then use the wave invariants associated to the systole to show that within this class of metrics, each metric can be uniquely identified by its spectrum.

1. Introduction

The *spectrum* of a (connected) closed Riemannian manifold (M, g) is defined to be the sequence $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty$ consisting of the eigenvalues (counting multiplicities) of the associated Laplace operator Δ_g acting on $L^2(M, d\nu_g)$. The inverse spectral problem is concerned with the extent to which one can recover the geometry of a Riemannian manifold from its spectrum. Two classes of spectral invariants that are widely used in inverse spectral geometry are the heat invariants and the wave invariants. The heat invariants are defined via the asymptotic expansion of the trace of the heat semi-group at its singularity at $t = 0$:

$$\mathrm{Trace}(e^{-t\Delta_g}) = \sum_{k=0}^{\infty} e^{-t\lambda_k} \overset{t\searrow 0+}{\sim} (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k(M, g) t^k,$$

where n is the dimension of M. The coefficients $a_k(M, g)$ of this asymptotic expansion are known as the *heat invariants* of the Riemannian manifold and they are clearly spectral invariants. For each k, the heat invariant $a_k(M, g)$ can be expressed as $a_k(M, g) = \int_M u_k(x) d\nu_g$, where $u_k(x)$ is a universal homogeneous polynomial of degree $2k$ in the coefficients of the curvature tensor R and its higher order derivatives. For instance, $a_0(M, g) = \mathrm{vol}(M, g)$ and $a_1(M, g) = \frac{1}{6}\int_M \mathrm{Scal}(x) d\nu_g$, where Scal denotes the scalar curvature. Therefore, the short-time asymptotics of the heat trace reveals that the dimension, volume and total scalar curvature

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of a Riemannian manifold are encoded in its spectrum. The expressions for the higher heat invariants are more convoluted. For example, $a_2(M,g) = \frac{1}{360}\int_M 2(|R|^2 - |\operatorname{Ric}|^2) + 5\operatorname{Scal}^2 \, d\nu_g$ (cf. [Pol]). In spite of the fact that the heat invariants are determined by local geometric data, they have been used to prove spectral uniqueness results. For instance, Tanno used the first four heat invariants to establish that a round sphere of dimension $1 \leq n \leq 6$ is uniquely determined by its spectrum among all Riemannian manifolds [T1]. It is perhaps surprising that four decades later it is still unresolved whether this is true for round spheres in dimension 7 and higher (cf. [T2]).

The wave invariants of a Riemannian manifold arise by considering the asymptotic behavior of the trace of the wave group at its singularities. The *wave group* associated to (M,g) is the family of operators $U_g(t) = e^{it\sqrt{\Delta_g}} : L^2(M,\nu_g) \to L^2(M,\nu_g)$, indexed by $t \in \mathbb{R}$ and defined via the functional calculus. That is, for each $t \in \mathbb{R}$, $e^{it\sqrt{\Delta_g}}$ acts on the λ-eigenspace of Δ_g via multiplication by the scalar $e^{it\sqrt{\lambda}}$, and we extend this to all of $L^2(M,\nu_g)$ by linearity. The wave group is the quantum mechanical analogue of the geodesic flow and its orbits $u(t,x) = U_g(t)f(x)$ satisfy the wave equation $\frac{\partial^2}{\partial t^2}u(t,x) + \Delta_g u(t,x) = 0$. The trace of the wave group, denoted by $\operatorname{Trace}(U_g(t))$, is a tempered distribution on $\mathbb{R}$ defined by

$$\langle \operatorname{Trace}(U_g(t)), \varphi \rangle = \operatorname{Trace} \int U_g(t)\varphi \, dt,$$

and one can see that it is the Fourier transform of the "spectral distribution" $\sigma(t) = \sum_{j=1}^{\infty} \delta(t - \sqrt{\lambda_j})$. Therefore, the distribution $\operatorname{Trace}(U_g(t))$ is completely determined by the spectrum of (M,g) and is given by $\operatorname{Trace}(U_g(t)) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}$.

As with the heat semi-group, one considers the asymptotic behavior of the trace of the wave group at its singularities. Interestingly, the singular support of the trace of the wave group, denoted $\operatorname{SingSupp}(\operatorname{Trace}(U_g(t)))$, is a subset of the length spectrum of our manifold, whereby the *length spectrum* of (M,g) we mean the set $\operatorname{Spec}_L(M,g)$ consisting of the lengths of the smoothly closed geodesics in (M,g) [Ch, DuGu]. It is a major open problem to determine whether this containment, known as the *Poisson relation*, is actually an equality. Indeed, equality in the Poisson relation—which is known to hold generically [DuGu, p. 61]—would show that the length spectrum of a manifold can be recovered from its spectrum.

Now, if we let $\Phi : \mathbb{R} \times SM \to SM$ signify the geodesic flow on the unit tangent bundle of our Riemannian manifold and let $\Phi_t(\cdot) = \Phi(t,\cdot)$, then a length $\tau \in \operatorname{Spec}_L(M,g)$ is said to be *clean* if

(1) the fixed-point set of Φ_τ, denoted $\operatorname{Fix}(\Phi_\tau)$, is a disjoint union of finitely many closed manifolds;
(2) for each $u \in \operatorname{Fix}(\Phi_\tau)$ the fixed point set of $D_u\Phi_\tau$ is precisely equal to $T_u \operatorname{Fix}(\Phi_\tau)$.

Otherwise, we will say that τ is *unclean* or *dirty*. Under the assumption that the length $\tau \in \operatorname{Spec}_L(M,g)$ is clean, Duistermaat and Guillemin determined that there is an interval I containing τ on which the wave trace can be expressed as a sum of compactly supported distributions

$$\mathrm{Trace}(U_g(t)) = R(t-\tau) + \beta^{\mathrm{even}}(t-\tau) + \beta^{\mathrm{odd}}(t-\tau),$$

where $R(x)$ is smooth on a neighborhood of 0 and the distributions $\beta^{\mathrm{even}}(x)$ and $\beta^{\mathrm{odd}}(x)$ are singular at 0 [DuGu, Theorem 4.5]. Since the distributions β^{even} and β^{odd} are compactly supported their Fourier transforms are given by the smooth functions α^{even} and α^{odd}, respectively. Furthermore, Duistermaat and Guillemin showed that α^{even} and α^{odd} have the following asymptotic expansions:

$$\alpha^{\mathrm{even}}(s) \overset{s\to+\infty}{\sim} \sum_{k=0}^{\infty} \mathrm{Wave}_k^{\mathrm{even}}(\tau) s^{(D_{\mathrm{even}}-2k-1)/2}$$

and

$$\alpha^{\mathrm{odd}}(s) \overset{s\to+\infty}{\sim} \sum_{k=0}^{\infty} \mathrm{Wave}_k^{\mathrm{odd}}(\tau) s^{(D_{\mathrm{odd}}-2k-1)/2},$$

where D_{even} (respectively, D_{odd}) equals the maximum taken over the dimensions of the even-dimensional (respectively, odd-dimensional) components of $\mathrm{Fix}(\Phi_\tau)$ (see [DuGu, Theorem 4.5] or Theorem 2.4). We note that the faster α^{even} (respectively, α^{odd}) decays at infinity the less singular β^{even} (respectively, β^{odd}) is at 0. The coefficients $\mathrm{Wave}_k^{\mathrm{even}}(\tau)$ and $\mathrm{Wave}_k^{\mathrm{odd}}(\tau)$ in the asymptotic expansions above are complex numbers known as the *k-th wave invariants* of the clean length τ. In contrast with the heat invariants, the wave invariants are semi-global in nature and the trace formula implies that a clean length τ is in the singular support of the trace of the wave group if and only if at least one of its wave-invariants is non-zero.

We will agree to say that a Riemannian manifold is *clean* if each length in its length spectrum is clean. And, in this case, the asymptotic behavior of the wave trace at its singularities provides a wealth of spectral invariants. It can be seen that "cleanliness" is a generic property, so that generically we have wave invariants at our disposal to address the inverse spectral problem. Indeed, let M be a closed manifold and $\mathcal{M}(M)$ denote the space of all smooth Riemannian metrics on M (equipped with the C^∞-topology). A metric $g \in \mathcal{M}(M)$ is said to be *bumpy* if each smoothly closed geodesic γ with respect to g has the property that the space of periodic Jacobi fields along γ is spanned by $J(t) = \gamma'(t)$. Equivalently, the metric g is *bumpy* if for each $u \in SM$ such that $\Phi_\tau(u) = u$ for some $\tau \neq 0$, we have that 1 is the only root of unity that is an eigenvalue of $D_u\Phi_\tau$ and it occurs with multiplicity one. A bumpy metric $g \in \mathcal{M}(M)$ has the property that for any length $\tau \in \mathrm{Spec}_L(M,g)$ there are finitely many geometrically distinct closed geodesics of length τ (see [A, Theorem 2] and [An, Section 4]). Therefore, we may conclude that all bumpy metrics are clean. Now, the bumpy metric theorem of Abraham [A, Theorem 1] states that the set of bumpy metrics on M contains a residual set (see [An] for a complete proof), which establishes that cleanliness is a generic property.

Given that homogeneous spaces are far from generic (i.e., bumpy) and serve as important model spaces in geometry, this article is motivated by the following questions:

(1) To what extent is "cleanliness" a common trait among homogeneous manifolds?

(2) Among those homogeneous spaces that are clean, to what extent can wave invariants be employed to distinguish these spaces via the spectrum?

As a test case, we consider the family of left-invariant naturally reductive metrics on SO(3), which we will denote by $\mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$. In a sense, the naturally reductive metrics on SO(3) are close relatives of the metric of constant curvature on SO(3) (see Section 3), and in Theorem 5.14 we provide necessary and sufficient conditions for a naturally reductive left-invariant metric on SO(3) to be clean. From this we are able to deduce that cleanliness is a generic property within $\mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$.

1.1. **Theorem.** *Within the class of naturally reductive left-invariant metrics on* SO(3) *the clean metrics form a residual set. In particular, the bi-invariant metric on* SO(3) *is clean. However, the collection of unclean or dirty metrics contains certain normal homogeneous metrics.*

For the dirty metrics in Theorem 1.1 and those observed by Gornet [Gt], the issue is the existence of a length τ that satisfies condition (1) of cleanliness, but fails to satisfy condition (2). In our case, satisfying condition (2) will hinge on the behavior of the Poincaré map along so-called Type II geodesics (cf. Remark 5.16), which (up to translation by the isometry group) turn out to be iterates of certain one-parameter subgroups of SO(3). We pause to note that we are not aware of any examples where cleanliness fails at condition (1).

Although the previous theorem tells us that there are left-invariant naturally reductive metrics on SO(3) with dirty lengths, we will see that for any left-invariant naturally reductive metric g on SO(3) the length of the shortest non-trivial closed geodesic, denoted $\tau_{\min}(g)$, is always clean and "audible."

1.2. **Theorem.** *Let g be a naturally reductive left-invariant metric on* SO(3)*. Then, $\tau_{\min}(g)$ is clean and appears in the singular support of* $\mathrm{Trace}(U_g(t))$*.*

We note that in this setting a closed geodesic of length $\tau_{\min}(g)$ is always non-contractible, so $\tau_{\min}(g)$ is also the systole of the metric.

In Theorem 1.2 we reach the conclusion that $\tau_{\min}(g)$ is in the singular support of $\mathrm{Trace}(U_g(t))$ by noticing that for each naturally reductive metric g on SO(3) exactly one of its 0-th wave invariants is non-zero (see Proposition 5.18). It will follow from the wave-trace formula that $\dim \mathrm{Fix}(\Phi_{\tau_{\min}(g)})$ is spectrally determined. This along with Corollary 5.19—which establishes that the volume of any left-invariant naturally reductive metric g on SO(3) is a function of $\tau_{\min}(g)$, $\dim \mathrm{Fix}(\Phi_{\tau_{\min}(g)})$, and $\mathrm{Wave}_0^{\bullet}(\tau_{\min}(g))$, where $\bullet$ denotes the parity of $\dim \Phi_{\tau_{\min}(g)}$—will allow us to conclude that within this class each metric g can be completely recovered from the asymptotic expansion of the wave trace at the "audible" singularity $\tau_{\min}(g)$.

1.3. **Theorem.** *Within the class of left-invariant naturally reductive metrics on* SO(3) *each metric is uniquely determined by its Laplace spectrum.*

This theorem is an improvement of [GS, Theorem 4.1] in the case of SO(3), where (in collaboration with Gordon) we demonstrated that within the class of naturally reductive left-invariant metrics on any simple Lie group each metric is spectrally isolated. Of particular interest, however, is the following interpretation of Theorem 1.3 in terms of physical chemistry.

Consider a three-dimensional rigid body $\mathbf{W}$ with its center of mass located at the origin. We recall that the *moment of inertia* of $\mathbf{W}$ about the axis $\mathbb{R}\mathbf{v}$ determined by the unit vector $\mathbf{v} \in \mathbb{R}^3$ is given by the scalar $\langle \mathbb{I}(\mathbf{v}), \mathbf{v} \rangle$, where $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}^3$ is the *moment of inertia tensor* associated to $\mathbf{W}$ and $\langle \cdot, \cdot \rangle$ is the standard euclidean inner product. The quantity $\langle \mathbb{I}(\mathbf{v}), \mathbf{v} \rangle$ measures the resistance of the rigid body to rotation about the axis $\mathbb{R}\mathbf{v}$. For example, a figure skater who wishes to increase their angular momentum during a spin does so by bringing in their arms, which reduces the moment of inertia about the axis of rotation. The moment of inertia tensor $\mathbb{I}$ is symmetric with respect to the standard inner product and, therefore, there is an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$, with $\langle \mathbb{I}(\mathbf{e}_j), \mathbf{e}_j \rangle = I_j$ for $0 < I_1 \leq I_2 \leq I_3$. The numbers I_1, I_2 and I_3 are known as the *principal moments of inertia* of the body and the vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are named the *principal axes*.

Now, the moments of inertia $0 < I_1 \leq I_2 \leq I_3$ determine a left-invariant metric $g_{(I_1,I_2,I_3)}$ on SO(3) as follows. Let $B(\cdot, \cdot)$ denote the Killing form on $\mathfrak{so}(3)$ and let $\Theta_1, \Theta_2, \Theta_3$ denote the usual orthonormal basis of $\mathfrak{so}(3)$ with respect to the inner product $-B$ (see 5.3). For each triple $0 < I_1 \leq I_2 \leq I_3$ we obtain a self-adjoint map $\mathbb{I}_{I_1,I_2,I_3} : (\mathfrak{so}(3), -B) \to (\mathfrak{so}(3), -B)$ given by $\Theta_j \mapsto \frac{1}{I_j}\Theta_j$, for $j = 1, 2, 3$. We then let $g_{(I_1,I_2,I_3)}$ denote the left-invariant metric on SO(3) induced by the inner product $-B(\mathbb{I}_{I_1,I_2,I_3}(\cdot), \cdot)$ on $\mathfrak{so}(3)$. Then, according to Proposition 5.2, the map $(I_1, I_2, I_3) \mapsto g_{(I_1,I_2,I_3)}$ is a bijection onto the isometry classes of left-invariant metrics on SO(3). With respect to the metric $g_{(I_1,I_2,I_3)}$, classical mechanics tells us that the geodesics in SO(3) describe the free rotations of the rigid body $\mathbf{W}$ about its center of mass (cf. [GuSt, Section 28]). Furthermore, in the event that $\mathbf{W}$ is a molecule, Schrödinger's equation tells us that the eigenvalues associated to the Laplacian of $g_{(I_1,I_2,I_3)}$ describe the rotational spectrum (or energy levels) of the molecule.

A molecule with moments of inertia $0 < I_1 \leq I_2 \leq I_3$ is said to be *spherical* in the event that all the moments of inertia are equal (e.g., methane), *symmetric* in the case where exactly two of the moments of inertia are identical (e.g., benzene and chloromethane) and *asymmetric* when the moments of inertia are all distinct (e.g., water). It follows from 5.3 and Proposition 5.2 that the left-invariant metrics on SO(3) describing the free rotations and energy levels of the spherical and symmetric molecules are precisely the left-invariant naturally reductive metrics. Consequently, we have the following corollary of Theorem 1.3.

1.4. **Corollary.** *Within the class of spherical and symmetric molecules, the rotational spectrum of a molecule determines its moments of inertia.*

In light of the discussion above it would appear to be an interesting problem to study whether the left-invariant metrics on SO(3) can be mutually distinguished via their spectra. More generally, one can ask whether homogeneous 3-manifolds can be mutually distinguished via their spectra.[1] An affirmative answer to this question would stand in stark contrast to the state of affairs for homogeneous spaces of higher dimension. For example, Schueth has demonstrated the existence of continuous families of isopsectral left-invariant metrics on classical Lie groups of sufficiently large rank [Sch]. We note that the results in this paper rely on our ability to explicitly compute the geodesic flow and analyze the Poincaré map of the naturally reductive metrics on SO(3). This appears to be infeasible for an arbitrary left-invariant metric on SO(3) and most other homogeneous 3-manifolds; hence, a different approach appears to be needed to address this problem.

The outline of this article is as follows. In Section 2 we review the trace formula of Duistermaat and Guillemin and discuss the ingredients needed to compute the 0-th wave invariant associated to a clean length. In Section 3 we recall the definition of a naturally reductive metric, review the classification of left-invariant naturally reductive metrics on simple Lie groups due to D'Atri and Ziller and say a few words about geodesics on such spaces. In particular, we recall a necessary and sufficient condition for a geodesic with respect to a naturally reductive left-invariant metric to be closed. While on the topic of closed geodesics we note in Proposition 3.11 that there are no geodesic lassos in a homogeneous space, a fact previously known to hold for naturally reductive spaces and left-invariant metrics on Lie groups. In Section 4 we study the derivative of the geodesic flow. In particular, we review Ziller's method for computing the Poincaré map along closed geodesics in naturally reductive spaces. The balance of the paper, which is contained in Section 5, is devoted to proving Theorems 1.2 and 1.3. And, we conclude Section 5 with a few comments on the feasibility of establishing equality in the Poisson relation for the clean left-invariant naturally reductive metrics on SO(3) by making use of the 0-th wave invariants.

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2. Wave invariants and the Duistermaat-Guillemin Measure

In this section we will outline how one can compute the 0-th wave invariants associated to a clean length. In particular, we will review the method of Brummelhuis, Paul and Uribe for constructing the *Duistermaat-Guillemin measure* on clean fixed point sets of the geodesic flow. Throughout we will adopt the following notation.

2.1. Notation.

(1) (M, g) will denote a closed Riemannian manifold
(2) Δ_g will denote the associated Laplacian and $\mathrm{Spec}_\Delta(M, g)$ its spectrum;

[1] The existence of isospectral hyperbolic 3-manifolds (e.g., [Vi, R]) and the presence of a pair of isospectral flat 3-manifolds [DoRo] shows that this type of rigidity cannot hold within the larger class of locally homogeneous 3-manifolds.

(3) $\mathrm{Spec}_L(M,g)$ will denote the length spectrum of (M,g); i.e., it is the set consisting of the lengths of all smoothly closed geodesics in (M,g);
(4) TM will denote the tangent bundle;
(5) $q : TM \to \mathbb{R}$ will be given by $q(X_p) = g(X_p, X_p)^{\frac{1}{2}}$. q is smooth on the punctured tangent bundle $TM - \{0\}$ and $q(tX) = |t|q(x)$ for all $t \in \mathbb{R}$;
(6) $SM = q^{-1}(1)$ is the unit tangent bundle;
(7) We will let Ω denote the standard symplectic form on TM induced by the Sasaki metric $\tilde{g}$ corresponding to g (see [Sa, Chp. 2 Sec. 4]).
(8) H_q will denote the Hamiltonian vector field associated to q; i.e., $dq(\cdot) = \Omega(H_q, \cdot)$;
(9) For each $\tau \in \mathbb{R}$ we will let $\widetilde{\Phi}_\tau$ denote the time τ map of the geodesic flow on TM and we will let Φ_τ denote its restriction to SM;
(10) For each $\tau \in \mathbb{R}$ we let $\mathrm{Fix}(\Phi_\tau)$ denote the fixed point set of Φ_τ.

2.2. **Definition.** A length $\tau \in \mathrm{Spec}_L(M,g)$ is said to be *clean* if

(1) $\mathrm{Fix}(\Phi_\tau)$ is a disjoint union of finitely many closed manifolds $\Theta_1, \ldots, \Theta_r$ of dimension $d_1, \ldots, d_r$, respectively;
(2) For each $u \in \mathrm{Fix}(\Phi_\tau)$ we have $\ker(D_u\Phi_\tau - Id_u) = T_u \mathrm{Fix}(\Phi_\tau)$. That is, $J(t)$ is a periodic Jacobi field along a geodesic of length τ if and only if $(J(0), J'(0))$ is tangent to $\mathrm{Fix}(\Phi_\tau)$.

Otherwise, we say that τ is *unclean* or *dirty.* In the event that all lengths $\tau \in \mathrm{Spec}_L(M,g)$ are clean, we will say that (M,g) is a *clean* manifold.

As is shown in [DuGu], if τ is a clean length, then each component Θ_j of $\mathrm{Fix}(\Phi_\tau)$ admits a canonical positive measure μ_j^τ, which we will refer to as the *Duistermaat-Guillemin Measure* (or *density*). We will now review the construction of the Duistermaat-Guillemin measure as discussed in the appendix of [BPU].

Constructing the Duistermaat-Guillemin Measure. For simplicity we will assume that $\Theta = \mathrm{Fix}(\Phi_\tau)$ is connected and we will let $\widetilde{\Theta} = \{tX_p : X_p \in F \text{ and } t > 0\}$. We will exploit the symplectic structure of the tangent bundle to construct a canonical measure $\tilde{\mu}^\tau$ on $\widetilde{\Theta}$ and obtain a canonical measure on Θ be dividing by the measure $|dq|$ (in the transverse direction).

Indeed, one can check that $\widetilde{\Theta}$ is a clean fixed point set of $\widetilde{\Phi}_\tau$. Now, let $z \in \widetilde{\Theta}$ and consider $T = Id_z - D_z\Phi_\tau : V \to V$, where $V \equiv T_uTM$. Following [BPU, p. 524-525] we can construct a density on $T_z\widetilde{\Theta}$ as follows.

- Let $\mathcal{E} = \{e_1, \ldots, e_k\}$ be a basis for $W \equiv T_z\widetilde{\Theta}$;
- Let $W^\Omega = \{v \in V : \Omega(w,v) = 0 \text{ for each } w \in W\}$ be the Ω-orthogonal complement of W in V.
- Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ be a basis for a complement of W^Ω satisfying
$$\Omega(e_i, f_j) = \delta_{ij}.$$
- Let $\mathcal{V} = \{v_1, \ldots, v_{2n-k}\}$ be a basis for a complement of W in V.

With the above notation we have the following lemma.

2.3. **Lemma** (Lemma A.2 [BPU]).

(1) *$\ker(T) = W$ and the image of T equals W^{Ω}, so that $T\mathcal{V} \cup \mathcal{F}$ is a basis for V.*
(2) *Let $\varphi \in |V|^{1/2}$ be an arbitrary half-density on V. Then the DG-density $\tilde{\mu}^{\tau}$ on $W \equiv T_z\widetilde{\Theta}$ is given by*

$$\tilde{\mu}^{\tau}(\mathcal{E}) = \frac{\varphi(\mathcal{V} \wedge \mathcal{E})}{\varphi(T\mathcal{V} \wedge \mathcal{F})} = \frac{1}{|\alpha(u)|^{1/2}},$$

where we abuse notation and have $\mathcal{E} = e_1 \wedge \cdots \wedge e_k$, $\mathcal{F} = f_1 \wedge \cdots \wedge f_k$, $\mathcal{V} = v_1 \wedge \cdots \wedge v - 2n - k$, $T\mathcal{V} = Tv_1 \wedge \cdots \wedge Tv_{2n-k}$ and $\alpha(u) \neq 0$ satisfies $T\mathcal{V} \wedge \mathcal{F} = \alpha(z)\mathcal{V} \wedge \mathcal{E}$.

It then follows that if we let $\nu_{\tilde{g}\restriction\Theta}$ denote the Riemannian density on Θ induced by the Sasaki metric $\tilde{g}$ on TM, then the *Duistermaat-Guillemin measure* μ^{τ} on Θ is given by

$$\mu^{\tau} = \frac{1}{|\alpha|^{1/2}} \nu_{\tilde{g}\restriction\Theta},$$

where for each $z \in \Theta$ the function $\alpha(z)$ is computed as in the preceding lemma.

The Duistermaat-Guillemin Trace Formula. The trace of the wave group $\mathrm{Trace}(U_g(t)) = \sum_{j=0}^{\infty} e^{it\sqrt{\lambda_j}}$ is a spectrally determined tempered distribution on $\mathbb{R}$ that is equal to the Fourier transform of the "spectral distribution" $\sigma(t) = \sum_{j=1}^{\infty} \delta(t - \sqrt{\lambda_j})$. As we noted in the introduction, Chazarain [Ch] and the pair of Duistermaat and Guillemin [DuGu] independently established the so-called Poisson relation which states that the singular support of this distribution is contained in the length spectrum of the Riemannian manifold (M, g). It is a long-standing open problem to determine whether these sets are actually equal. The trace formula of Duistermaat and Guillemin describes the nature of the singularities of $\mathrm{Trace}(U_g(t))$ that occur at clean lengths and, in the event the manifold is clean, it establishes that equality in the Poisson relation is synonymous with each length in the length spectrum having at least one non-vanishing wave-invariant.

2.4. **Theorem** (Theorem 4.5 [DuGu]). *Suppose that $\tau \in \mathrm{Spec}_L(M, g)$ is clean and let D_{even} (resp. D_{odd}) denote the maximum dimension of an even-dimensional (resp. odd-dimensional) component of $\mathrm{Fix}(\Phi_{\tau})$. Then we have the following.*

(1) *There is an open interval $I \subset \mathbb{R}$ such that $I \cap \mathrm{Spec}_L(M, g) = \{\tau\}$;*
(2) *On the interval I, $\mathrm{Trace}(U_g(t))$ is the sum of compactly supported distributions:*

$$\mathrm{Trace}(U_g(t)) = \beta^{\mathrm{even}}(t - \tau) + \beta^{\mathrm{odd}}(t - \tau) + R(t - \tau),$$

where R is smooth in a neighborhood of 0 and $\beta^{\mathrm{even}}(x)$ and $\beta^{\mathrm{odd}}(x)$ are singular at 0. Furthermore, the Fourier transforms of $\beta^{\mathrm{even}}(x)$ and $\beta^{\mathrm{odd}}(x)$ are smooth functions $\alpha^{\mathrm{even}}(s)$ and $\alpha^{\mathrm{odd}}(s)$ having the following asymptotic behavior behavior: $\alpha^{\mathrm{even}}(s) \overset{s \to -\infty}{\sim} 0$, $\alpha^{\mathrm{odd}}(s) \overset{s \to -\infty}{\sim} 0$

$$\alpha^{\text{even}}(s) \overset{s\to+\infty}{\sim} \sum_{k=0}^{\infty} \mathrm{Wave}_k^{\text{even}}(\tau) s^{\frac{D_{\text{even}}-2k-1}{2}}$$
$$\alpha^{\text{odd}}(s) \overset{s\to+\infty}{\sim} \sum_{k=0}^{\infty} \mathrm{Wave}_k^{\text{odd}}(\tau) s^{\frac{D_{\text{odd}}-2k-1}{2}},$$

where $\mathrm{Wave}_k^{\text{even}}(\tau), \mathrm{Wave}_k^{\text{odd}}(\tau) \in \mathbb{C}$ *for all* k.

(3) *Letting* $\Theta_1, \dots, \Theta_s$ *denote the components of* $\mathrm{Fix}(\Phi_\tau)$ *of dimension* D_{even} *and* $\mu_1^\tau, \dots, \mu_s^\tau$ *denote the corresponding Duistermaat-Guillemin measures, we see that*

$$\mathrm{Wave}_0^{\text{even}}(\tau) = (\frac{1}{2\pi i})^{(D_{\text{even}}-1)/2} \sum_{j=1}^{s} i^{-\sigma_j} \int_{\Theta_j} d\mu_j^\tau, \tag{2.5}$$

where σ_j *equals the Morse Index (in the space of closed loops) of a geodesic* γ_j *with* $\gamma_j'(t) \in \Theta_j$ *(see* [DuGu, p. 69-70]*). And, an analogous expression holds for* $\mathrm{Wave}_0^{\text{odd}}(\tau)$.

(4) τ *is in the singular support of* $\mathrm{Trace}(U_g(t))$ *if and only if there is a non-negative integer* k *such that at least one of* $\mathrm{Wave}_k^{\text{even}}(\tau)$ *and* $\mathrm{Wave}_k^{\text{odd}}(\tau)$ *is non-zero.*

2.6. **Remark.**

(1) From (1) we see that the length spectrum of a clean manifold is a countable and discrete subset of $\mathbb{R}$. However, in general, this need not be the case. In fact, the length spectrum of a manifold can even be uncountable [SS].
(2) In our proofs of Theorems 1.2 and 1.3, we will make use of the 0-th wave invariants associated to $\tau_{\min}$. And, from Equation 2.5 we see that in order to compute the 0-th wave invariants it is generally necessary to compute the Morse index. However, as we will only need the absolute value of these invariants, it will not be necessary to compute the Morse index. Indeed, as we will see in Corollary 5.13, for each $g \in \mathcal{M}_{\text{Nat}}(\mathrm{SO}(3))$, the submanifold $\mathrm{Fix}(\Phi_{\tau_{\min}(g)})$ will have at most two components, and in the case where it has two components it will be clear that the Morse index coming from each component will be identical. Nevertheless, it is possible to compute these indices, an exercise that we omit.

2.7. **Definition.** The constants $\mathrm{Wave}_k^{\text{even}}(\tau)$ and $\mathrm{Wave}_k^{\text{odd}}(\tau)$ in the asymptotic expansion above are known as the k-th *wave invariants* associated to the length τ, for $k \in \mathbb{N} \cup \{0\}$.

2.8. **Corollary** ([DuGu])**.** *If* (M, g) *is a clean manifold, then the singular support of its wave group equals the length spectrum if and only if each* $\tau \in \mathrm{Spec}_L(M, g)$ *has a non-zero wave invariant.*

2.9. **Example** (Recovering the length spectrum of a CROSS)**.** A Riemannian manifold (M, g) is said to be a C_ℓ-manifold, for some $\ell > 0$, if every non-trivial geodesic is closed and has the same minimal period ℓ [Be]. It is then clear that each τ in the length spectrum of (M, g) is clean, since $\mathrm{Fix}(\Phi_\tau) = SM$. Therefore, since $\mathrm{Fix}(\Phi_\tau) = SM$ is odd dimensional and connected, we see that for each $\tau \in \mathrm{Spec}_L(M, g)$ the wave invariant $\mathrm{Wave}_0^{\mathrm{odd}}(\tau)$ is non-zero. Hence, the length spectrum of any C_ℓ-manifold is encoded in its Laplace spectrum. In particular, the length spectrum of a compact rank-one symmetric space (i.e., a CROSS) can be recovered from its spectrum.

3. Naturally reductive metrics and their Geodesics

3.1. **Classification of Naturally Reductive Metrics on Lie Groups.** Let (M, g) be a connected homogeneous Riemannian manifold. Choose a base point $p_0 \in M$. Let H be a transitive group of isometries of (M, g), and let K be the isotropy group of p_0. Now, suppose the Lie algebra $\mathfrak{h}$ of H decomposes into a direct sum $\mathfrak{h} = \mathfrak{K} + \mathfrak{p}$, where $\mathfrak{K}$ is the Lie algebra of K and $\mathfrak{p}$ is an $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{K}$. Given a vector $X \in \mathfrak{h}$ we obtain a Killing field X^* on M by $X_p^* \equiv \frac{d}{dt}|_{t=0} \exp_H tX \cdot p$ for $p \in M$. The map $X \mapsto X^*$ is an antihomomorphism of Lie algebras. We may identify $\mathfrak{p}$ with $T_{p_0}M$ by the linear map $X \mapsto X_{p_0}^*$. Thus, the homogeneous Riemannian metric g on M corresponds to an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$. For $X \in \mathfrak{g}$, write $X = X_{\mathfrak{K}} + X_{\mathfrak{p}}$ with $X_{\mathfrak{K}} \in \mathfrak{K}$ and $X_{\mathfrak{p}} \in \mathfrak{p}$. Recall that for $X, Y \in \mathfrak{p}$,

$$(\nabla_{X^*} Y^*)_{p_0} = -\frac{1}{2}([X, Y]_{\mathfrak{p}}^*)_{p_0} + U(X, Y)_{p_0}^*, \tag{3.1}$$

where $U : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ is the symmetric bilinear map defined by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{p}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{p}}\rangle.$$

3.2. **Definition.** Let (M, g) be a Riemannian homogeneous space and let H be a transitive group of isometries of (M, g), so that $M = H/K$.

(1) (M, g) is said to be *reductive* (with respect to H) if there is an $\mathrm{Ad}(K)$-invariant complement $\mathfrak{p}$ of $\mathfrak{K}$ in $\mathfrak{h}$.

(2) (M, g) is said to be *naturally reductive* (with respect to H) or *H-naturally reductive*, if there exists an $\mathrm{Ad}(K)$-invariant complement $\mathfrak{p}$ of $\mathfrak{K}$ (as above) such that

$$\langle [Z, X]_{\mathfrak{p}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{p}}\rangle = 0,$$

or equivalently $U \equiv 0$. That is, for any $Z \in \mathfrak{p}$ the map $[Z, \cdot]_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is skew symmetric with respect to $\langle \cdot, \cdot \rangle$.

(3) (M, g) is said to be *normal homogeneous* if there is an $\mathrm{Ad}(H)$-invariant inner product Q on $\mathfrak{h}$ such that

$$Q(\mathfrak{p}, \mathfrak{K}) = 0 \text{ and } Q \restriction \mathfrak{p} = \langle \cdot, \cdot \rangle.$$

At our preferred point p_0, the Levi-Civita connection ∇ of a naturally reductive space (M, g) is given by

$$(\nabla_v X^*)(e) = \begin{cases} [X, v] & \text{if } X \in \mathfrak{K} \\ \frac{1}{2}[X, v]_{\mathfrak{p}} & \text{if } X \in \mathfrak{p}. \end{cases}$$

In subsequent sections of this article, it will be useful to recall that there is a metric connection $\widetilde{\nabla}$ whose geodesics coincide with those of ∇, but whose torsion tensor $T^{\widetilde{\nabla}}$ and curvature tensor $R^{\widetilde{\nabla}}$ are both $\widetilde{\nabla}$-parallel. The relationship between ∇ and $\widetilde{\nabla}$ is given by $\nabla_X Y = \widetilde{\nabla}_X Y - \frac{1}{2}T^{\widetilde{\nabla}}(X,Y)$. At p_0, we notice that $\widetilde{\nabla}$, $T^{\widetilde{\nabla}}$ and $R^{\widetilde{\nabla}}$ can be expressed in terms of the Lie bracket. Indeed, for $v \in T_{p_0}M \equiv \mathfrak{p}$ and $X \in \mathfrak{h}$ we have

$$(\widetilde{\nabla}_v X^*)(e) = \begin{cases} [X,v] & \text{if } X \in \mathfrak{K} \\ [X,v]_{\mathfrak{p}} & \text{if } X \in \mathfrak{p} \end{cases}$$

and for $X, Y, Z \in \mathfrak{p}$ we have $T^{\widetilde{\nabla}}(X,Y) = -[X,Y]_{\mathfrak{p}}$ and $R^{\widetilde{\nabla}}(X,Y)Z = -[[X,Y]_{\mathfrak{K}}, Z]$.

3.3. **Remark.**

(1) Any homogeneous Riemannian manifold is reductive. This is essentially a consequence of the fact that for any Riemannian manifold (M,g) (not necessarily homogeneous) and $p \in M$ the subgroup of the full isometry group of (M,g) fixing p is compact in the compact open topology [Hel, Theorem IV.2.5]. The reader can consult [KS] for a complete proof.
(2) M being reductive implies $[\mathfrak{K},\mathfrak{p}] \subset \mathfrak{p}$.
(3) Normal homogeneous metrics are naturally reductive. Indeed, with respect to the Killing form B the map $[Z,\cdot] : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric. Then, for $X, Y, Z \in \mathfrak{p}$ we have
$$\langle [Z,X]_{\mathfrak{p}}, Y\rangle = \langle [Z,X], Y\rangle = -\langle X, [Z,Y]\rangle = -\langle X, [Z,Y]_{\mathfrak{p}}\rangle.$$
(4) Note that if $H_1 \leq H_2$ are two transitive groups of isometries on (M,g), then the metric can be naturally reductive with respect to H_2 while failing to be naturally reductive with respect to H_1 and vice versa. See [DZ, p.20] for an example.
(5) As is noted in [DZ, p. 5], a theorem of Kostant implies that in order to find all naturally reductive metrics on a manifold M one must (a) find all groups H which act transitively on M; (b) for each such group H, find all $\mathrm{Ad}(H)$-invariant bilinear forms Q on $\mathfrak{h}$ such that $Q \restriction \mathfrak{p}$ is positive definite, where $\mathfrak{p} = \mathfrak{K}^{\perp} \equiv T_{p_0}M$. The normal homogeneous metrics are obtained from positive definite Q on $\mathfrak{h}$.

Naturally reductive spaces are a generalization of symmetric spaces. Although the geodesic symmetries of naturally reductive metrics need not be isometries, they are (up to sign) volume preserving [D]. Moreover, every geodesic in a naturally reductive space $M = H/K$ is the orbit of a one-parameter subgroup of the transitive group H. In fact, every geodesic through our base point p_0 is of the form $\exp_H(tX) \cdot p_0$, where $X \in \mathfrak{p}$, and it follows from Equation 3.1 and Remark 3.3(1) that the naturally reductive spaces are precisely the homogeneous Riemannian manifolds with this property.

In [DZ], D'Atri and Ziller addressed the problem of classifying the naturally reductive left-invariant metrics on compact Lie groups. Recalling that for any subgroup K of G the natural action of $G \times K$ on G is defined by $(g, k) \cdot x = gxk^{-1}$, D'Atri and Ziller's classification of such metrics is as follows.

3.4. **Theorem** ([DZ] Theorems 3 and 7)**.** *Let G be a connected compact simple Lie group and let g_0 be the bi-invariant Riemannian metric on G induced by the negative of the Killing form B. Let $K \leq G$ be a connected subgroup with Lie algebra $\mathfrak{K} = \mathfrak{K}_0 \oplus \mathfrak{K}_1 \oplus \cdots \oplus \mathfrak{K}_r$, where $\mathfrak{K}_0 = Z(\mathfrak{K})$ is the center of $\mathfrak{K}$ and $\mathfrak{K}_1, \ldots, \mathfrak{K}_r$ are the simple ideals in $\mathfrak{K}$. Let $\mathfrak{u}$ be a g_0-orthogonal complement of $\mathfrak{K}$ in $\mathfrak{g}$. Given any $\alpha, \alpha_1, \ldots, \alpha_r > 0$ and an arbitrary inner product h on $\mathfrak{K}_0$, then the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$ given by*

$$\alpha g_0 \restriction \mathfrak{u} \oplus h \restriction \mathfrak{K}_0 \oplus \alpha_1 g_0 \restriction \mathfrak{K}_1 \oplus \cdots \oplus \alpha_r g_0 \restriction \mathfrak{K}_r \tag{3.5}$$

induces a left-invariant metric $g_{\alpha,\alpha_1,\ldots,\alpha_r,h}$ on G. Then:

1. *$g_{\alpha,\alpha_1,\ldots,\alpha_r,h}$ is naturally reductive with respect to the natural action of $G \times K$ on G;*
2. *every left-invariant naturally reductive metric on G arises in this fashion;*
3. *$g_{\alpha,\alpha_1,\ldots,\alpha_r,h}$ is normal homogeneous if and only if $h \leq \alpha g_0 \restriction \mathfrak{K}$.*
4. *$\mathrm{Isom}(g_{\alpha,\alpha_1,\ldots,\alpha_r,h})^0$, the connected isometry group, is given by $G \times N_G(K)^0$, where $N_G(K)$ denotes the normalizer of K in G.*

3.6. **Remark.**

1. If g is naturally reductive with respect to $G \times K$, then it is also naturally reductive with respect to $G \times xKx^{-1}$ for any $x \in G$. Conjugating K corresponds to changing the choice of base point p_0 in G.
2. There is a finite collection $\mathcal{K}$ of connected subgroups of a simple Lie group G such that up to isometry every left-invariant naturally reductive metric on G is $G \times K$ naturally reductive for some $K \in \mathcal{K}$ [GS, Corollary 3.7].
3. A Lie group G can admit metrics naturally reductive with respect to $H \times K$ where $H, K < G$, but which are not left-invariant [DZ, p. 12–14]. Such metrics are sometimes called semi-invariant.
4. If G is an arbitrary connected compact Lie group it is known that left-invariant metrics induced by inner products of the form given by Equation 3.5 are naturally reductive, where we allow g_0 to denote *any* bi-invariant metric on G. However, it is unknown whether (up to isometry) this list is exhaustive (see [DZ, Theorem 1 and p. 20]).

3.2. **Geodesics.** In our proof of Theorems 1.1 and 1.2 we will need an explicit description of the closed geodesics of an arbitrary left-invariant naturally reductive metric on SO(3). Therefore, since the geodesics through e with respect to a metric on G as in Equation 3.5 are of the form $\exp_{G\times K}(tX) \cdot e$, where X is an element of $\mathfrak{p}$ (the $\mathrm{Ad}(\Delta K)$-invariant complement of $\Delta\mathfrak{K}$ in $\mathfrak{g} \times \mathfrak{K}$), it will be beneficial to review the recipe provided by D'Atri and Ziller for constructing $\mathfrak{p}$.

3.7. **Constructing an** $\mathrm{Ad}(\Delta K)$**-invariant Complement.** To begin we let $A : \mathfrak{K}_o \to \mathfrak{K}_o$ denote the g_0-symmetric endomorphism satisfying $h(X,Y) = g_0(AX,Y)$ for each $X, Y \in \mathfrak{K}_0$. Then as is described in [DZ, p. 9-11] there are two cases to consider:

(1) α is not an eigenvalue of A and $\alpha_j \neq \alpha$ for each $j = 1, \dots, r$.

In this case we consider the symmetric bi-linear form Q on $\mathfrak{g} \times \mathfrak{K}$ given by

$$Q = \beta\, g \restriction \mathfrak{g} \oplus 0 + \bar{h} \restriction 0 \oplus \mathfrak{K}_0 + \beta_1\, g \restriction 0 \oplus \mathfrak{K}_1 + \cdots + \beta_r\, g \restriction 0 \oplus \mathfrak{K}_r,$$

where $\beta = \alpha$, $\beta_j = \frac{\beta\alpha_j}{\alpha - \alpha_j}$, and $\bar{h}(X,Y) = g_0(\overline{A}X,Y)$ is defined by the g_0-symmetric endomorphism $\overline{A} : \mathfrak{K}_0 \to \mathfrak{K}_0$ satisfying $A = \beta\overline{A}(\overline{A} + \beta I)^{-1}$. Q can be seen to be non-degenerate on $\mathfrak{g} \times \mathfrak{K}$ and $\Delta\mathfrak{K}$. We then take $\mathfrak{p}$ to be the Q-orthogonal complement of $\Delta\mathfrak{K}$ which is given by

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_r, \tag{3.8}$$

where

(a) $\mathfrak{p}_1 = \{(X,0) : X \in \mathfrak{u}\}$;
(b) $\mathfrak{q}_0 = \{(\overline{A}X, -\beta X) : X \in \mathfrak{K}_0\}$;
(c) $\mathfrak{q}_j = \{(\beta_j X, -\beta X) : X \in \mathfrak{K}_j\}$ for $j = 1, \dots, r$.

From this one may conclude that the metric $g_{\alpha,\alpha_1,\dots,\alpha_r,h}$ is naturally reductive.

(2) α is an eigenvalue of A *or* $\alpha_j = \alpha$ for some $j = 1, \dots, r$.

We find the $\mathrm{Ad}(\Delta K)$-invariant complement $\mathfrak{p}$ of $\Delta\mathfrak{K}$ in $\mathfrak{g} \times \mathfrak{K}$ by considering a proper subgroup $K' \leq K$ with respect to which the metric $g_{\alpha,\alpha_1,\dots,\alpha_r,h}$ falls into the previous case. Indeed, consider the Lie algebra

$$\mathfrak{K}'' = \mathfrak{K}_0'' \oplus (\oplus_{\alpha_j = \alpha} \mathfrak{K}_j),$$

where $\mathfrak{K}_0'' = \{X \in \mathfrak{K}_0 : AX = \alpha X\}$. Then we let $\mathfrak{K}'$ denote the g_0-orthogonal complement of $\mathfrak{K}''$ in $\mathfrak{K}$ and let K' denote the corresponding connected proper subgroup of K. One can check that

$$\mathfrak{K}' = \mathfrak{K}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \mathfrak{K}_j),$$

where $\mathfrak{K}_0'$ is the g_0-orthogonal complement of $\mathfrak{K}_0''$ in $\mathfrak{K}_0$. We can then view the metric $g_{\alpha,\alpha_1,\dots,\alpha_r,h}$ as being induced by the inner product

$$\alpha g_0 \restriction \mathfrak{u}' \oplus h \restriction \mathfrak{K}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \alpha_j g_0 \restriction \mathfrak{K}_j),$$

where $\mathfrak{u}' = \mathfrak{u} \oplus \mathfrak{K}''$ is the g_0 orthogonal complement of $\mathfrak{K}'$. The metric then falls into the previous case with respect to K' and we take $\mathfrak{p}$ to be the corresponding complement of $\Delta\mathfrak{K}'$ in $\mathfrak{g} \times \mathfrak{K}'$:

$$\mathfrak{p} = \mathfrak{p}_1' \oplus \mathfrak{q}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \mathfrak{q}_j), \tag{3.9}$$

where

(a) $\mathfrak{p}_1' = \{(X,0) : X \in \mathfrak{u}'\}$;

(b) $\mathfrak{q}_0' = \{(\overline{A}X, -\beta X) : X \in \mathfrak{K}_0'\}$;
(c) $\mathfrak{q}_j = \{(\beta_j X, -\beta X) : X \in \mathfrak{K}_j\}$ for $j = 1, \dots, r$.

However, one can check that $\mathfrak{p}$ is also an $\mathrm{Ad}(\Delta K)$-invariant complement of $\Delta\mathfrak{K}$ in $\mathfrak{g} \times \mathfrak{K}$ and we can then see that the metric is naturally reductive with respect to $G \times K$.

For convenience we summarize our discussion of geodesics with respect to a left-invariant naturally reductive metric on a simple Lie group.

3.10. **Proposition.** *Let G be a simple Lie group and K a connected subgroup. Now, let $g_{\alpha,\alpha_1,\dots,\alpha_r,h}$ be a $G \times K$ naturally reductive metric on G and $\mathfrak{p} \leq \mathfrak{g} \times \mathfrak{k}$ the $\mathrm{Ad}(\Delta K)$-invariant complement given by 3.7. Then the geodesics through $g \in G$ with respect to $g_{\alpha,\alpha_1,\dots,\alpha_r,h}$ are of the form*

$$\exp_{G\times K}(t\,\mathrm{Ad}(g)X, Y) \cdot g = g \exp_G(tX) \exp_G(-tY),$$

where $(X, Y) \in \mathfrak{p}$, and such a geodesic is smoothly closed if and only if $\exp_G(tX) = \exp_G(tY)$ for some $t > 0$.

Since, as we remarked earlier, the geodesics in a naturally reductive space are integral curves of Killing fields, we see there are no geodesic lassos in a naturally reductive space (i.e., all self-intersections of a geodesic are smooth). Although it is not needed elsewhere in the paper, we observe that every homogeneous Riemannian manifold has this property.

3.11. **Proposition.** *Let (M, g) be a homogeneous Riemannian manifold and $\gamma : \mathbb{R} \to M$ a geodesic. If $\gamma(t_0) = \gamma(t_1)$, then $\gamma'(t_0) = \gamma'(t_1)$. That is, any self-intersection of a geodesic in a homogeneous space is smooth.*

Proof. As noted earlier in Remark 3.3(1), Kowalski and Szenthe have shown that any homogeneous Riemannian manifold (M, g) is reductive with respect to any connected Lie group H acting transitively via isometries on (M, g). Let $H \leq \mathrm{Isom}(M, g)$ be a connected group acting transitively on M with isotropy group K, and let $\mathfrak{p}$ be the attendant $\mathrm{Ad}(K)$-invariant complement of $\mathfrak{K}$ in $\mathfrak{h}$. Then we recall that $\mathfrak{p}$ may be identified with $T_{p_0}M$ via the map $X \mapsto X^*_{p_0}$, where for any $X \in \mathfrak{h}$, X^* is the Killing filed $X^*_p \equiv \frac{d}{dt}|_{t=0} \exp_G(tX) \cdot p_0$, which is a complete vector field. We now recall that it follows from Noether's theorem (cf. [Tak, Theorem 1.3]) that if Z is a Killing field and $\gamma(t)$ is a geodesic on a Riemannian manifold (N, h), then $h(Z_{\gamma(t)}, \gamma'(t))$ is constant. Now, let γ be a geodesic in (M, g) such that $\gamma(t_0) = \gamma(0) = p_0$, for some $t_0 \neq 0$, and let X^* be a killing vector field on M, then we have

$$g(\gamma'(0), X^*_{p_0}) = g(\gamma'(0), X^*_{\gamma(0)}) = g(\gamma'(t_0), X^*_{\gamma(t_0)}) = g(\gamma'(t_0), X^*_{p_0}).$$

Therefore, since every vector in $T_{p_0}M$ is of the form $X^*_{p_0}$ for some Killing field X^*, we conclude that $\gamma'(0) = \gamma'(t)$. □

3.12. **Remark.** In the case of left-invariant metrics on Lie groups, this proposition was previously demonstrated to the author by Dorothee Schueth in 2008.

4. The Poincaré map of naturally reductive metrics

We recall that given a Riemannian manifold (M, g) the geodesic flow is the map $\Phi : \mathbb{R} \times TM \to TM$ given by

$$\Phi(t, v) = \frac{d}{dt}\gamma_v(t),$$

where γ_v is the unique geodesic with $\gamma_v'(0) = v$. Throughout we will set $\Phi_t(v) = \Phi(t, v)$. Of particular interest to us is the derivative of Φ_τ. If for each $v \in TM$ we let $T_vTM = \mathcal{H}_v \oplus \mathcal{V}_v$ be the decomposition into the horizontal and vertical spaces, then for any $(A, B) \in T_vTM$ we have

$$\Phi_{t*}(A, B) = (Y(t), \nabla Y(t)),$$

where $Y(t)$ is the Jacobi field along γ_v such that $Y(0) = A$ and $\nabla Y(0) = B$ (see [Sa, p. 56]). If the geodesic γ_v is periodic of period τ, then we set

$$P = \Phi_{\tau *} : T_vTM \to T_vTM.$$

Since $\gamma_v'(t)$ and $t\gamma_v'(t)$ are Jacobi fields along γ_v we see that

$$P(v, 0) = (v, 0) \text{ and } P(0, v) = (\tau v, v).$$

Hence, in order to understand P we must analyze how it behaves on the orthogonal complement of $(v, 0)$ and $(0, v)$; that is, we seek to understand

$$P : E \oplus E \to E \oplus E,$$

where $E = \{u \in T_pM : \langle u, v\rangle = 0\}$. This map is called the (linearized) *Poincaré map* and from the above if Y is a Jacobi field with initial data $(Y(0), \nabla Y(0)) \in E \oplus E$, then

$$P(Y(0), \nabla Y(0)) = (Y(\tau), \nabla Y(\tau)).$$

In the case of (compact) naturally reductive manifolds the Poincaré map has been completely determined by Ziller as follows.

Let $M = H/K$ be a naturally reductive space and as before let $\mathfrak{p} \leq \mathfrak{h}$ be an $\mathrm{Ad}(K)$-invariant complement. For any unit vector $v \in \mathfrak{p} \equiv T_{p_0}M$ we let $\gamma_v(t)$ be the unit speed geodesic given by $\exp_H(tv) \cdot p_0$. Now, let $v \in \mathfrak{p}$ be a unit vector such that the geodesic $\gamma_v(t)$ is closed and set $E = \{u \in \mathfrak{p} : \langle u, v\rangle = 0\}$. Then the restriction of the maps $B(\cdot) = -[v, [v, \cdot]_{\mathfrak{K}}]$ and $T(\cdot) = -[v, \cdot]_{\mathfrak{p}}$ to E are symmetric and skew-symmetric, respectively. Now let E_0 denote the 0-eigenspace of $B : E \to E$ and E_1 be the sum of its non-zero eigenspaces, and we express E_0 as the orthogonal direct sum $E_0 = E_2 \oplus E_3$, where $E_2 = \{X \in E_0 : T(X) \in E_1\}$. Then as in [Z2, p. 579] we define the following subspaces of $E \oplus E$:

1. $V_1 = \{(X, \frac{1}{2}[X, v]_{\mathfrak{p}}) : X \in E_1 \oplus E_3\}$
2. $V_2 = \{(0, X) : X \in E_1\}$
3. $V_3 = \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_2\}$
4. $V_4 = \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_3\} = \{(X, -\frac{1}{2}T(X)) : X \in E_3\}$
5. $V_5 = \{(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) : X \in E_2, Z \in E_1 \text{ and } B(Z) = T(X) \equiv [X, v]_{\mathfrak{p}}\}$

4.1. **Remark.**

(1) In [Z2] there is an omission in the definition of V_5 (cf. [Z1, p. 73]).
(2) We note that since $B : E_1 \to E_1$ is an isomorphism, V_5 is non-trivial if and only if E_2 is non-trivial. In particular, for each $X \in E_2$, there exists a unique $Z \in E_1$ such that $B(Z) = T(X)$.
(3) It will be useful later to notice that $E_1 \leq [\mathfrak{K}, v]$. Indeed, following [Z1, p. 72], we recall that $B : E \to E$ is a self-adjoint map. Let $X_1, \ldots, X_q$ be an orthonormal basis of eigenvectors with eigenvalues $\lambda_1, \ldots, \lambda_q$, and set $Z_i \equiv [v, X_i]_{\mathfrak{K}} \in \mathfrak{K}$. Then $\lambda_i X_i = B(X_i) = [Z_i, v]$ and for $\lambda_i \neq 0$ we get $X_i = \frac{1}{\lambda_i}[Z_i, v] \in [\mathfrak{K}, v]$, which establishes the claim.

With the notation as above we have the following theorem due to Ziller.

4.2. **Theorem.** *Let $(M = H/K, g)$ be a (compact) naturally reductive space and let $\gamma_v(t) = \exp_H(tv) \cdot p_0$ be a smoothly closed unit speed geodesic in M of length τ with $\gamma_v'(0) = v \in \mathfrak{p} \equiv T_{p_0}M$. Then*

(1) *(*[Z2, Theorem 1]*)* $E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$
(2) *(*[Z1, Theorem 1]*) The Poincaré map $P : E \oplus E \to E \oplus E$ along γ_v is described as follows:*
   (a) $P \restriction V_1 \oplus V_2 \oplus V_3 = \mathrm{Id}$*;*
   (b) $P(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) = (\Psi(X), \Psi(\frac{1}{2}[v, X]_{\mathfrak{p}})) = (\Psi(X), \frac{1}{2}[v, \Psi(X)]_{\mathfrak{p}})$*, for* $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$*, where Ψ is the isometry* $e^{\mathrm{ad}(\tau v)} = \mathrm{Ad}(\exp_H(\tau v))$*, we recall that because γ_v is a geodesic it is given by* $\exp_H(tv) \cdot p_0$ *and since it is closed of length τ we have that* $\exp_H(\tau v) \in K$*;*
   (c) $P(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) = \tau(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) + (Z, \frac{1}{2}[v, Z]_{\mathfrak{p}})$*, for* $(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) \in V_5$*.*

4.3. **Remark.** The compactness condition in the above was used by Ziller to establish that a Jacobi filed $J(t)$ along γ_v with $J(0) \in V_5$ must have unbounded length, which is used to show that $V_5 \cap (V_1 \oplus V_2 \oplus V_3 \oplus V_4)$ is trivial [Z2, p. 579-80]. However, this argument only really requires completeness, which is enjoyed by all naturally reductive spaces since geodesics are precisely the orbits of one-parameter groups of isometries. Therefore, the above is true for all naturally reductive manifolds.

The following observation is an immediate consequence of the previous proposition.

4.4. **Corollary.** *Let $\gamma_v(t)$ be a closed unit speed geodesic as above and let $Y(t)$ be a Jacobi field along γ_v. Then $Y(t)$ is periodic if and only if $Y(t)$ has the following initial conditions:*

$$(Y(0), \nabla Y(0)) \in V_1 \oplus V_2 \oplus V_3 \oplus V_4^{\mathrm{per}} \oplus \mathrm{Span}_{\mathbb{R}}\{(v, 0)\},$$

where $V_4^{\mathrm{per}} \equiv \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_3 \text{ and } \psi(X) = X\} \leq V_4$*.*

5. Distinguishing naturally reductive metrics on $SO(3)$ via the spectrum

Let G be an arbitrary compact semi-simple Lie group with bi-invariant metric g_0 induced by the negative of the Killing form B on T_eG. Now for any left-invariant metric g on G there is a linear transformation $\Omega : T_eG \to T_eG$ that is self-adjoint with respect to $-B$ and such that for any $v, w \in T_eG$ we have $\langle v, w\rangle = -B(\Omega(v), w)$, where $\langle \cdot, \cdot\rangle$ is the restriction of g to T_eG.

5.1. **Definition.** With the notation as above, the eigenvalues $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ of Ω are called the *eigenvalues of the metric g*.

5.2. **Proposition** ([BFSTW] Proposition 3.2)**.** *Two left-invariant metrics g_1 and g_2 on* SO(3) *are isometric if and only if g_1 and g_2 have the same eigenvalues counting multiplicities.*

5.3. **Notation and Remarks.** We will now establish notation and collect some facts that will prove useful throughout the remainder of this section.

(1) For the remainder of this section we will let G denote the Lie group SO(3), $\mathfrak{g}$ denote its Lie algebra $\mathfrak{so}(3)$, and g_0 will denote the bi-invariant metric on SO(3) induced by $-B$, where B denotes the Killing form. Additionally, we will let exp denote the exponential map $\exp_G : \mathfrak{g} \to G$.

(2) With Proposition 5.2 in mind we let

$$\Theta_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},\ \Theta_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ \Theta_3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denote the standard g_0-orthonormal basis of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. Then for any choice of positive constants c_1, c_2 and c_3 the self-adjoint map $\Omega : (\mathfrak{so}(3), -B) \to (\mathfrak{so}(3), -B)$ given by $\Omega(\Theta_j) = c_j\Theta_j$ defines a left-invariant metric $g_{(c_1,c_2,c_3)}$ on SO(3) and, by Proposition 5.2, these account for all of the left-invariant metrics on SO(3) up to isometry. Now, since SO(2) is the only non-trivial connected proper subgroup of SO(3) it follows from Theorem 3.4 that up to isometry the left-invariant naturally reductive metrics on SO(3) are the metrics $g_{(\alpha,\alpha,A)}$ given by:

$$g_{(\alpha,\alpha,A)} = \alpha g_0 \restriction \mathfrak{u} \oplus A g_0 \restriction \mathfrak{K}, \tag{5.4}$$

where $\mathfrak{K} = \mathfrak{so}(2) = \mathrm{Span}(\Theta_3)$ and $\mathfrak{u} = \mathfrak{K}^{\perp_0} = \mathrm{Span}\{\Theta_1, \Theta_2\}$ is the orthogonal complement of $\mathfrak{K}$ with respect to g_0. We set $K = \exp_H(\mathfrak{K})$.

(3) Let $\mathfrak{p}$ denote the $\mathrm{Ad}(K)$-invariant complement of $\Delta\mathfrak{K} \leq \mathfrak{g} \times \mathfrak{K}$ discussed in 3.7. Then we have the following.

(a) If $\alpha = A$, then by Equation 3.9 we see $\mathfrak{p} = \mathfrak{g} \oplus 0$. In which case

$$\mathfrak{p} = \mathrm{Span}\{\frac{1}{\sqrt{\alpha}}(\Theta_1, 0), \frac{1}{\sqrt{\alpha}}(\Theta_2, 0), \frac{1}{\sqrt{\alpha}}(\Theta_3, 0)\}$$

and

$$\Delta\mathfrak{K} = \mathrm{Span}\{D = (\Theta_3, \Theta_3)\},$$

where by $\mathrm{Span}\{A_1, \dots, A_k\}$ we denote the linear span of $A_1, \dots, A_k$ over $\mathbb{R}$.

(b) If $\alpha \neq A$, then by Equation 3.8 $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0$, where $\mathfrak{p}_1 = \{(X, 0) : X \in \mathfrak{u} = \mathfrak{K}^{\perp_0}\}$ and $\mathfrak{q}_0 = \{(\overline{A}Z, -\alpha Z) : Z \in \mathfrak{K}\}$ for $\overline{A} = \frac{A\alpha}{\alpha - A}$. In which case

$$\mathfrak{p} = \mathrm{Span}\{Z_1 = (\frac{1}{\sqrt{\alpha}}\Theta_1, 0), Z_2 = (\frac{1}{\sqrt{\alpha}}\Theta_2, 0), Z_3 = \frac{1}{\sqrt{\overline{A}(\overline{A}+\alpha)}}(\overline{A}\Theta_3, -\alpha\Theta_3)\}$$

and

$$\Delta\mathfrak{K} = \mathrm{Span}\{D = (\Theta_3, \Theta_3)\}.$$

It is clear that the adjoint action of $\Delta K \leq G \times K$ on $\mathfrak{p}$ fixes Z_3 and acts as the group of rotations on $\mathrm{Span}_{\mathbb{R}}\{Z_1, Z_2\} = \mathfrak{p}_1$.

(4) For any $(V, W) \in \mathfrak{p}$, where $\mathfrak{p}$ is as above, the geodesic $\gamma_{(V,W)}(t)$ with $\gamma_{(V,W)}(0) = e$ and $\gamma'_{(V,W)}(0) = V - W$ is given by

$$\gamma_{(V,W)}(t) = \exp(tV)\exp(-tW).$$

The geodesic $\gamma_{(V,W)}$ is a one-parameter subgroup of $\mathrm{SO}(3)$ if and only if $V, W \in \mathfrak{so}(3)$ are linearly dependent.

(5) For any compact Lie group endowed with a bi-invariant metric the sectional curvature of a 2-plane σ in the Lie algebra spanned by two orthonormal vectors X and Y is given by $\mathrm{Sec}(\sigma) = \frac{1}{4}\|[X, Y]\|^2$. Consequently, with respect to the metric g_0, the Lie group $\mathrm{SO}(3)$ has constant sectional curvature $\frac{1}{8}$ and is double covered by $S^3(2\sqrt{2})$, the round 3-sphere of radius $2\sqrt{2}$. It follows that the geodesics in $(\mathrm{SO}(3), g_0)$ are all closed, have a common (primitive) length $\ell_0 \equiv 2\sqrt{2}\pi$.

(6) It follows from the previous remark that any two primitive geodesics through a given point of $\mathrm{SO}(3)$ with respect to $g_0 = g_{(1,1,1)}$ have only one point in common or have exactly the same image. Furthermore, since g_0 is bi-invariant, its geodesics through e coincide with the one-parameter subgroups of $\mathrm{SO}(3)$. Given a vector $X \in \mathfrak{g} = \mathfrak{so}(3)$ we then define its *period* to be $\mathrm{Per}(X) = \frac{\ell_0}{\|X\|_0}$, so $\mathrm{Per}(X)$ is the amount of time it takes for the one-parameter subgroup $\exp(tX)$ to return to the identity element for the first time.

(7) It will be useful to observe that $\mathrm{vol}(g_{(\alpha,\alpha,A)}) = \alpha\sqrt{A}V_0$, where $V_0 \equiv \mathrm{vol}(g_{(1,1,1)}) = \frac{1}{2}\mathrm{vol}(S^3(2\sqrt{2})) = 16\sqrt{2}\pi^2$.

We now describe the closed geodesics of an arbitrary naturally reductive metric on $\mathrm{SO}(3)$ and compute the length spectrum.

5.5. **Theorem.** *Consider the naturally reductive metric $g_{(\alpha,\alpha,A)}$ on $\mathrm{SO}(3)$ and let ℓ_0 be as in 5.3(5).*

(1) *If $\alpha = A$, then the closed geodesics through the identity are precisely the one-parameter subgroups of $\mathrm{SO}(3)$ and the non-trivial primitive geodesics are all of length $\sqrt{A}\ell_0$.*

(2) *If $A \neq \alpha$, then the geodesic $\gamma_{(V,W)}$ is closed if and only if one of the following holds:*

(a) *$(V, W) \in \mathfrak{p}_1$, in which case $\gamma_{(V,W)}$ is a one-parameter subgroup of $\mathrm{SO}(3)$ with primitive length $\sqrt{\alpha}\ell_0$.*

(b) $(V,W) \in \mathfrak{q}_0$, *in which case* $\gamma_{(V,W)}$ *is a one-parameter subgroup of* SO(3) *with primitive length* $\sqrt{A}\ell_0$.

(c) $(V,W) = (X + \overline{A}Z, -\alpha Z) \in \mathfrak{p}$, *where* $X \neq 0 \in \mathfrak{u}$ *and* $Z \neq 0 \in \mathfrak{K}$ *and there exist* $p, q \in \mathbb{N}$ *relatively prime integers such that:*
  (i) $\frac{q^2}{p^2} > \frac{A^2}{(A-\alpha)^2}$
  (ii) $\|X\|_0^2 = \sigma(p,q,\alpha,A)\|Z\|_0^2$, *where* $\sigma(p,q,\alpha,A) \equiv \frac{q^2\alpha^2}{p^2} - \frac{A^2\alpha^2}{(\alpha-A)^2}$.

In this case we see that the closed geodesic $\gamma_{(V,W)}$ *is not a one-parameter subgroup and its primitive length is given by* $\sqrt{\alpha}\ell_0[q^2 + p^2\frac{A}{\alpha-A}]^{\frac{1}{2}}$, *which is always strictly larger than* $\sqrt{\alpha}\ell_0$.

Consequently, the length spectrum of $g_{(\alpha,\alpha,A)}$ *is given by*

$$\mathrm{Spec}_L(g_{(\alpha,\alpha,A)}) = \begin{cases} \{k\sqrt{\alpha}\ell_0 : k \in \mathbb{N}\} \cup \{0\} & \alpha = A \\ \{0\} \cup \{k\sqrt{\alpha}\ell_0, k\sqrt{A}\ell_0, k\tau : k \in \mathbb{N} \text{ and } \tau > 0 \text{ with } \mathcal{E}_{\tau,\alpha,A} \neq \varnothing\} & A \neq \alpha, \end{cases}$$

where for each $\tau > 0$ *we let* $\mathcal{E}_{\tau,\alpha,A}$ *denote the finite collection of relatively prime ordered pairs* $(p,q) \in \mathbb{N} \times \mathbb{N}$ *satisfying* $\frac{q}{p} > |\frac{A}{A-\alpha}|$ *and* $\sqrt{\alpha}\ell_0[q^2 + p^2\frac{A}{\alpha-A}]^{\frac{1}{2}} = \tau$.

5.6. **Definition.** Let $g_{(\alpha,\alpha,A)}$ be a naturally reductive metric on SO(3) with $\alpha \neq A$.

(1) A geodesic of the form given in Theorem 5.5(2a) or a translate thereof is said to be of *Type I.*
(2) A geodesic of the form given in Theorem 5.5(2b) or a translate thereof is said to be of *Type II.*
(3) A geodesic of the form given in Theorem 5.5(2c) or a translate thereof is said to be of *Type III.*

5.7. **Remark.** Theorem 5.5 shows us that if $\alpha \neq A$, then the shortest non-trivial closed geodesic with respect to $g_{(\alpha,\alpha,A)}$ is always of Type I or Type II. Therefore, since (primitive) one-parameter subgroups of SO(3) are homotopically non-trivial, it follows that the systole with respect to any metric in $\mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$ coincides with the length of the shortest non-trivial closed geodesic. We also note that it is easy to show that a prime geodesic of Type III is homotopically trivial if and only if $p+q$ is even.

5.8. **Remark.** In the case where $A \leq \alpha$ the primitive geodesics of Type I and II are shorter than the primitive geodesics of Type III. However, when $A > \alpha$, this need not be the case. For example, if we let $\alpha = 1$ and $A = 10$, then $(p,q) = (1,2)$ gives rise to a primitive geodesic that is not a one-parameter subgroup and is of length $\ell_0\sqrt{4 + \frac{10}{9}}$. However, if $A > \alpha$ and $(A-\alpha)^2 < \alpha$, then the prime geodesics of Type I and II will still be shorter than the prime geodesics of Type III.

Proof of Theorem 5.5. For any vector $U \in TG$ we will let $\|U\|_0$ (respectively $\|U\|$) denote its length with respect to the metric g_0 (respectively $g_{(\alpha,\alpha,A)}$).

In the case where $\alpha = A$ we recall from 5.3 that $\mathfrak{p} = \mathfrak{p}_1' = \mathfrak{g} \oplus 0$. Hence, the geodesics $\gamma_{(V,0)}(t) = \exp(tV)$ are one-parameter subgroups of G and the primitive non-trivial geodesics are of length $\sqrt{A}\ell_0 = \sqrt{\alpha}\ell_0$ with respect to $g_{(A,A,A)}$. Thus establishing (1).

In the case where $\alpha \neq A$ we recall that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0$, where $\mathfrak{p}_1 = \{(X, 0) : X \in \mathfrak{K}^{\perp_0}\}$ and $\mathfrak{q}_0 = \{(\overline{A}Z, -\alpha Z) : Z \in \mathfrak{K}\}$ (since $\mathfrak{K}$ is abelian). To find the closed geodesics and their lengths we consider the following three cases.

Case I: $(V, W) = (X, 0) \in \mathfrak{p}_1$ for some $X \neq 0 \in \mathfrak{K}^{\perp_0}$.

In this case the geodesic $\gamma_{(V,W)}(t) = \exp(tX)$ is a non-trivial one-parameter subgroup of $\mathrm{SO}(3)$. Consequently, it is closed and has primitive length

$$\begin{aligned} L(\gamma_{(V,W)}) &= \mathrm{Per}(X) \cdot \|X\| \\ &= \mathrm{Per}(X)\sqrt{\alpha}\|X\|_0 \\ &= \sqrt{\alpha}\ell_0. \end{aligned}$$

Case II: $(V, W) = (\overline{A}Z, -\alpha Z) \in \mathfrak{q}_0$ for some $Z \neq 0 \in \mathfrak{K}$

In this case the geodesic $\gamma_{(V,W)}(t) = \exp_G(t(\overline{A}+\alpha)Z)$ is a non-trivial one-parameter subgroup of $\mathrm{SO}(3)$. Consequently, it is closed and has primitive length

$$\begin{aligned} L(\gamma_{(V,W)}) &= \mathrm{Per}((\overline{A} + \alpha)Z) \cdot \|(\overline{A} + \alpha)Z\| \\ &= \mathrm{Per}((\overline{A} + \alpha)Z)\sqrt{A}\|(\overline{A} + \alpha)Z\|_0 \\ &= \sqrt{A}\ell_0. \end{aligned}$$

Case III: $(V, W) = (X + \overline{A}Z, -\alpha Z)$, where $X \neq 0 \in \mathfrak{K}^{\perp_0}$ and $Z \neq 0 \in \mathfrak{K}$.

The geodesic $\gamma_{(V,W)}(t) = \exp(t(X + \overline{A}Z)) \exp(t\alpha Z)$ is clearly not a one-parameter subgroup of $\mathrm{SO}(3)$, and it is closed if and only if there is a $t_0 > 0$ such that

$$\exp(t_0(X + \overline{A}Z)) = \exp(-t_0\alpha Z). \tag{5.9}$$

As noted in 5.3(5), the images of two non-trivial one-parameter subgroups $\exp(tX_1)$ and $\exp(tX_2)$ in $\mathrm{SO}(3)$ either have only the identity element in common or are identical, and the latter occurs if and only if X_1 and X_2 are linearly dependent. Therefore, since $X + \overline{A}Z$ and αZ are linearly independent we see that Equation 5.9 holds if and only if there is a $t_0 > 0$ such that

$$e^{t_0(X+\overline{A}Z)} = e^{-t_0\alpha Z} = e, \tag{5.10}$$

which is equivalent to the existence of relatively prime integers $p, q \in \mathbb{N}$ such that $p\,\mathrm{Per}(\alpha Z) = q\,\mathrm{Per}(X+\overline{A}Z)$. Writing out the period of αZ and $X+\overline{A}Z$ explicitly we find that Equation 5.10 holds if and only if there exist relatively prime $p, q \in \mathbb{N}$ such that

(1) $\sigma(p,q,\alpha,A) \equiv \alpha^2(\frac{q^2}{p^2} - \frac{A^2}{(\alpha-A)^2}) > 0$;
(2) $\|X\|_0^2 = \sigma(p,q,\alpha,A)\|Z\|_0^2$.

The function σ has the property that $\sigma(p,q,\alpha,A) = \sigma(\tilde{p},\tilde{q},\alpha,A)$ if and only if $\frac{q}{p} = \frac{\tilde{q}}{\tilde{p}}$ and clearly $\sigma(p,q,\alpha,A) > 0$ is equivalent to $\frac{q^2}{p^2} > \frac{A^2}{(A-\alpha)^2}$.

Now, let $X \neq 0 \in \mathfrak{u}$, $Z \neq 0 \in \mathfrak{K}$ and let $p,q \in \mathbb{N}$ be relatively prime integers such that Equation 5.10 holds. Then $\gamma_{(V,W)}$ is closed and its primitive length is given by

$$
\begin{aligned}
L(\gamma_{(X+\overline{A}Z,-\alpha Z)})^2 &= [q\,\mathrm{Per}(X+\overline{A}Z)\|X+(\overline{A}+\alpha)Z\|]^2 \\
&= [\frac{q\ell_0}{\|X+\overline{A}Z\|_0}\|X+(\overline{A}+\alpha)Z\|]^2 \\
&= \frac{q^2\ell_0^2}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2}\|X+(\overline{A}+\alpha)Z\|^2 \\
&= \frac{q^2\ell_0^2}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2}(\|X\|^2+(\overline{A}+\alpha)^2\|Z\|^2) \\
&= \frac{q^2\ell_0^2}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2}(\alpha\|X\|_0^2+(\overline{A}+\alpha)^2A\|Z\|_0^2) \\
&= \frac{q^2\ell_0^2}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2}(\alpha\|X\|_0^2+(\frac{\alpha^2}{\alpha-A})^2A\|Z\|_0^2) \\
&= q^2\ell_0^2\frac{(\alpha\|X\|_0^2+(\frac{\alpha^2}{\alpha-A})^2A\|Z\|_0^2)}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2} \\
&= q^2\ell_0^2\frac{(\alpha\|X\|_0^2+\frac{A\alpha^4}{(\alpha-A)^2}\|Z\|_0^2)}{\|X\|_0^2+\overline{A}^2\|Z\|_0^2} \\
&= \alpha q^2\ell_0^2\cdot\frac{\|X\|_0^2+\frac{A\alpha^3}{(\alpha-A)^2}\|Z\|_0^2}{\|X\|_0^2+\frac{A^2\alpha^2}{(\alpha-A)^2}\|Z\|_0^2} \\
&= \alpha q^2\ell_0^2\cdot\frac{\|X\|_0^2+\frac{A\alpha^3}{(\alpha-A)^2\sigma(p,q,\alpha,A)}\|X\|_0^2}{\|X\|_0^2+\frac{A^2\alpha^2}{(\alpha-A)^2\sigma(p,q,\alpha,A)}\|X\|_0^2} \\
&= \alpha q^2\ell_0^2\cdot\frac{\|X\|_0^2+\frac{A\alpha^3}{(\alpha-A)^2}\|Z\|_0^2}{\|X\|_0^2+\frac{A^2\alpha^2}{(\alpha-A)^2}\|Z\|_0^2} \\
&= \alpha q^2\ell_0^2\cdot\frac{\|X\|_0^2+\frac{A\alpha^3}{(\alpha-A)^2\sigma(p,q,\alpha,A)}\|X\|_0^2}{\|X\|_0^2+\frac{A^2\alpha^2}{(\alpha-A)^2\sigma(p,q,\alpha,A)}\|X\|_0^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha q^2\ell_0^2 \cdot \frac{1+\frac{A\alpha p^2}{q^2(\alpha-A)^2-p^2A^2}}{1+\frac{A^2p^2}{q^2(\alpha-A)^2-p^2A^2}}\\
&= \alpha q^2\ell_0^2\cdot(1+\frac{p^2A(\alpha-A)}{q^2(\alpha-A)^2})\\
&= \alpha\ell_0^2\cdot(q^2+p^2\frac{A}{(\alpha-A)}).
\end{aligned}
$$

In the event that $\alpha > A$ it is clear that this geodesic will have length strictly greater than $\sqrt{\alpha}\ell_0$. To handle the case where $\alpha < A$ we note that $\alpha\ell_0^2\cdot(q^2+p^2\frac{A}{(\alpha-A)})$ is greater than $\alpha\ell_0^2$ if and only if $\frac{A}{A-\alpha} = |\frac{A}{A-\alpha}| < \frac{q^2-1}{p^2}$. But we recall that $p,q\in\mathbb{N}$ were chosen so that $\frac{q}{p} > |\frac{A}{A-\alpha}| = \frac{A}{A-\alpha} > 1$, and notice that for $q>p$ we have $\frac{q^2-1}{p^2} > \frac{q}{p}$. Hence, for $A\neq\alpha$, we see that $L(\gamma_{(V,W)}) > \sqrt{\alpha}\ell_0$.

Cases I-III establish statement (2) of the theorem and the statement concerning the length spectrum of an arbitrary naturally reductive metric $g_{(\alpha,\alpha,A)}$ is now immediate. We conclude the proof by showing that the set $\mathcal{E}_{\tau,\alpha,A}$ is finite.

Indeed, in the case where $A<\alpha$, we see that $\mathcal{E}_{\tau,\alpha,A}$ is a subset of the intersection of an ellipse with the integer lattice in $\mathbb{R}^2$, which implies it is finite. In the event that $A>\alpha$, the points $(p,q)\in\mathcal{E}_{\tau,\alpha,A}$ are a subset of the intersection of the integral lattice with the hyperbola

$$\frac{y^2}{\tau^2/\alpha\ell_0^2} - \frac{x^2}{\tau^2(A-\alpha)/\alpha\ell_0^2A} = 1$$

having asymptotes $y=\pm\sqrt{\frac{A}{A-\alpha}}x$. Now, suppose $\mathcal{E}_{\tau,\alpha,A}$ is infinite, then, since $\frac{q}{p} > |\frac{A}{A-\alpha}| = \frac{A}{A-\alpha} > 1$, we see that q must become arbitrarily large. Then, since the hyperbola is asymptotic to $y=\sqrt{\frac{A}{A-\alpha}}x$, we see that the expression $|p-\sqrt{\frac{A-\alpha}{A}}q|$ can be made arbitrarily small in $\mathcal{E}_{\tau,\alpha,A}$. However, $\frac{q}{p} > \frac{A}{A-\alpha} > 1$ implies

$$p < \frac{A-\alpha}{A}q < \sqrt{\frac{A-\alpha}{A}}q$$

for any $(p,q)\in\mathcal{E}_{\tau,\alpha,A}$, which implies the quantity $|p-\sqrt{\frac{A-\alpha}{A}}q|$ cannot be made arbitrarily small. So, we see $\mathcal{E}_{\tau,\alpha,A}$ is finite. □

For any τ in the length spectrum of a symmetric metric $g_{(\alpha,\alpha,\alpha)}$ on SO(3), we see that $\mathrm{Fix}(\Phi_\tau)$ is the entire unit tangent bundle and it follows that such metrics are clean. We now wish to examine the "cleanliness" of the other naturally reductive metrics on SO(3). Towards this end we begin by examining the fixed point sets of the geodesic flow for naturally reductive metrics that are not symmetric.

5.11. **Lemma.** *Consider the naturally reductive metric $g_{(\alpha,\alpha,A)}$ on* SO(3) *where $\alpha\neq A$ and let $G\times K = \mathrm{SO}(3)\times\mathrm{SO}(2)$ be the connected component of the identity in the isometry group of*

$g_{(\alpha,\alpha,A)}$. *We let* $v = c_1Z_1 + c_2Z_2 + c_3Z_3 \in \mathfrak{p} \equiv T_eG$ *be a unit vector where* $Z_1, Z_2, Z_3 \in T_eG$ *is the orthonormal basis given in 5.3(3).*

(1) *If* $c_1^2 + c_2^2 = 1$, *then* $(G \times K) \cdot v \simeq \mathrm{SO}(3) \times S^1$ *and this* 4*-dimensional submanifold of* $T^1\,\mathrm{SO}(3)$ *accounts for all the unit speed primitive geodesics of Type I, all of which have length* $\sqrt{\alpha}\ell_0$. *The manifold* $(G \times K) \cdot v$ *is said to be a Type I component.*

(2) *If* $c_3 = \pm 1$, *then* $(G \times K) \cdot v \simeq \mathrm{SO}(3)$ *and the* 3*-dimensional submanifold* $(G \times K) \cdot v \cup (G \times K) \cdot (-v)$ *of* $T^1\,\mathrm{SO}(3)$ *accounts for all the unit speed primitive geodesics of Type II, all of which have length* $\sqrt{A}\ell_0$. *The manifold* $(G \times K) \cdot v$ *is said to be a Type II component.*

(3) *Let* $\tau > 0$ *be such that* $\mathcal{E}_{\tau,\alpha,A}$ *is non-empty. For each* $(p, q) \in \mathcal{E}_{\tau,\alpha,A}$ *fix a unit vector* $v_{(p,q)} = c_1Z_1 + c_2Z_2 + c_3Z_3$, *where* $c_1^2 + c_2^2 = \frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}$ *and* $c_3^2 = \frac{1}{\sigma(p,q,\alpha,A)+1}$. *Then* $(G\times K)\cdot v_{(p,q)} \simeq \mathrm{SO}(3)\times S^1_{(p,q)}$, *where* $S^1_{(p,q)} = \{xZ_1+yZ_2+zZ_3 : x^2+y^2 = \frac{\sigma}{\sigma+1} z = c_3\}$, *and the* 4*-dimensional submanifold* $\cup_{(p,q)\in\mathcal{E}_{\tau,\alpha,A}}(G \times K)\cdot(\pm v_{(p,q)})$ *accounts for the unit speed primitive geodesics of Type III having length* τ. *The manifold* $(G \times K) \cdot v_{(p,q)}$ *is said to be a Type III component.*

Proof. We recall that the isotropy group of the identity element corresponding to the natural action of $G \times K$ on $\mathrm{SO}(3)$ is $\Delta K = \mathrm{SO}(3)$, and as we noted in 5.3(3) the isotropy action of ΔK on $\mathfrak{p} \equiv T_eG$ acts via rotations on $\mathfrak{p}_1 = \mathrm{Span}_{\mathbb{R}}\{Z_1, Z_2\}$ and fixes $\mathfrak{q}_0 = \mathrm{Span}_{\mathbb{R}}\{Z_3\}$. The lemma now follows from Theorem 5.5. □

5.12. **Lemma.** *For any* $B > 0$, *there are finitely many* $0 < \tau < B$ *such that* $\mathcal{E}_{\tau,\alpha,A}$ *is non-empty.*

Proof. This follows immediately from the fact that a Type III geodesic has length of the form $\sqrt{\alpha}\ell_0[q^2 + p^2\frac{A}{\alpha-A}]^{\frac{1}{2}}$, where $p, q, \in \mathbb{N}$, and the values of this function form a discrete subset of $\mathbb{R}$. □

Using Theorem 5.5 and Lemmas 5.11 and 5.12 the following is immediate.

5.13. **Corollary.** *Let* $g_{(\alpha,\alpha,A)}$ *be a naturally reductive metric on* $\mathrm{SO}(3)$ *with unit tangent bundle* $T^1\,\mathrm{SO}(3)$ *and corresponding geodesic flow* $\Phi_t : T^1\,\mathrm{SO}(3) \to T^1\,\mathrm{SO}(3)$, $t \in \mathbb{R}$. *Then for each* τ *in the length spectrum of* $g_{(\alpha,\alpha,A)}$ *we see that* $\mathrm{Fix}(\Phi_\tau)$ *is a union of finitely many (homogeneous) submanifolds of* $T^1\,\mathrm{SO}(3)$ *and for each* $u \in \mathrm{Fix}(\Phi_\tau)$ *the connected component of* $\mathrm{Fix}(\Phi_\tau)$ *containing* u *is given by* $\mathrm{Isom}(g_{(\alpha,\alpha,A)})^0 \cdot u$, *where* $\mathrm{Isom}(g_{(\alpha,\alpha,A)})^0$ *denotes the connected component of the identity in the isometry group. In particular, we have the following:*

(1) $\alpha = A$ *if and only if* $\tau = \sqrt{\alpha}\ell_0$ *is the length of the shortest non-trivial closed geoedesic and* $\mathrm{Fix}(\Phi_\tau) = T^1\,\mathrm{SO}(3)$ *is* 5*-dimensional.*

(2) $A < \alpha$ *if and only if* $\tau = \sqrt{A}\ell_0$ *is the length of the shortest non-trivial closed geodesic and* $\mathrm{Fix}(\Phi_\tau) \simeq \mathrm{SO}(3) \cup \mathrm{SO}(3)$ *is* 3*-dimensional. In which case all geodesics of length* $\tau = \sqrt{A}\ell_0$ *are of Type II.*

(3) *$A > \alpha$ if and only if $\tau = \sqrt{\alpha}\ell_0$ is the length of the shortest non-trivial closed geodesic and* $\mathrm{Fix}(\Phi_\tau) \simeq \mathrm{SO}(3) \times S^1$ *is 4-dimensional. In which case all geodesics of length $\tau = \sqrt{\alpha}\ell_0$ are of Type I.*

We now give an explicit description of the naturally reductive metrics on SO(3) which fail to satisfy the clean intersection hypothesis of Duistermaat and Guillemin.

5.14. **Theorem.** *The naturally reductive metric $g_{(\alpha,\alpha,A)}$ is unclean if and only if $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$, where $\mathbb{Q}_+$ denotes the positive rational numbers. Moreover, if we express $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ as $A = \frac{2\alpha j}{k}$, where $k, j \in \mathbb{N}$ are relatively prime, then $\tau \in \mathrm{Spec}_L(g_{(\alpha,\alpha,A)})$ is unclean if and only if $\tau = mk\sqrt{A}\ell_0$ for some $m \in \mathbb{N}$.*

5.15. **Corollary.** *The length of the shortest non-trivial closed geodesic with respect to a left-invariant naturally reductive metric on* SO(3) *is clean.*

Proof. Let $g_{(\alpha,\alpha,A)} \in \mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$ and $\tau_{\min}$ denote the length of its shortest non-trivial closed geodesic. If $A \leq \alpha$, then $\tau_{\min} = \sqrt{A}\ell_0$ and in the event that $A > \alpha$ we see that $\tau_{\min} = \sqrt{\alpha}\ell_0$. Now, let $\tau \in \mathrm{Spec}_L(g_{(\alpha,\alpha,A)})$ be a dirty length. Then, by Theorem 5.14, we have that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ and, if we express A as $\frac{2j}{k}\alpha$, where j, k are relatively prime, then $\tau = mk\sqrt{A}\ell_0$ for some positive integer m. It follows that if $A < \alpha$, then $k \geq 3$ and, therefore, $\tau = mk\sqrt{A}\ell_0 > \tau_{\min} = \sqrt{A}\ell_0$. Similarly, if $A > \alpha$, then $\tau = mk\sqrt{A}\ell_0 > \tau_{\min} = \sqrt{\alpha}\ell_0$. Therefore, $\tau_{\min}$ is always clean. □

Proof of Theorem 5.14. In Corollary 5.13 we have already established that for each $\tau \in \mathrm{Spec}_L(g_{(\alpha,\alpha,A)})$ the fixed point set $\mathrm{Fix}(\Phi_\tau)$ is the disjoint union of finitely many *homogeneous* submanifolds $N_1, \ldots, N_q$. Hence, our objective is to show that for each $\tau \in \mathrm{Spec}_L(g_{(\alpha,\alpha,A)})$, each $j = 1, \ldots, q \equiv q(\tau)$ and each $u \in N_j$ we have

$$\ker(D_u\Phi_\tau - \mathrm{Id}_u) = T_u(N_j).$$

That is, we must show that the periodic Jaocbi fields $Y(t)$ along the geodesic $\gamma_v(t)$ are precisely those whose initial conditions satisfy $(Y(0), \nabla Y(0)) \in T_v(N_j)$. Since $g_{(\alpha,\alpha,A)}$ is a homogeneous metric, it is enough to verify this for some $v \in T_eG \cap \mathrm{Fix}(\Phi_\tau)$. And, since the connected components are homogeneous, Corollary 4.4 informs us that $\ker(D_v\Phi_\tau - \mathrm{Id}_v) = T_v(\mathrm{Fix}(\Phi_\tau))$ if and only if $V_4^{\mathrm{per}} = V_4^{\mathrm{iso}}$.

In the case where $A = \alpha$, it is clear that the metric is clean since all geodesics are closed and have the same primitive length ℓ_0. Therefore, the remainder of our discussion will focus on the case where $A \neq \alpha$.

Suppose that $A \neq \alpha$. Now, let $\mathfrak{p} \equiv T_eG$ denote the $\mathrm{Ad}(\Delta K)$-invariant complement of $\Delta\mathfrak{K} = \mathrm{Span}\{D\}$ in $\mathfrak{g} \times \mathfrak{K}$. Then, following 5.3(3), the collection $\{Z_1, Z_2, Z_3\}$ forms a g-orthonormal basis for $\mathfrak{p}$. Hence, any unit vector $v \in \mathfrak{p} \equiv T_eG$ is of the form $c_1Z_1 + c_2Z_2 + c_3Z_3$, where $c_1^2 + c_2^2 + c_3^2 = 1$. By Theorem 5.5 the geodesic $\gamma_v(t) = \exp_{G\times K}(tv) \cdot e$ is closed if and only if one of the following hold:

A	B	$[A,B]_{\Delta\mathfrak{K}}$	$[A,B]_{\mathfrak{p}}$
Z_1	Z_2	$\frac{1}{\sqrt{2}(A+\alpha)}D$	$\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_3$
Z_1	Z_3	0	$-\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_2$
Z_2	Z_3	0	$\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_1$
Z_1	D	0	$-\frac{1}{\sqrt{2}}Z_2$
Z_2	D	0	$\frac{1}{\sqrt{2}}Z_1$
Z_3	D	0	0

FIGURE 1. The Lie Bracket in $\mathfrak{g}\times\mathfrak{K}=\mathfrak{p}\oplus\Delta\mathfrak{K}$

(1) $c_1^2+c_2^2=1$ (i.e., γ_v is of Type I);
(2) $c_3=\pm 1$ (i.e., γ_v is of Type II);
(3) $c_1^2+c_2^2=\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}$ and $c_3=\pm\sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$ for some choice of $p,q\in\mathbb{N}$ relatively prime with $\frac{q^2}{p^2}>(\frac{A}{A-\alpha})^2$ (i.e., γ_v is of Type III).

In the case where γ_v is closed we must determine the fixed point set of the associated Poincaré map $P: E\oplus E\to E\oplus E$, where (as in Section 4) $E=\{u\in\mathfrak{p}:\langle u,v\rangle=0\}$. By Corollary 4.4, this means we must determine the subspaces $V_1,\dots,V_4^{\mathrm{per}},V_5\le E\oplus E$. In particular, as noted above, we want to determine whether $V_4^{\mathrm{per}}=V_4^{\mathrm{iso}}$. Towards this end, in Figure 1 we have collected information concerning Lie brackets in $\mathfrak{g}\times\mathfrak{K}=\mathfrak{p}\oplus\Delta\mathfrak{K}$ that will be useful in our computations. We now examine the behavior of the Poincaré map associated to the three types of closed geodesics listed above.

Case I: $v=c_1Z_1+c_2Z_2$ with $c_1^2+c_2^2=1$.

By Theorem 5.5 and Corollary 5.13 we see that $v\in\mathrm{Fix}(\Phi_\tau)$ if and only if $\tau=k\sqrt{\alpha}\ell_0$ for $k\in\mathbb{N}$, in which case the connected component of $\mathrm{Fix}(\Phi_\tau)$ containing v is the 4-dimensional manifold $(G\times K)\cdot v\simeq \mathrm{SO}(3)\times S^1$.

Fix $\tau=k\sqrt{\alpha}\ell_0$. Since $v=c_1Z_1+c_2Z_2$ with $v_1^2+c_2^2=1$ we see that $E=\mathrm{Span}\{c_2Z_1-c_1Z_2,Z_3\}$. We now compute the eigenspaces of the self-adjoint map $B:E\to E$ given by $B(\cdot)=-[v,[v,\cdot]_{\Delta\mathfrak{K}}]$. We have

$$\begin{aligned} B(Z_3) &= -[c_1Z_1+c_2Z_2,[c_1Z_1+c_2Z_2,Z_3]_{\Delta\mathfrak{K}}]\\ &= -[c_1Z_1+c_2Z_2,0]\\ &= 0\end{aligned}$$

and

$$\begin{aligned} B(c_2Z_1 - c_1Z_2) &= [c_1Z_1 + c_2Z_2, [Z_1, Z_2]_{\Delta\mathfrak{K}}] \\ &= [c_1Z_1 + c_2Z_2, \frac{1}{\sqrt{2}(\overline{A}+\alpha)}D] \\ &= \frac{c_1}{\sqrt{2}(\overline{A}+\alpha)}[Z_1, D] + \frac{c_2}{\sqrt{2}(\overline{A}+\alpha)}[Z_2, D] \\ &= -\frac{c_1}{\sqrt{2}(\overline{A}+\alpha)}Z_2 + \frac{c_2}{\sqrt{2}(\overline{A}+\alpha)}Z_1 \\ &= -\frac{1}{\sqrt{2}(\overline{A}+\alpha)}(c_2Z_1 - c_1Z_2). \end{aligned}$$

Hence, $E_0 = \mathrm{Span}\{Z_3\}$ and $E_1 = \mathrm{Span}\{c_2Z_1 - c_1Z_2\}$. Now, let $T : E \to E$ be the skew-symmetric map $T(\cdot) = -[v, \cdot]_{\mathfrak{p}}$. Then

$$\begin{aligned} T(Z_3) &= -c_1[Z_1, Z_3]_{\mathfrak{p}} - c_2[Z_2, Z_3]_{\mathfrak{p}} \\ &= c_1\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_2 - c_2\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_1 \\ &= -\frac{\sqrt{A}}{\alpha\sqrt{2}}(c_2Z_1 - c_1Z_2), \end{aligned}$$

which is an element of E_1, and by skew-adjointness we have $T(c_2Z_1 - c_1Z_3) = \frac{\sqrt{A}}{\alpha\sqrt{2}}Z_3$ which is an element of E_0. Therefore, $E_2 = E_0$ and $E_3 = 0$ which implies $E = E_1 \oplus E_2$. We then find that

$$E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_5.$$

In particular, $V_4 = 0$. Consequently, we conclude that the fixed vectors of P coincide with the isotropic Jacobi fields. It then follows that

$$\ker(D_v\Phi_\tau - \mathrm{Id}_v) = T_v\,\mathrm{Fix}(\Phi_\tau).$$

Case II: $v = \pm Z_3$

By Theorem 5.5 and Corollary 5.13 we see that $v \in \mathrm{Fix}(\Phi_\tau)$ if and only if $\tau = k\sqrt{A}\ell_0$ for $k \in \mathbb{N}$, in which case the connected component of $\mathrm{Fix}(\Phi_\tau)$ containing v is the 3-dimensional manifold $(G \times K) \cdot v \simeq \mathrm{SO}(3)$.

Fix $\tau = k\sqrt{A}\ell_0$, form some $k \in \mathbb{N}$. Since $v = \pm Z_3$, we find that $E = \mathrm{Span}\{Z_1, Z_2\}$. It is then clear that $B \equiv 0$, and we conclude that $E_0 = E$ and $E_1 = 0$. The skew-adjoint map $T : E \to E$ is given by the following:

$$T(Z_1) = -[Z_3, Z_1]_{\mathfrak{p}} = -\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_2$$

and

$$T(Z_2) = -[Z_3, Z_2]_{\mathfrak{p}} = \frac{\sqrt{A}}{\alpha\sqrt{2}}Z_1.$$

Hence, $E_2 \equiv \{\Theta \in E_0 : T(\Theta) \in E_1\} = 0$, $E_3 = E_0 = E$ and we conclude that

$$E \oplus E = V_1 \oplus V_4.$$

Therefore, since V_1 is 2-dimensional and the connected component of $\mathrm{Fix}(\Phi_\tau)$ containing v is 3-dimensional we see that $T_v(\mathrm{Fix}(\Phi_\tau)) = V_1 \oplus \mathrm{Span}\{(v, 0)\}$, which implies $V_4^{\mathrm{iso}} = 0$. This last equality can also be seen by recalling that $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$ gives rise to a non-trivial isotropic Jacobi field along γ_v if and only if $X \neq 0 \in E_3$ is such that $T(X) \in [\Delta\mathfrak{K}, v]$. However, since $[\Delta\mathfrak{K}, v] = 0$ and $T : E \to E$ is an isomorphism, no such vector exists and we see that $V_4^{\mathrm{iso}} = 0$. Hence, if P has non-trivial fixed vectors in V_4 (i.e., $V_4^{\mathrm{per}} \neq 0$), they will not lie in $T_v(\mathrm{Fix}(\Phi_\tau))$.

We now recall that $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$ is fixed by P if and only if $\Psi(X) = X$, where $\Psi : E \to E$ is given by $\Psi = e^{\mathrm{ad}(k\sqrt{A}\ell_0 v)}$. Now, since Z_1 and Z_2 span E and $v = Z_3$, it follows that $\mathrm{ad}\, v = -T$; therefore,

$$\Psi = e^{-k\sqrt{A}\ell_0 T}.$$

With respect to the basis $\{Z_1, Z_2\}$ of E we see that $-k\sqrt{A}\ell_0 T$ is represented by the following matrix

$$\begin{pmatrix} 0 & -\theta(\alpha, A) \\ \theta(\alpha, A) & 0 \end{pmatrix},$$

where $\theta(\alpha, A) = \frac{kA\ell_0}{\alpha\sqrt{2}} = \frac{kA\pi}{\alpha}$. Hence, with respect to the basis $\{Z_1, Z_2\}$, Ψ has the following matrix

$$\begin{pmatrix} \cos\theta(\alpha, A) & \sin\theta(\alpha, A) \\ -\sin\theta(\alpha, A) & \cos\theta(\alpha, A) \end{pmatrix}.$$

Therefore, Ψ has a fixed vector if and only if $\theta(\alpha, A) \in 2\pi\mathbb{N}$, which is equivalent to $A \in \frac{2\alpha}{k}\mathbb{N}$. This implies that $\ker(D_v\Phi_\tau - \mathrm{Id}_v) \neq T_v(\mathrm{Fix}(\Phi_\tau))$ if and only if $A \in \frac{2\alpha}{k}\mathbb{N}$. Since $k \in \mathbb{N}$ is arbitrary, we may conclude that in the case where $A \neq \alpha$ we have $\ker(D_v\Phi_\tau - \mathrm{Id}_v) \neq T_v(\mathrm{Fix}(\Phi_\tau))$ if and only if $A = \frac{2\alpha j}{k}$, where j and k are relatively prime, and $\tau = m(k\sqrt{A}\ell_0)$ for some $m \in \mathbb{N}$.

Case III: $v = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$, where $c_1^2 + c_2^2 = \frac{\sigma(p,q,\alpha,A)}{1+\sigma(p,q,\alpha,A)}$ and $c_3 = \pm\sqrt{\frac{1}{1+\sigma(p,q,\alpha,A)}}$ for unique $p, q \in \mathbb{N}$ relatively prime such that $\frac{q^2}{p^2} > (\frac{A}{\alpha - A})^2$.

By Theorem 5.5 and Corollary 5.13, we see that in this case $v \in \mathrm{Fix}(\Phi_\tau)$ if and only if $\tau = k\sqrt{\alpha}\ell_0(q^2 + p^2\frac{A}{(\alpha - A)})^{\frac{1}{2}}$ for $k \in \mathbb{N}$, in which case the connected component of $\mathrm{Fix}(\Phi_\tau)$ containing v is the 4-dimensional manifold $(G \times K) \cdot v \simeq \mathrm{SO}(3) \times S^1$.

Fix $\tau = k\sqrt{\alpha}\ell_0(q^2+p^2\frac{A}{(\alpha-A)})^{\frac{1}{2}}$ for some $k \in \mathbb{N}$ and notice that $E = \mathrm{Span}\{c_2Z_1 - c_1Z_2, c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3\}$. To find the eigenspaces of $B : E \to E$ we observe that

$$\begin{aligned}
B(c_2Z_1 - c_1Z_2) &= -[v,[v,c_2Z_1 - c_1Z_2]_{\Delta\mathfrak{K}}] \\
&= -[c_1Z_1 + c_2Z_2 + c_3Z_3, -(c_1^2+c_2^2)[Z_1,Z_2]_{\Delta\mathfrak{K}}] \\
&= -[c_1Z_1c_2Z_2 + c_3Z_3, -\frac{(c_1^2+c_2^2)}{\sqrt{2(A+\alpha)}}D] \\
&= \frac{(c_1^2+c_2^2)}{\sqrt{2(A+\alpha)}}(c_1[Z_1,D] + c_2[Z_2,D] + c_3[Z_3,D]) \\
&= \frac{(c_1^2+c_2^2)}{\sqrt{2(A+\alpha)}}(c_2Z_1 - c_1Z_2)
\end{aligned}$$

and

$$\begin{aligned}
B(c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3) &= -[v,[v,c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3]_{\Delta\mathfrak{K}}] \\
&= -[c_1Z_1 + c_2Z_2 + c_3Z_3, c_1c_2c_3[Z_1,Z_2]_{\Delta\mathfrak{K}} - c_1c_2c_3[Z_1,Z_2]_{\Delta\mathfrak{K}}] \\
&= 0.
\end{aligned}$$

Hence, $E_0 = \mathrm{Span}\{c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3\}$ and $E_1 = \mathrm{Span}\{c_2Z_1 - c_1Z_2\}$. We now determine E_2 and E_3 by computing $T : E \to E$:

$$\begin{aligned}
T(c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3) &= -[c_1Z_1 + c_2Z_2 + c_3Z_3, c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3]_{\mathfrak{p}} \\
&= c_1[Z_1,Z_3]_{\mathfrak{p}} + c_2[Z_2,Z_3]_{\mathfrak{p}} \\
&= \frac{\sqrt{A}}{\alpha\sqrt{2}}(c_2Z_1 - c_1Z_2)
\end{aligned}$$

and we also see that

$$T(c_2Z_1 - c_1Z_2) = -\frac{\sqrt{A}}{\alpha\sqrt{2}}(c_1c_3Z_1 + c_2c_3Z_2 - (c_1^2+c_2^2)Z_3).$$

It follows that $E_2 = \{X \in E_0 : T(X) \in E_1\} = E_0$ and $E_3 = 0$, which allows us to see that $E = E_1 \oplus E_2$. Therefore, $V_4 = 0$ and

$$E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_5.$$

Hence, the only fixed vectors of P come from isotropic Jacobi fields and we have

$$\ker(D_v\Phi_\tau - I_v) = T_v(\mathrm{Fix}(\Phi_\tau)).$$

Cases I - III now clearly imply the theorem. Indeed, when $\alpha \neq A$, we see that the cleanliness of $\tau \in \mathrm{Spec}_L(g_{(\alpha,\alpha,A)}$ hinges on the behavior of the Poincaré map along geodesics of length τ having Type II. The conclusion of Case II, then gives us the main statement of the theorem.

$\square$

5.16. **Remark.** It is clear from the proof of Theorem 5.14 that the cleanliness of a metric is dictated by the behavior of the Poincaré map along Type II geodesics.

Proof of Theorem 1.1. The space of naturally reductive left-invariant metrics on SO(3) is identified with $\mathcal{A} = \{(\alpha, A) : \alpha, A > 0\} \subset \mathbb{R}^2$. Now for each $r \in \mathbb{Q}_+$ let $\mathcal{A}_r \equiv \{(\alpha, A) : A = r\alpha\}$. Then it follows from Theorem 5.14 that the class of clean metrics in $\mathcal{A}$ is given by $\mathcal{C} = \cap_{r \neq 1 \in \mathbb{Q}_+}(\mathcal{A} - \mathcal{A}_r)$, which is a residual set containing the bi-invariant metrics $\mathcal{A}_1$.

The final statement follows from the fact that the normal homogeneous metrics on SO(3) are identified with the set $\mathcal{N} = \{(\alpha, A) : \alpha \leq A\}$ and, by Theorem 5.14, we see that $\mathcal{N} \cap (\mathcal{A} - \mathcal{C}) \neq \varnothing$. ∎

5.17. **Proposition.** *Let $g = g_{(\alpha,\alpha,A)}$ be a left-invariant naturally reductive metric on $G = \mathrm{SO}(3)$ with corresponding Sasaki metric $\tilde{g}$ on the tangent bundle. Let τ be a clean length in the length spectrum of g and $d\mu^\tau$ denote the corresponding DG-measure on $\mathrm{Fix}(\Phi_\tau)$ as in Section 2. And, the set $\mathcal{E}_{\tau,\alpha,A}$ and function $\sigma(p.q, \alpha, A)$ are as in Theorem 5.5.*

(1) *If $\alpha = A$, then $\mathrm{Fix}(\Phi_{\tau_{\min}}) = T^1\,\mathrm{SO}(3) = \mathrm{SO}(3) \times S^2_1$ and $d\mu^\tau = d\nu_{\tilde{g}\restriction \mathrm{Fix}(\Phi_\tau)}$. That is, $d\mu_\tau$ is the Riemannian density on $\mathrm{Fix}(\Phi_\tau)$ that is induced by the restriction of the Sasaki metric. And, we have*

$$\int_{\mathrm{Fix}(\Phi_\tau)} d\mu^\tau = \mathrm{vol}(g) \cdot \mathrm{vol}(S^2_1) = 4\pi\,\mathrm{vol}(g)$$

(2) *For $\alpha \neq A$ the components of $\mathrm{Fix}(\Phi_\tau)$ are of Type I, II or III (see Lemma 5.11).*

  (a) *Suppose $\Theta \subset \mathrm{Fix}(\Phi_\tau)$ is a component of Type I. Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \restriction \Theta \equiv \frac{1}{\sqrt{\tau}} d\nu_{\tilde{g}\restriction\Theta}$ and*

$$\int_\Theta d\mu^\tau = \frac{1}{\sqrt{\tau}}\,\mathrm{vol}(g)\,\mathrm{vol}(S^1_1) = \frac{2\pi}{\sqrt{\tau}}\,\mathrm{vol}(g).$$

  (b) *Suppose $\Theta \subset \mathrm{Fix}(\Phi_\tau)$ is a component of Type II. Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \restriction \Theta \equiv \frac{1}{\tau} d\nu_{\tilde{g}\restriction\Theta}$ and*

$$\int_\Theta d\mu^\tau = \frac{1}{\tau}\,\mathrm{vol}(g).$$

  (c) *Suppose $\Theta \subset \mathrm{Fix}(\Phi_\tau)$ is a component of Type III, so that $\Theta = (G \times K) \cdot v_{(p,q)}$ for $(p, q) \in \mathcal{E}_{\tau,\alpha,A}$ and $v_{(p,q)} \in T_eG$ as in Lemma 5.11(3). Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \restriction \Theta \equiv \frac{1}{\sqrt{\tau}} d\nu_{\tilde{g}\restriction\Theta}$ and*

$$\int_\Theta d\mu^\tau = \frac{2\pi}{\sqrt{\tau}}\sqrt{\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}}\,\mathrm{vol}(g).$$

Proof. For this proof, the reader will find it useful to refer to the exposition of the Trace formula in Section 2 and the corresponding notation. By Corollary 5.13, any component Θ of $\mathrm{Fix}(\Phi_\tau)$ is homogeneous. Therefore, it is enough to compute the value of the DG-density at a single point $z \in \Theta$.

Case I: $\alpha = A$

With respect to this metric, the geodesic in $G = \mathrm{SO}(3)$ are all closed and have a common primitive period. Therefore we see that $\Theta = \mathrm{Fix}(\Phi_\tau) = T^1\,\mathrm{SO}(3) = (G \times G)\cdot z$ for any unit vector $z \in T_eG$. Now, fix a unit vector $z \in T_eG$.

Let X_1, X_2, X_3 be an orthonormal basis for $T_e\,\mathrm{SO}(3)$ then

- $\mathcal{E} = \{e_1 = (X_1, 0), e_2 = (X_2, 0), e_3 = (X_3, 0), e_4 = (0, X_1), e_5 = (0, X_2), e_6 = (0, X_3)\}$ is a basis for $W \equiv T_z\Theta = T_z\widetilde{\Theta} = V$.
- W^Ω is trivial.
- $\mathcal{F} = \{f_1 = (0, X_1), f_2 = (0, X_2), f_3 = (0, X_3), f_4 = (X_1, 0), f_5 = (X_2, 0), f_6 = (X_3, 0)\}$ is a basis for a complement of $\mathcal{W}^\Omega$ such that
$$\Omega(e_i, f_j) = \delta_{ij}.$$
- As the complement of W is trivial we take $\mathcal{V} = \varnothing$ and it follows that $T\mathcal{V} = \varnothing$.
- We then see that
$$\begin{aligned} T\mathcal{V} \wedge \mathcal{F} &= \mathcal{F} \\ &= (0, X_1) \wedge (0, X_2) \wedge (0, X_3) \wedge (X_1, 0) \wedge (X_2, 0) \wedge (X_3, 0) \\ &= (-1)^{3\cdot 3}(X_1, 0) \wedge (X_2, 0) \wedge (X_3, 0) \wedge (0, X_1) \wedge (0, X_2) \wedge (0, X_3) \\ &= (-1)\mathcal{E} \\ &= (-1)\mathcal{V} \wedge \mathcal{E} \end{aligned}$$

Therefore, by Lemma 2.3, for any half-density $\varphi \in |V|^{1/2}$ we have the DG-Density is given by
$$\tilde{\mu}^\tau(\mathcal{E}) = \frac{\varphi(\mathcal{V} \wedge \mathcal{E})}{\varphi(T\mathcal{V} \wedge \mathcal{F})} = \frac{1}{|-1|^{1/2}} = 1.$$

It then follows that
$$\mu^\tau = \nu_{\tilde{g}\restriction\Theta}.$$

Case II: $\alpha \neq A$

Suppose τ is a clean length in the length spectrum of g, then a connected component Θ of $\mathrm{Fix}(\Phi_\tau)$ is of Type I, II or III. We will now compute the restriction of μ^τ to Θ in each of these cases.

Subcase IIA: Θ is a Type I component

In this case $\Theta = (G \times K)\cdot z \simeq \mathrm{SO}(3) \times S^1$ for any unit vector $z = c_1 Z_{+} c_2 Z_2 \in T_G \equiv \mathfrak{p}$. Then we observe the following.

- $\mathcal{E} = \{e_1 = (Z_1, 0), e_2 = (Z_2, 0), e_3 = (Z_3, 0), e_4 = (0, Z_1), e_5 = (0, Z_2), \}$ is a basis for $W = T_z\widetilde{\Theta}$.
- $W^\Omega = \mathrm{Span}\{(Z_3, 0)\}$
- $\mathcal{F} = \{f_1 = (0, Z_1), f_2 = (0, Z_2), f_3 = (0, Z_3), f_4 = (-Z_1, 0), f_5 = (-Z_2, 0)\}$ is a basis for a complement of W^Ω such that $\Omega(e_i, f_j) = \delta_{ij}$.

- $\mathcal{V} = \{(0, Z_3)\}$ is a basis for a complement of W
- $T\mathcal{V} = \{(\tau Z_3, 0)\}$
- And, we obtain

$$\begin{aligned} T\mathcal{V} \wedge \mathcal{F} &= (\tau Z_3, 0) \wedge ((0, Z_1) \wedge (0, Z_2) \wedge (0, Z_3) \wedge (-Z_1, 0) \wedge (-Z_2, 0) \\ &= \tau(Z_3, 0) \wedge ((0, Z_1) \wedge (0, Z_2) \wedge (0, Z_3) \wedge (-Z_1, 0) \wedge (-Z_2, 0) \\ &= -\tau(0, Z_3) \wedge (Z_3, 0) \wedge ((0, Z_1) \wedge (0, Z_2) \wedge (Z_1, 0) \wedge (Z_2, 0) \\ &= -\tau(0, Z_3) \wedge (Z_1, 0) \wedge (Z_2, 0) \wedge (Z_3, 0) \wedge ((0, Z_1) \wedge (0, Z_2) \\ &= -\tau \mathcal{V} \wedge \mathcal{E} \end{aligned}$$

Therefore, by Lemma 2.3 for any half-density $\varphi \in |V|^{1/2}$ we have

$$\tilde{\mu}^\tau(\mathcal{E}) = \frac{\varphi(\mathcal{V} \wedge \mathcal{E})}{\varphi(T\mathcal{V} \wedge \mathcal{F})} = \frac{1}{\sqrt{\tau}}.$$

It then follows, by homogeneity of Θ, that

$$\mu^\tau \restriction \Theta = \frac{1}{\sqrt{\tau}} \nu_{\tilde{g} \restriction \Theta}.$$

Subcase IIB: Θ is a Type II component

In this case $\Theta = (G \times K) \cdot z \simeq \mathrm{SO}(3)$ for $v = \pm Z_3 \in T_e G \equiv \mathfrak{p}$. Then we observe the following.

- $\mathcal{E} = \{e_1 = (Z_1, 0), e_2 = (Z_2, 0), e_3 = (Z_3, 0), e_4 = (0, Z_3)\}$ is a basis for $W = T_z \widetilde{\Theta}$.
- $W^\Omega = \mathrm{Span}\{(Z_1, 0), (Z_2, 0)\}$
- $\mathcal{F} = \{f_1 = (0, Z_1), f_2 = (0, Z_2), f_3 = (0, Z_3), f_4 = (-Z_3, 0)\}$ is a basis for a complement of W^Ω such that $\Omega(e_i, f_j) = \delta_{ij}$.
- $\mathcal{V} = \{(0, Z_1), (0, Z_2)\}$ is a basis for a complement of W
- $T\mathcal{V} = \{(\tau Z_1, 0), (\tau Z_2, 0)\}$
- We then see that

$$\begin{aligned} T\mathcal{V} \wedge \mathcal{F} &= (\tau Z_1, 0) \wedge (\tau Z_2, 0) \wedge (0, Z_1) \wedge (0, Z_2) \wedge (0, Z_3) \wedge (-Z_3, 0) \\ &= \tau^2 (0, Z_1) \wedge (0, Z_2) \wedge (Z_1, 0) \wedge (Z_2, 0) \wedge (0, Z_3) \wedge (-Z_3, 0) \\ &= -\tau^2 (0, Z_1) \wedge (0, Z_2) \wedge (Z_1, 0) \wedge (Z_2, 0) \wedge (0, Z_3) \wedge (Z_3, 0) \\ &= \tau^2 (0, Z_1) \wedge (0, Z_2) \wedge (Z_1, 0) \wedge (Z_2, 0) \wedge (Z_3, 0) \wedge (0, Z_3) \\ &= \tau^2 \mathcal{V} \wedge \mathcal{E} \end{aligned}$$

Therefore, by Lemma 2.3 for any half-density $\varphi \in |V|^{1/2}$ we have

$$\tilde{\mu}^\tau(\mathcal{E}) = \frac{\varphi(\mathcal{V} \wedge \mathcal{E})}{\varphi(T\mathcal{V} \wedge \mathcal{F})} = \frac{1}{\tau}.$$

The homogeneity of Θ, then allows us to conclude that

$$\mu^\tau \restriction \Theta = \frac{1}{\tau} \nu_{\tilde{g} \restriction \Theta}.$$

Subcase IIC: Θ is a Type III component

In this case there exists $(p,q)\in\mathcal{E}_{\tau,\alpha,A}$ and a unit vector $z=v_{(p,q)}=c_1Z_1+c_2Z_2+c_3Z_3\in T_eG\equiv\mathfrak{p}$, where $c_1^2+c_2^2=\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}$ and $c_3=\pm\sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$ such that $\Theta=(G\times K)\cdot v_{(p,q)}\simeq \mathrm{SO}(3)\times S^1_r$, where $r=\sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$. Without loss of generality we may assume that $c_1=\sqrt{\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}}$, $c_2=0$ and $c_3=\sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$.

Now, let $z=c_1Z_1+c_3Z_3\in\Theta\subset\widetilde{\Theta}$ and let $v,v^\perp\in T_z\widetilde{\Theta}$ be given by $v=c_1Z_1+c_3Z_3$ and $v^\perp=c_3Z_1-c_1Z_3$. Then

- $\mathcal{E}=\{e_1=(v,0),e_2=(Z_2,0),e_3=(v^\perp,0),e_4=(0,v),e_5=(0,Z_2)\}$ is a basis for $W=T_z\widetilde{\Theta}$.
- $W^\Omega=\mathrm{Span}\{(v^\perp,0)\}$
- $\mathcal{F}=\{f_1=(0,v),f_2=(0,Z_2),f_3=(0,v^\perp),f_4=(-v,0),f_5=(-Z_2,0)\}$ is a basis for a complement of W^Ω such that $\Omega(e_i,f_j)=\delta_{ij}$.
- $\mathcal{V}=\{(0,v^\perp)\}$ is a basis for a complement of W
- $T\mathcal{V}=\{(\tau v^\perp,0)\}$
- We then see that

$$\begin{aligned}
T\mathcal{V}\wedge\mathcal{F} &= (\tau v^\perp,0)\wedge(0,v)\wedge(0,Z_2)\wedge(0,v^\perp)\wedge(-v,0)\wedge(-Z_2,0)\\
&= \tau(v^\perp,0)\wedge(0,v)\wedge(0,Z_2)\wedge(0,v^\perp)\wedge(v,0)\wedge(Z_2,0)\\
&= -\tau(0,v^\perp)\wedge(v^\perp,0)\wedge(0,v)\wedge(0,Z_2)\wedge(v,0)\wedge(Z_2,0)\\
&= -\tau(0,v^\perp)\wedge(v,0)\wedge(Z_2,0)\wedge(v^\perp,0)\wedge(0,v)\wedge(0,Z_2)\\
&= -\tau\mathcal{V}\wedge\mathcal{E}
\end{aligned}$$

Therefore, by Lemma 2.3 for any half-density $\varphi\in|V|^{1/2}$ we have

$$\tilde{\mu}^\tau(\mathcal{E})=\frac{\varphi(\mathcal{V}\wedge\mathcal{E})}{\varphi(T\mathcal{V}\wedge\mathcal{F})}=\frac{1}{\sqrt{\tau}}.$$

From the homogeneity of Θ, we conclude that the DG-measure is give by

$$\mu^\tau\restriction\Theta=\frac{1}{\sqrt{\tau}}\nu_{\tilde{g}\restriction\Theta}.$$

The theorem now follows from Cases I and II. $\square$

5.18. **Proposition.** *Let $g=g_{(\alpha,\alpha,A)}$ be a left-invariant naturally reductive metric on $\mathrm{SO}(3)$. Let $\tau_{\min}=\tau_{\min}(g)$ denote the length of the shortest closed geodesic with respect to g and σ denote the Morse index of any smooth closed geodesic with respect to g having length $\tau_{\min}$.*

(1) *If $\alpha=A$, then $\tau_{\min}=\sqrt{\alpha}\ell_0$ is clean, $\mathrm{Fix}(\Phi_{\tau_{\min}})$ is a connected manifold of dimension 5, and all of the closed geodesics of length $\tau_{\min}$ have Morse index $\sigma=0$. Furthermore,*

$$\mathrm{Wave}_0^{\mathrm{odd}}(\tau_{\min})=-\frac{1}{\pi}\,\mathrm{vol}(g)$$

and $\mathrm{Wave}_k^{\mathrm{even}}(\tau_{\min})=0$ for $k\geq 0$.

(2) *If* $A < \alpha$, *then* $\tau_{\min} = \sqrt{A}\ell_0$ *is clean,* $\mathrm{Fix}(\Phi_{\tau_{\min}})$ *is a a manifold of dimension 3 having two connected components, and the closed geodesics of length* $\tau_{\min}$ *all have the same Morse index* σ. *Furthermore,*

$$\mathrm{Wave}_0^{\mathrm{odd}}(\tau_{\min}) = \frac{i^{-(\sigma+1)}}{\pi}\frac{\mathrm{vol}(g)}{\tau_{\min}}.$$

$\mathrm{Wave}_k^{\mathrm{even}}(\tau_{\min}) = 0$ *for* $k \geq 0$.

(3) *If* $A > \alpha$, *then* $\tau_{\min} = \sqrt{\alpha}\ell_0$ *is clean,* $\mathrm{Fix}(\Phi_{\tau_{\min}})$ *is a connected manifold of dimension 4, and the closed geodesics of length* $\tau_{\min}$ *all have common Morse index* σ. *Furthermore,*

$$\mathrm{Wave}_0^{\mathrm{even}}(\tau_{\min}) = \left(\frac{1}{2\pi i}\right)^{3/2} i^{-\sigma}\frac{2\pi}{\sqrt{\tau_{\min}}}\,\mathrm{vol}(g).$$

$\mathrm{Wave}_k^{\mathrm{odd}}(\tau_{\min}) = 0$ *for* $k \geq 0$.

Proof.

(1) By Corollary 5.13 we see that $\tau_{\min} = \sqrt{\alpha}\ell_0$ and $\mathrm{Fix}(\Phi_{\tau_{\min}})$ is the unit tangent bundle with respect to g. It is also clear that the geodesics of length $\tau_{\min}$ must have Morse index 0. It then follows from Theorem 2.4 and Proposition 5.17 that

$$\begin{aligned}\mathrm{Wave}_0^{\mathrm{odd}}(\tau_{\min}) &= \left(\frac{1}{2\pi i}\right)^{\frac{5-1}{2}} i^{-\sigma}\int_{T^1\,\mathrm{SO}(3)} d\nu_{\tilde{g}\restriction T^1\,\mathrm{SO}(3)}\\ &= -\frac{1}{4\pi^2}4\pi\,\mathrm{vol}(g)\\ &= -\frac{1}{\pi}\,\mathrm{vol}(g)\end{aligned}$$

And, since $\mathrm{Fix}(\Phi_{\tau_{\min}})$ has no even-dimensional components, we see that $\mathrm{Wave}_k^{\mathrm{even}}(\tau_{\min}) = 0$ for any $k \geq 0$.

(2) By Corollary 5.13 we see that $\mathrm{Fix}(\Phi_{\tau_{\min}}) = \Theta_+ \cup \Theta_- \simeq \mathrm{SO}(3) \cup \mathrm{SO}(3)$, where $\Theta_\pm = (G \times K)\cdot(\pm Z_3) = \mathrm{SO}(3) \times \{\pm Z_3\} \subset T\,\mathrm{SO}(3)$. It is clear that the geodesics of this length are translates of each other or the reverse parametrization, thereofre they all have a common Morse index σ It then follows from Theorem 2.4 and Proposition 5.17 that

$$\begin{aligned}\mathrm{Wave}_0^{\mathrm{odd}}(\tau_{\min}) &= \left(\frac{1}{2\pi i}\right)^{\frac{3-1}{2}} i^{-\sigma}\left(\int_{\Theta_+} d\mu_{\tau_{\min}} + \int_{\Theta_-} d\mu_{\tau_{\min}}\right)\\ &= \frac{1}{2\pi i}i^{-\sigma}\frac{2}{\tau_{\min}}\,\mathrm{vol}(g)\\ &= \frac{i^{-(\sigma+1)}}{\pi\tau_{\min}}\,\mathrm{vol}(g)\end{aligned}$$

And, since $\mathrm{Fix}(\Phi_{\tau_{\min}})$ has no even-dimensional components, we see that $\mathrm{Wave}_k^{\mathrm{even}}(\tau_{\min}) = 0$ for any $k \geq 0$

(3) By Corollary 5.13 we see that $\mathrm{Fix}(\Phi_{\tau_{\min}}) = (G \times K) \cdot v \simeq \mathrm{SO}(3) \times S^1$ for a unit vector $v = c_1 Z_1 + c_2 Z_2 \in T_e G$. The geodesics of length $\tau_{\min}$ are clearly all translates of each other by the isometry group, so they have a common Morse index σ. It then follows from Theorem 2.4 and Proposition 5.17 that

$$\begin{aligned} \mathrm{Wave}_0^{\mathrm{even}}(\tau_{\min}) &= \left(\frac{1}{2\pi i}\right)^{\frac{4-1}{2}} i^{-\sigma} \int_{SO(3)\times S^1} d\mu_{\tau_{\min}} \\ &= \left(\frac{1}{2\pi i}\right)^{\frac{3}{2}} i^{-\sigma} \frac{2\pi}{\sqrt{\tau_{\min}}} \mathrm{vol}(g) \end{aligned}$$

And, since $\mathrm{Fix}(\Phi_{\tau_{\min}})$ has no odd-dimensional components, we see that $\mathrm{Wave}_k^{\mathrm{even}}(\tau_{\min}) = 0$ for any $k \geq 0$

$\square$

Proof of Theorem 1.2. Let g be a left-invariant naturally reductive metric on $\mathrm{SO}(3)$. By Proposition 5.18 the length $\tau_{\min} = \tau_{\min}(g)$ is clean and one of $\mathrm{Wave}_0^{\mathrm{even}}(\tau_{\min})$ or $\mathrm{Wave}_0^{\mathrm{odd}}(\tau_{\min})$ is non-zero. Therefore, by Theorem 2.4(4), $\tau_{\min}$ is in the singular support of the trace of the wave group associated to g. And, being the smallest non-zero element in $\mathrm{SingSupp}(\mathrm{Trace}(U_g(t)))$ we see that $\tau_{\min}$ can be recovered from the spectrum of g $\square$

It follows immediately from Theorem 1.2 and Proposition 5.18 that we have the following result which states that the volume of a left-invariant naturally reductive metric g on $\mathrm{SO}(3)$ can be recovered from the asymptotic expansion of the trace of its wave group at the singularity $\tau_{\min}(g)$.

5.19. **Corollary.** *There is a function $f(\cdot,\cdot,\cdot)$ such that for any $g \in \mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$ we have*

$$\mathrm{vol}(g) = f(\dim \mathrm{Fix}(\Phi_{\tau_{\min}(g)}), \mathrm{Wave}_0^{\bullet}(\tau_{\min}(g)), \tau_{\min}(g)),$$

where $\bullet$ denotes the parity of $\dim \mathrm{Fix}(\Phi_{\tau_{\min}(g)})$.

Proof of Theorem 1.3. Let $g = g_{(\alpha,\alpha,A)}$ be a left-invariant naturally reductive metric on $\mathrm{SO}(3)$ and let $\tau_{\min} = \tau_{\min}(g)$ denote the length of the shortest non-trivial closed geodesic with respect to g. By Theorem 1.2 $\tau_{\min}$ is clean and is determined by the spectrum of g. Furthermore, using the asymptotic expansion of the wave trace at the singularity $\tau_{\min}$ we conclude that the dimension of the manifold $\mathrm{Fix}(\Phi_{\tau_{\min}})$ is determined by the spectrum of g and takes on the values $3, 4$ or 5. We will now show that in each of these cases α and A can be expressed in terms of the spectrally determined data $\tau_{\min}$ and $\mathrm{vol}(g) = \alpha\sqrt{A}V_0$, where V_0 is defined as in 5.3.

Case I: $\dim \mathrm{Fix}(\Phi_{\tau_{\min}}) = 5$

It follows from Proposition 5.18 that $\alpha = A$ and $\tau_{\min} = \sqrt{\alpha}\ell_0$, so that $\alpha = A = \left(\frac{\tau_{\min}}{\ell_0}\right)^2$.

Case II: $\dim \mathrm{Fix}(\Phi_{\tau_{\min}}) = 4$

In this case, Proposition 5.18 implies that $A > \alpha$ and $\tau_{\min} = \sqrt{\alpha}\ell_0$. Therefore, $\alpha = \left(\frac{\tau_{\min}}{\ell_0}\right)^2$. Also, since $\mathrm{vol}(g) = \alpha\sqrt{A}V_0$ we see that $A = \left(\frac{\mathrm{vol}(g)\ell_0^2}{\tau_{\min}^2 V_0}\right)^2$

Case III: $\dim \mathrm{Fix}(\Phi_{\tau_{\min}}) = 3$

Here, Proposition 5.18 tells us that $A < \alpha$ and $\tau_{\min} = \sqrt{A}\ell_0$. It follows that $A = \left(\frac{\tau_{\min}}{\ell_0}\right)^2$ and $\alpha = \frac{\mathrm{vol}(g)\ell_0}{\tau_{\min}V_0}$. □

5.20. **Remark.** In the introduction we noted that in [Sut] we produced examples of normal homogeneous manifolds of the form $\mathrm{SU}(n)/H_1$ and $\mathrm{SU}(n)/H_2$, which are isospectral. Hence, there is no hope of showing that within the class of all naturally reductive manifolds each space is uniquely determined by its spectrum.

The 0-th wave invariants and the Poisson Relation. In light of comments made in the Introduction it is natural to wonder whether it is possible to use the 0-th wave invariants to establish equality in the Poisson relation for the clean metrics in $\mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$. Let $g_{(\alpha,\alpha,A)}$ be such a clean metric. When $\alpha = A$ we have a bi-invariant metric on SO(3), which is a CROSS, and as we noted in Example 2.9 the Poisson relation is an equality for all CROSSes. So, we consider the case where $g_{(\alpha,\alpha,A)}$ is clean and $\alpha \neq A$.

First, we fix an element τ in the length spectrum of $g_{(\alpha,\alpha,A)}$ and recall that $\mathrm{Fix}(\Phi_\tau)$ consists of components of Type I, II and III. We observe that $\mathrm{Fix}(\Phi_\tau)$ cannot contain components of Type I and Type II simultaneously. Indeed, if this were the case, then we could find natural numbers m and n such that $\tau = m\sqrt{\alpha}\ell_0 = n\sqrt{A}\ell_0$, which would imply that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$, contradicting the fact that the metric $g_{(\alpha,\alpha,A)}$ is clean (see Theorem 5.14). It is also the case that Type I components cannot occur along with Type III components. For otherwise, there exist natural numbers m and n such that $\tau = m\sqrt{\alpha}\ell_0 = n\sqrt{\alpha}\ell_0(q^2 + p^2\frac{A}{\alpha - A})^{1/2}$, which implies that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ and leads us to conclude that the metric $g_{(\alpha,\alpha,A)}$ is actually unclean, which is a contradiction.

Now, let $\tau = n\sqrt{\alpha}\ell_0$ be the length of Type I geodesic. Then, the previous paragraph dictates that $\mathrm{Fix}(\Phi_\tau)$ consists of the lone Type I component. Therefore, since the Type I component is of dimension 4 we see that $\mathrm{Wave}_0^{\mathrm{even}}(\tau) \neq 0$. Therefore, the length of any Type I geodesic is contained in the singular support of the trace of the wave group.

To analyze lengths arising from Type II and Type III geodesics, we recall that the Type II and Type III components are all of dimension 3. If the only odd-dimensional components in $\mathrm{Fix}(\Phi_\tau)$ are the Type II components $\Theta_1 = (G \times K) \cdot v$ and $\Theta_2 = (G \times K) \cdot -v$, where v is the initial velocity of some unit speed Type II geodesic (see Lemma 5.11), then since the Morse index associated to these components is clearly the same, we conclude that $\mathrm{Wave}_0^{\mathrm{odd}}(\tau)$ is non-zero and, therefore, τ is also in the singular support of the trace of the wave group.

An issue arises in using the 0-th wave invariants when Type III components occur in $\mathrm{Fix}(\Phi_\tau)$. Indeed, one can show that the conjugate points along a Type III geodesics are as follows.

5.21. **Proposition.** *Suppose $\gamma_{v_{(p,q)}}$ is a Type III geodesic with $v_{(p,q)} = c_1Z_1 + c_2Z_2 + c_3Z_3 \in \mathfrak{p} \equiv T_eG$, where $c_1^2 + c_2^2 = \frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}$ and $c_3 = \pm\sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$ for a unique $(p,q) \in \mathcal{E}_{\tau,\alpha,A}$. And, let $a(p,q,\alpha,A) = \sqrt{\varphi^2 + \frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}\frac{1}{2(\overline{A}+\alpha)}}$, where $\varphi \equiv \frac{1}{\alpha}\sqrt{\frac{A}{2}}$ and $\overline{A} \equiv \frac{A\alpha}{\alpha - A}$.*

1. *If $\alpha > A$, then $\gamma_{v_{(p,q)}}(t_0)$, $t_0 > 0$, is conjugate to $e = \gamma_{v_{(p,q)}}(0)$ along $\gamma_{v_{(p,q)}}$ if and only if $t_0 \neq 0 \in \frac{2\pi}{a(p,q,\alpha,A)}\mathbb{N}$. And, in this case the conjugate point has multiplicity one.*
2. *If $\alpha < A$, then $\gamma_{v_{(p,q)}}(t_0)$, $t_0 > 0$, is conjugate to $e = \gamma_{v_{(p,q)}}(0)$ along $\gamma_{v_{(p,q)}}$ if and only if $t_0 \neq 0 \in \frac{2\pi}{a(p,q,\alpha,A)}\mathbb{N}$ or $t_0 = \frac{4\alpha^2}{(A-\alpha)\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}}$. And, in this case the conjugate point has multiplicity one.*

Proof. We omit the long computation, which makes use of Ziller's recasting of the Jacobi equation for naturally reductive metrics [Z2] and our explicit understanding of the Poincaré map. □

Using the previous proposition one can compute the Morse index associated to each Type III component $(G \times K) \cdot v_{(p,q)}$ contained in $\mathrm{Fix}(\Phi_\tau)$. This, in conjunction with computations in the spirit of those used to establish Proposition 5.17(2c), allows one to compute the contribution of each Type III component to the wave invariant $\mathrm{Wave}_0^{\mathrm{odd}}(\tau)$. However, some inspection will demonstrate that these contributions behave rather erratically making it difficult to rule out the possibility of cancellations, in general. Therefore, the best we can say at the moment is the following.

5.22. **Proposition.** *Let $g_{(\alpha,\alpha,A)}$ be a clean left-invariant naturally reductive metric on $\mathrm{SO}(3)$ and τ an element in the length spectrum of $g_{(\alpha,\alpha,A)}$. If τ is a multiple of $\sqrt{\alpha}\ell_0$, or τ is a multiple of $\sqrt{A}$ for which $\mathrm{Fix}(\Phi_\tau)$ contains no Type III components, then τ is in the singular support of the trace of the wave group of $g_{(\alpha,\alpha,A)}$.*

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Dartmouth College, Department of Mathematics, Hanover, NH 03755
E-mail address: craig.j.sutton@dartmouth.edu