Math 123: Topics in Analysis
Automorphic forms, representations and C*-algebras

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References

[K] Introduction to the representation theory of groups, by E. Kowalski.
[tfb²] Crossed products of C*-algebras, by D. P. Williams.

Other sources were used, including P. Garrett’s vignettes and W. Casselman’s essays.

Syllabus

I. Overview of representation theory, automorphic forms and applications
- 3 lectures + 1 guest lecture
1. Group representations and harmonic analysis on homogeneous spaces
2. Selberg’s $\frac{1}{4}$ Conjecture
3. L-functions and applications

II. Waveforms for cocompact lattices of $SL(2, \mathbb{R})$ - 14 lectures
4. Maass forms and the spectral problem, unbounded operators
5. Differential operators and Lie algebras
6. The Cartan decomposition and $K$-bi-invariant functions
7. Discreteness of the spectrum

III. Admissible and unitary representations of $SL(2, \mathbb{R})$ - 7 lectures
8. Admissible $(g, K)$-modules
9. Irreducible $(g, K)$-modules for $SL(2, \mathbb{R})$
10. Unitarizability of admissible representations
12. The unitary dual of $SL(2, \mathbb{R})$ and the solution of the spectral problem

IV. The noncommutative geometry point of view - 3 lectures
11. Induced representations and Frobenius reciprocity for finite groups
13. Parabolic induction in the $C^*$-algebraic framework and applications

Week 1

Lecture 1. Topological groups: Haar measure(s), modular function.
Group representations: examples, unitary representations, continuous representations.
The left regular representation $\lambda_G$ is unitary and continuous. Irreducible representations.
Case of the torus: $\hat{T} = \{\chi_n, n \in \mathbb{Z}\}$ with $\chi_n$ acting on $C_n = \mathbb{C}$ for every $n$. The $L^2$-theory of Fourier series gives a decomposition of the regular representation into irreducibles:

$$L^2(\hat{T}) = \sum_{n \in \mathbb{Z}} \nolimits C_n.$$ 

References: [tfb², §1.3] and [K, §3.3-3.4].
Lecture 2. Fourier transform: \( L^2(\mathbb{R}) \simeq \int_{\xi \in \mathbb{R}} C_{\xi} \, d\xi \) as a unitary representation of \( \mathbb{R} \).

General case of locally compact abelian groups: unirreps are one-dimensional, form a locally compact group (the Pontrjagyn dual \( \hat{G} \)) and Fourier theory yields:

\[
L^2(G) \simeq \int_{\chi \in \hat{G}} C_{\chi} \, d\chi.
\]

Definition of unitary equivalence, the unitary dual of a locally compact group:

\[
\hat{G}_u = \{\text{irreducible continuous unitary representations of } G\}/\text{unitary equivalence}.
\]

Example: \( \widehat{\text{SO}(3)} = \{H_\ell, \ell \in \mathbb{N}\} \) where \( \dim H_\ell = 2\ell + 1 \). Summary of Peter-Weyl theory: for \( G \) compact, the regular representation decomposes as

\[
L^2(G) \simeq \sum_{\pi \in \hat{G}} \dim(H_\pi) H_\pi
\]

and the inversion formula reads

\[
f = \sum_{\pi \in \hat{G}} \dim(H_\pi) \left(\text{Tr } \pi \ast f\right).
\]

General case of real reductive groups: Harish-Chandra proved the existence of (and determined) a measure \( \mu \) on \( \hat{G} \) such that, for \( f \in C_c(G) \),

\[
f = \int_{\pi \in \hat{G}} (\Theta_\pi \ast f) \, d\mu(\pi)
\]

where each \( \Theta_\pi \) is a distribution on \( G \), generalizing the trace.

Case of \( \text{SL}(2, \mathbb{R}) \) (Bargmann, 1947): the unitary dual consists of

- the discrete series;
- the principal series and the limits of discrete series;
- the complementary series and the trivial representation.

The concrete Plancherel formula for \( f \in C_c(\text{SL}(2, \mathbb{R})) \) is

\[
f = \sum_{n \in \mathbb{Z}} |n| (\Theta_n \ast f) + \frac{1}{4} \int_{-\infty}^{+\infty} (\Theta_{\nu_1} \ast f) \nu_1 \tanh \left(\frac{\pi\nu_1}{2}\right) \, d\nu_1 + \frac{1}{4} \int_{-\infty}^{+\infty} (\Theta_{\nu_2} \ast f) \nu_2 \coth \left(\frac{\pi\nu_2}{2}\right) \, d\nu_2.
\]

Some representations (the complementary series) do not appear in the Plancherel formula. The ones that do are called tempered and form a closed subspace \( \hat{G}_r \subset \hat{G} \).

The discrete series behave like representations of compact groups: the factor \( |n| \) should be interpreted as a formal dimension for these representations, which are characterised by the fact that they are actual subrepresentations of the regular and that their matrix coefficients are square-integrable. The other tempered representations are only weakly contained in the regular and have almost square-integrable matrix coefficients.

Lecture 3. Quasi-regular representations: if \( G \) acts on a space \( X \) that carries a \( G \)-invariant Borel measure \( \mu \), then \( L^2(X, \mu) \) is a unitary representation of \( G \). Criterion for the existence of a \( G \)-invariant measure on a homogeneous space \( G/H \): \( \Delta_G|_H = \Delta_H \). In
particular, this is satisfied when $G$ is reductive and $H$ is discrete. Case of $G = \text{SL}(2, \mathbb{R})$ and $H = \Gamma(N)$ or a congruence subgroup.

**Theorem** (Gelfand, Graev, Piatetski-Shapiro). As a unitary representation of $G$, $L^2(\Gamma \backslash G) \simeq \mathcal{H}_1 \oplus \mathcal{H}_2$

where

- $\mathcal{H}_1$ is a direct sum indexed by a countable subset of $\hat{G}$:
  $$\mathcal{H}_1 = \bigoplus_{\pi} m_{\pi} \mathcal{H}_{\pi}$$

- $\mathcal{H}_2$ is a direct integral of principal series representations of $G$:
  $$\mathcal{H}_2 = \int_{\nu \in \mathbb{R}} m_{\Gamma} \mathcal{H}_{\nu} \, d\nu$$

where $m_{\Gamma}$ only depends on $\Gamma$. In fact, $m_{\Gamma} = 0$ is $\Gamma$ is a cocompact lattice. In general, it is equal to the number of cusps of $\Gamma$.

Selberg conjectured in 1965 that no complementary series occurs in $\mathcal{H}_1$ if $\Gamma = \Gamma(N)$. $K$-fixed vectors in spherical representations are smooth and eigenfunctions of the hyperbolic Laplace operator. Conversely, to an automorphic form $f$ with eigenvalue $\lambda$, one can associate a representation $\pi_f$ and $\pi_f$ is in the complementary series if and only if $\lambda < \frac{1}{4}$. Selberg’s $\frac{1}{4}$ Conjecture is still open in general but Selberg proved that $\text{Sp} \Delta \subset \left[ \frac{3}{16}, +\infty \right)$.

**Reference:** [K, §7.4].

**Week 2**

**Lecture 4.** Reciprocal sums of primes numbers, Dirichlet’s Arithmetic Progression. Euler product for Riemann’s $\zeta$ function, estimates near 1. Dirichlet characters and associated $L$-series:

$$\sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$  

Proof of a special case of Dirichlet’s Theorem: $\sum_{p \equiv 1[4]} \frac{1}{p}$ diverges.

Maass cusp forms. Periodicity and Fourier expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi n x}$$

with $a_n(y) = c_n \sqrt{y} K_\nu(2\pi ny)$ where $K_\nu$ is a Bessel function. The corresponding $L$-function

$$L(s, f) = \sum_{n \geq 1} \frac{c_n}{n^s}$$

satisfies a functional equation.

**References:** [B, §1.9].
Lecture 5. Given a weight $k \in \mathbb{Z}$, the Maass operators on the Poincaré plane $\mathcal{H}$ are

$$R_k = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{k}{2} = \left( z - \bar{z} \right) \frac{\partial}{\partial z} + \frac{k}{2}$$

and

$$L_k = -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{k}{2} = \left( z - \bar{z} \right) \frac{\partial}{\partial \bar{z}} - \frac{k}{2}.$$ 

The weight $k$ non-Euclidean Laplacian is

$$\Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x}.$$ 

They are related via

$$-L_{k+2}R_k - \frac{k}{2} \left( 1 + \frac{k}{2} \right) = \Delta_k = -R_{k-2}L_k - \frac{k}{2} \left( 1 - \frac{k}{2} \right).$$

For each $k \in \mathbb{Z}$, the group $G = \text{GL}(2, \mathbb{R})^+$ acts on the right on $C^\infty(\mathcal{H})$ by

$$f |_{k+2} g = \left( \frac{cz + d}{cz + d} \right)^k f \left( \frac{az + b}{cz + d} \right)$$

where $g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$. The Maass operators satisfy the following equivariance relations:

$$(R_k f) |_{k+2} g = R_k (f |_{k} g)$$

$$(L_k f) |_{k-2} g = L_k (f |_{k} g)$$

$$(\Delta_k f) |_{k} g = \Delta_k (f |_{k} g).$$

We will study the operators $\Delta_k$ in the context of Hilbert spaces, that is as unbounded operators. Adjoint of a densely defined operator.

References: [R, Chap. 13], [B, §2.1].


A densely defined operator $T$ is symmetric if $T \subset T^*$, that is

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in \mathcal{D}(T)$. It is self-adjoint if $T = T^*$.

Symmetric operators may or may not have self-adjoint extensions: example of $i \frac{d}{dx}$ on $L^2([0,1])$, with various domains, after Rudin [R, Chap. 13], [B, §2.1].

The measure $\frac{dx \wedge dy}{y^2}$ is SL(2, $\mathbb{R}$)-invariant (use Bruhat decomposition to shorten the verification). Green’s formula for the Euclidean Laplacian $\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$:

$$\int_{\Omega} (g \Delta^e f - f \Delta^e g) \, dx \wedge dy = \int_{\partial \Omega} g \left( \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) - f \left( \frac{\partial g}{\partial x} dy - \frac{\partial g}{\partial y} dx \right).$$
Week 3

Lecture 7. The hyperbolic Laplacian \( \Delta_k, C_c^\infty(\mathcal{H}) \) is a symmetric operator on \( L^2(\mathcal{H}) \). Let \( \Gamma \) be a subgroup of \( \text{SL}(2, \mathbb{R}) \) acting discontinuously on \( \mathcal{H} \), \( \chi \in \text{Hom}(\Gamma, \mathbb{T}) \) a character, \( k \in \mathbb{Z} \) a weight and define \( C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \) as

\[
\left\{ f \in C_c^\infty(\mathcal{H}) \mid \forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \quad f(\gamma \cdot z) = \chi(\gamma) \left( \frac{cz+d}{|cz+d|} \right)^{-k} f(z) \right\}
\]

with the compatibility assumption \( \chi(-I_2) = (-1)^k \).

If \( f, g \in C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \), then \( f\bar{g} \) is \( \Gamma \)-invariant and one can define

\[
\langle f, g \rangle = \int_{\Gamma \setminus \mathcal{H}} f(z)\overline{g(z)} \frac{dx \, dy}{y^2}
\]

and complete \( C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \) into a Hilbert space, denoted by \( L^2(\Gamma \setminus \mathcal{H}, \chi, k) \).

Behaviour of the Maass operators:

\[
R_k : C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \rightarrow C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k + 2)
\]

\[
L_k : C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \rightarrow C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k - 2)
\]

\[
\Delta_k : C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \rightarrow C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k)
\]

and

\[
\langle R_k f, g \rangle = \langle f, -L_k g \rangle
\]

for \( f \) and \( g \) in spaces with appropriate weights.

It follows that \( \Delta_k \) is a symmetric operator on \( L^2(\Gamma \setminus \mathcal{H}, \chi, k) \).

**Spectral problem (v.1):** determine the spectrum of \( \Delta_k \) on \( L^2(\Gamma \setminus \mathcal{H}, \chi, k) \).

References: [B, §2.1].

Lecture 8. Definition of *Maass forms of weight* \( k \) as elements of \( C_c^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \cap \text{Sp}(\Delta_k) \).

Generalities on Iwasawa decomposition and decompositions of Haar measure.

In the case of \( G = \text{SL}(2, \mathbb{R}) \), every element \( g \) can be written uniquely as

\[
g = \begin{bmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{with } \theta \in \mathbb{R}/2\pi \mathbb{Z}, \quad x, y \in \mathbb{R}_+ \quad \text{and the Haar measure decomposes accordingly in these coordinates:}
\]

\[
dg = \frac{dx \, dy}{y^2} \, d\theta.
\]

Given a character \( \chi \in \text{Hom}(\Gamma, \mathbb{T}) \), consider

\[
L^2(\Gamma \setminus G, \chi) = \left\{ f \in L^2(G) \mid \forall \gamma \in \Gamma, \quad f(\gamma \cdot z) = \chi(\gamma) f(z) \right\}.
\]
It is a Hilbert space for the inner product
\[ \langle f_1, f_2 \rangle = \int_{\Gamma \setminus G} f_1(g) \overline{f_2(g)} \, d\hat{g} \]
and smooth functions constitute a dense subspace. Moreover, letting \( G \) act by right translation, \( L^2(\Gamma \setminus G, \chi) \) is a continuous unitary representation of \( G \).

**Spectral problem (v.2):** decompose \( L^2(\Gamma \setminus G, \chi) \) into irreducibles.

**References:** [B, §2.1]. See also Knapp’s books for Iwasawa decomposition and determination of measures.

**Week 4**

**Lecture 9.** \( K \)-isotypic decomposition of a unitary representation, proof in the case of \( \text{SL}(2, \mathbb{R}) \), by means of Fejér’s kernel.

Admissible representations. Harish-Chandra’s Admissibility Theorem: unitary irreducible representations of reductive groups are admissible. Consider the \( K \)-isotypic decomposition of \( L^2(\Gamma \setminus G, \chi) \):
\[
L^2(\Gamma \setminus G, \chi) = \sum_{k \in \mathbb{Z}} L^2(\Gamma \setminus G, \chi, k)
\]
where
\[
L^2(\Gamma \setminus G, \chi, k) = \{ f \in L^2(\Gamma) : \forall \gamma \in \Gamma, \forall \theta \in \mathbb{R}/2\pi \mathbb{Z}, \ f(\gamma g R_\theta) = \chi(\gamma) e^{ik\theta} f(g) \}.
\]
The map \( \sigma_k \) defined on \( C^\infty(\Gamma \setminus \mathcal{H}, \chi, k) \) by
\[
\sigma_k f(g) = (f|_{\Gamma \setminus \mathcal{H}})(i)
\]
is an isometric isomorphism of Hilbert spaces:
\[
\sigma_k : L^2(\Gamma \setminus \mathcal{H}, \chi, k) \xrightarrow{\sim} L^2(\Gamma \setminus G, \chi, k).
\]

**References:** [B, §2.1]. See also Katznelson for details about the Fejér Kernel.

**Lecture 10.** Image of Maass operators under the isomorphisms \( \sigma_k \):
\[
\sigma_{k+2} \circ R_k = R \circ \sigma_k
\]
\[
\sigma_{k-2} \circ L_k = L \circ \sigma_k
\]
\[
\sigma_k \circ \Delta_k = \Delta \circ \sigma_k
\]
where \( R, L \) and \( \Delta \) are given in the coordinates \( x, y, \theta \) of the Iwasawa decomposition (1) by:
\[
R = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta}
\]
\[
L = -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta}
\]
\[ \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta}. \]

Lie algebras: definition, Lie algebra \( \text{Lie}(A) \) associated with an associative algebra \( A \): \([a, b] = ab - ba.\)

The Lie algebra of a closed subgroup \( G \) of \( \text{GL}(n, \mathbb{R}) \):
\[
\mathfrak{g} = \left\{ x \in M_n(\mathbb{R}) , \forall t \in \mathbb{R}, e^{tx} \in G \right\}.
\]

Examples: using the relation \( \det(e^x) = e^{\text{Tr}x} \), one proves that
\[
\mathfrak{so}(n, \mathbb{R}) = \left\{ x \in M_n(\mathbb{R}) , x^\top + x = 0 \right\}
\]
\[
\mathfrak{sl}(n, \mathbb{R}) = \left\{ x \in M_n(\mathbb{R}) , \text{Tr}x = 0 \right\}
\]
\[
\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}).
\]

References: \([B, \S 2.2]\) and P. Garrett’s notes on Invariant differential operators.

Lecture 11. If a Lie group \( G \) acts smoothly on the right of a manifold \( \mathcal{M} \), then it acts on \( C^\infty(\mathcal{M}) \) via
\[
g \cdot f(m) = f(m \cdot g)
\]
and \( \mathfrak{g} \) acts by the differential operators \( X_x \) where
\[
X_x f(m) = \frac{d}{dt} \bigg|_{t=0} f(m \cdot e^{tx}).
\]

These two actions do not commute but they satisfy, for \( g \in G \) and \( x \in \mathfrak{g} \),
\[
gX_x g^{-1} = X_{\text{Ad}(g)x}
\]
where the adjoint representation \( \text{Ad} : G \to \text{End}(\mathfrak{g}) \) is defined by \( \text{Ad}(g)x = gxg^{-1} \). We admit (for now) the important fact that \( x \mapsto X_x \) is a Lie algebra morphism, that is,
\[
X_{[x,y]} = X_x X_y - X_y X_x.
\]

The universal enveloping algebra: there is an (associative) algebra \( \mathcal{U}(\mathfrak{g}) \) such that for every algebra \( A \),
\[
\text{Hom}_{\text{assoc.}}(\mathcal{U}(\mathfrak{g}), A) = \text{Hom}_{\text{Lie}}(\mathfrak{g}, \text{Lie}(A)).
\]
In other words, the functor \( \mathcal{U}(\cdot) \) is a left adjoint for \( \text{Lie}(\cdot) \).

Construction of \( \mathcal{U}(\mathfrak{g}) \): consider the ideal \( I \) in the tensor algebra \( \mathcal{T}(\mathfrak{g}) \) generated by elements of the form \( x \otimes y - y \otimes x - [x, y] \) and let
\[
\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I.
\]

The adjoint action \( G \ltimes \mathfrak{g} \) extends to an action \( G \ltimes \mathcal{U}(\mathfrak{g}) \) and the map \( x \mapsto X_x \) also extends to \( \mathcal{U}(\mathfrak{g}) \) by the universal property.

Killing form \( \kappa \), Cartan’s criterion for semisimplicity:
\[
\kappa(x, y) = 2n \text{Tr}(xy) - 2 \text{Tr}(x) \text{Tr}(y)
\]
on \(\mathfrak{gl}(n, \mathbb{R})\) (degenerate) and \(\mathfrak{sl}(n, \mathbb{R})\) (non-degenerate) so \(\mathfrak{sl}(n, \mathbb{R})\) is semisimple, and \(\mathfrak{gl}(n, \mathbb{R})\) is not. In addition, \(\kappa\) is \(G\)-invariant:

\[
\kappa(\text{Ad}(g)x, \text{Ad}(g)y) = \kappa(x, y)
\]

hence defines a \(G\)-equivariant identification \(\mathfrak{g} \simeq \mathfrak{g}^*\), where \(G\) acts on \(\mathfrak{g}^*\) via the contragredient of \(\text{Ad}\). Since \(\mathfrak{g}\) is finite-dimensional, one can consider the composition

\[
\alpha : \text{End}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\kappa} \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathcal{T}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}).
\]

The \textit{Casimir element} is

\[
\Omega = \alpha(\text{Id}_\mathfrak{g}).
\]

Since \(\alpha\) is \(G\)-equivariant, it is an element of \(Z(\mathfrak{g})\), that is, a \(G\)-invariant element in \(U(\mathfrak{g})\).

**References:** [B, §2.2] and P. Garrett’s notes on \textit{Invariant differential operators}. See also S. Sternberg’s notes on \textit{Lie algebras}.

**Week 5**

**Lecture 12.** Elements in the center of the universal enveloping algebra \(U(\mathfrak{g})\),

\[
Z(\mathfrak{g}) = \{ A \in U(\mathfrak{g}) : \text{Ad}(G)A = A \}
\]

define \(G\)-left-invariant differential operators on manifolds of the form \(G/H\).

**Case of \(SL(2, \mathbb{R})\):** the matrices

\[
H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

constitute a basis of \(\mathfrak{sl}(2, \mathbb{R})\) and satisfy the relations

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = 0.
\]

Under the identification \(\mathfrak{sl}(2, \mathbb{R})^* \simeq \mathfrak{sl}(2, \mathbb{R})\), the dual basis of \(\{H, X, Y\}\) is \(\{\frac{1}{2}H, Y, X\}\).

Therefore, the Casimir element can be expressed as

\[
\Omega = \frac{1}{2}H^2 + XY + YX
\]

where \textit{products are taken in} \(U(\mathfrak{g})\). Observe that \(X - Y \in \mathfrak{so}(2)\) so

\[
(X - Y) \cdot f = 0
\]

for any \(SO(2)\)-invariant function \(f\) on \(SL(2, \mathbb{R})\).

**References:** P. Garrett’s notes on \textit{Invariant differential operators}.

**Lecture 13.** The Casimir operator \(\Omega \in Z(\mathfrak{g})\) acts as \(2y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\).

\(KAK\) and \(K \exp(p)\) (Cartan) decompositions for \(GL(n, \mathbb{R})\) and \(SL(n, \mathbb{R})\).

**References:** [B, §2.2]. See also Knapp’s book.

**Lecture 14.** Comments on the Cartan motion group \(K \ltimes \mathfrak{g}/\mathfrak{k}\) associated with \(G\) and the Mackey-Higson-Afgoustidis analogy.
The convolution ring \( C^\infty_c(K\backslash G/K) \) of \( K \)-bi-invariant functions on \( G \) is commutative (Gelfand). If \((G,K) = (\text{SL}(2,\mathbb{R}),\text{SO}(2)) \) and \( \sigma \in \text{Hom}(K,\mathbb{C}^\times) \), the convolution ring

\[
C^\infty_c(K\backslash G/K,\sigma) = \{ f \in C^\infty_c(G), \ f(k_1gk_2) = \sigma(k_1)f(g)\sigma(k_2) \}
\]

is also commutative.

The Spectral Theorem: if \( T \) is a compact self-adjoint operators on a Hilbert space \( \mathcal{H} \), there exists a Hilbert basis of \( \mathcal{H} \) of eigenvectors and the eigenvalues \( \lambda_i \) satisfy \( \lim \lambda_i = 0 \).

References: [B, §2.2].

Week 6

Lecture 15. Compact operators are the limits of finite-rank operators. They form a closed two-sided ideal in \( B(\mathcal{H}) \). Hilbert-Schmidt operators: if \( K(x,y) \in L^2(X \times X) \), then the operator \( T \) defined on \( L^2(X) \) by

\[
Tf(x) = \int_X K(x,y)f(y) \, dy
\]

is compact. Every unitary representation \((\pi,\mathcal{H})\) of \( G \), yields a *-representation \( \tilde{\pi} \) of the convolution algebra \( C^\infty_c(G) \):

\[
\tilde{\pi}(\varphi)\xi = \int_G \varphi(g)\pi(g)\xi \, dg
\]

satisfies

\[
\tilde{\pi}(\varphi_1 \ast \varphi_2) = \tilde{\pi}(\varphi_1)\tilde{\pi}(\varphi_2) \quad \text{and} \quad \tilde{\pi}(\varphi^*) = \tilde{\pi}(\varphi)^*
\]

where \( \varphi^*(g) = \overline{\varphi(g^{-1})} \). In the case of the right quasi-regular representation \( \rho \) on \( L^2(\Gamma\backslash G, \chi) \),

\[
\rho(\varphi)f(g) = \int_G f(h)\varphi(g^{-1}h) \, dh.
\]

This is a Hilbert-Schmidt operator with kernel

\[
K(g,h) = \sum_{\gamma \in \Gamma} \chi(\gamma)\varphi(g^{-1}\gamma h).
\]

Moreover,

\[
\rho(\varphi) (L^2(\Gamma\backslash G, \chi)) \subset C^\infty(\Gamma\backslash G, \chi)
\]

and, if \( \varphi(R_{\theta}g) = e^{-ik\theta} \varphi(g) \), then

\[
\rho(\varphi) (L^2(\Gamma\backslash G, \chi)) \subset C^\infty(\Gamma\backslash G, \chi, k).
\]

References: [B, §2.3].

Lecture 16. Guest lecture by J. Voight: On the arithmetic significance of \( \lambda = \frac{1}{4} \).

References: see also [B, §Chap. I].
Lecture 17. Let $F$ be a closed $G$-invariant space of $L^2(\Gamma\backslash G, \chi)$, with $K$-isotypical decomposition
\[ F = \sum_{k \in \mathbb{Z}} \oplus F_k. \]
If $F_k \neq \{0\}$, then $\Delta$ has a non-zero eigenvector in $F_k^\infty = F_k \cap C^\infty(\Gamma\backslash G, \chi)$.

The representation $L^2(\Gamma\backslash G, \chi)$ of $G$ is semisimple: it decomposes as the direct sum of unitary irreducible representations of $G$.

References: [B, §2.3].

Week 7

Lecture 18. For $\sigma \in \widehat{SO(2)}$ and $\xi$ character of $C_c^\infty(K \backslash G/K, \sigma)$, let
\[ \mathcal{H}(\xi) = \{ f \in L^2(\Gamma \backslash G, \chi, k), \rho(\varphi)f = \xi(\varphi)f \text{ for all } \varphi \in C_c^\infty(K \backslash G/K, \sigma) \}. \]

The spaces $\mathcal{H}(\xi)$ are finite-dimensional, mutually orthogonal and
\[ L^2(\Gamma\backslash G, \chi, k) = \sum_\xi \mathcal{H}(\xi). \]

It follows that $L^2(\Gamma\backslash \mathcal{H}, \chi, k)$ decomposes as the Hilbert direct sum of eigenspaces for the weight $k$ Laplacian $\Delta_k$. One can also prove that
\[ \sum_{\lambda \in \text{Sp}(\Delta_k)} \lambda^{-2} \]
converges, from which it follows that $\Delta_k$ has a self-adjoint extension to $L^2(\Gamma\backslash \mathcal{H}, \chi, k)$.

References: [B, §2.3].

Lecture 19. Construction of smooth vectors: if $(\pi, \mathcal{H})$ is a representation on a Hilbert space and $\xi \in \mathcal{H}$, then $\tilde{\pi}(\varphi)\xi \in \mathcal{H}^\infty$ for $\varphi \in C_c^\infty(G)$. Using a Dirac sequence, it follows that smooth vectors are dense in $\mathcal{H}$.

Overview of the representation theory of compact groups: a locally compact group is compact if and only if it has finite Haar measure, which can be assumed equal to 1.

All representations on Hilbert spaces can unitarized: if $(\pi, \mathcal{H})$ is a representation on a Hilbert space, then $\pi$ is unitary for the inner product
\[ \langle \xi, \eta \rangle = \int_G \langle \pi(g)\xi, \pi(g)\eta \rangle_{\mathcal{H}} \, dg, \]
which defines the same topology.

If $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitary representations of a compact group $G$ that possess matrix coefficients $f_1$ and $f_2$ which are not orthogonal in $L^2(G)$, then there exists a non-trivial intertwiner $L : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, namely, if $f_i(g) = \langle \pi_i(g)\xi_i, \eta_i \rangle$,
\[ \xi_1 \mapsto \int_G \langle \pi_1(g)\xi_1, \eta_1 \rangle \pi_2(g^{-1})\eta_2 \, dg. \]

Peter-Weyl Theorem: if $G$ is a compact group,
(i) matrix coefficients of finite dimensional unitary representations are dense in $C(G)$ and $L^p(G)$ for $1 \leq p \leq \infty$;
(ii) unitary irreducible representations of $G$ are finite-dimensional;
(iii) all unitary representations are semisimple.

In other words,

$$L^2(G) \simeq \bigoplus_{\pi \in \hat{G}} V_\pi^* \otimes V_\pi \simeq \bigoplus_{\pi \in \hat{G}} \dim(V_\pi)V_\pi.$$

A representation $\pi$ of a (non-compact) group $G$ with maximal compact subgroup $K$ is said admissible if all its $K$-isotypical components are finite-dimensional. In other words,

$$\pi|_K \simeq \bigoplus_{\rho \in \hat{K}} m_\rho V_\rho$$

with all multiplicities $m_\rho$ finite. A famous theorem of Harish-Chandra says that unitary irreducible representations of Lie groups are admissible. We will prove in the case of $G = \text{SL}(2, \mathbb{R})$ that all the representations that occur in $L^2(\Gamma \backslash G, \chi)$ are admissible.

References: [B, §2.4].

Lecture 20. If $(\pi, \mathcal{H})$ is a unitary irreducible representation of $G = \text{SL}(2, \mathbb{R})$, then for each $k \in \mathbb{Z} \simeq \hat{K}$, the isotypical component $\mathcal{H}_k$ is an irreducible $C_c^\infty(K \backslash G/K, \sigma_k)$-module and has dimension at most 1.

Introductory example of $(\mathfrak{g}, K)$-module: trigonometric polynomials in $L^2(\mathbb{T})$. Action of $K$, action of $\mathfrak{g}$. General definition of $(\mathfrak{g}, K)$-modules.

References: [B, §2.4] and Casselman’s essays.

Week 8

Lecture 21. $K$-finite vectors of an admissible representation are smooth and everywhere dense; they form a $(\mathfrak{g}, K)$-module. Representations with isomorphic $(\mathfrak{g}, K)$-modules are said infinitesimally equivalent.

References: [B, §2.4], see also Knapp.

Lecture 22. The complexification $\mathfrak{g}_C$ of $\mathfrak{sl}(2, \mathbb{R})$ is generated by

$$R = \frac{1}{2} \begin{bmatrix} 1 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \frac{1}{2} \begin{bmatrix} 1 & -i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

subject to the relations


There is also a Casimir element $\Omega \in Z(\mathfrak{g}_C)$ defined by

$$-4\Omega = H^2 + 2RL + 2LR.$$ 

This element acts by a scalar on every irreducible admissible $(\mathfrak{g}, K)$-module. If $V$ is an irreducible admissible $(\mathfrak{g}, K)$-module and $k \in \mathbb{Z}$, let $V(k)$ denote the isotypical component of $V$ associated with $\sigma_k : R_\theta \mapsto e^{ik\theta}$. Then,
\( V(k) = \{ x \in V , Hx = kx \} \)

\[ R : V(k) \to V(k+2) \quad \text{and} \quad L : V(k) \to V(k-2). \]

If \( V(k) \ni x \neq 0 \), then \( \mathbb{C}R^n x = V(k+2n) \), and \( \mathbb{C}L^n x = V(k-2n) \) and
\[ V = \mathbb{C}x \oplus \bigoplus_{n>0} \mathbb{C}R^n x \oplus \bigoplus_{n>0} \mathbb{C}L^n x. \]

If \( \Omega \) acts by \( \lambda \) on \( V \), then for \( x \in V(k) \)
\[ LRx = \left( -\lambda - \frac{k}{2} \left( 1 + \frac{k}{2} \right) \right) x \quad \text{and} \quad RLx = \left( -\lambda + \frac{k}{2} \left( 1 - \frac{k}{2} \right) \right) x. \]

If \( V(k) \) contains a non-zero vector \( x \) such that \( Rx = 0 \) (resp. \( Lx = 0 \)), then
\[ \lambda = -\frac{k}{2} \left( 1 + \frac{k}{2} \right) \quad \text{(resp.} \lambda = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \text{)} \]

It follows that all the \( K \)-types of a given admissible irreducible \((g,K)\)-module have the same parity, giving a dichotomy between even and odd modules.

**Uniqueness results:**

- If \( \lambda \) is not of the form \( \frac{k}{2} \left( 1 - \frac{k}{2} \right) \) with \( k \) even (resp. odd), then there exists at most one isomorphism class of even (resp. odd) \((g,K)\)-modules on which \( \Omega \) acts by \( \lambda \). The \( K \)-types of such a module are all the even (resp. odd) integers.

- If \( \lambda = \frac{k}{2} \left( 1 - \frac{k}{2} \right) \) with \( k \in \mathbb{Z} \), then the \( K \)-types of an irreducible admissible \((g,K)\)-module with parity \( k \mod 2 \) on which \( \Omega \) acts by \( \lambda \) must be one of the following:
  \[ \Sigma^+(k) = \{ \ell \in \mathbb{Z} , \ l = k \mod 2 , \ \ell \geq k \} \]
  \[ \Sigma^-(k) = \{ \ell \in \mathbb{Z} , \ l = k \mod 2 , \ \ell \leq -k \} \]
  \[ \Sigma^0(k) = \{ \ell \in \mathbb{Z} , \ l = k \mod 2 , \ |\ell| < k \} \]

and there exists at most one isomorphism class of irreducible admissible \((g,K)\)-module with a given set of such \( K \)-types.

It remains to prove the existence and study the realizability of \((g,K)\)-modules corresponding to these situations.

**References:** [B, §2.4].

Lecture 23. (Generalized, non-unitary) principal series: for \( (\varepsilon,s) \in \{0,1\} \times \mathbb{C}, \)
\[ H^\infty(\varepsilon,s) = \left\{ f \in C^\infty(G) , \ f \left( \begin{bmatrix} u & t \\ 0 & u^{-1} \end{bmatrix} g \right) = [u]^\varepsilon [u]^{\nu+1} f(g) \right\} \subset \text{Ind}_{MAN}^G \sigma_\varepsilon \otimes \chi_\nu \otimes 1_N \]
where \( s = \frac{\nu+1}{2} \) and \( \sigma_\varepsilon(m) = m^\varepsilon \) for \( m \in \{ \pm 1 \} \cong M \), \( \chi_\nu(a) = a^\nu \) for \( a \in \mathbb{R}^\times \cong A \) and \( 1_N \) is the trivial representation of \( N \). A function in \( H^\infty(\varepsilon,s) \) is determined by its restriction
to $K$, which must be even or odd. Conversely, any even or odd function $f$ on $T$ extends to an element of $H^\infty(\varepsilon, s)$ by

$$f\left(\begin{bmatrix} \sqrt{y} & \frac{s}{\sqrt{y}} \\ 0 & \sqrt{y} \end{bmatrix} R_\theta\right) = y^s f(\theta).$$

Complete $H^\infty(\varepsilon, s)$ into a Hilbert space $\mathcal{H}(\varepsilon, s)$ for the norm associated with

$$\langle f_1, f_2 \rangle = \langle f_1|_K : f_2|_K \rangle_{L^2(K)}.$$ 

Action of $g$ on $K$-finite vectors: $H(\varepsilon, s)(K)$ is generated by functions of the form

$$f_\ell\left(\begin{bmatrix} \sqrt{y} & \frac{s}{\sqrt{y}} \\ 0 & \sqrt{y} \end{bmatrix} R_\theta\right) = y^s e^{i\ell\theta}$$

which satisfy

$$H f_\ell = \ell f_\ell , \quad R f_\ell = \left(s + \frac{\ell}{2}\right) f_{\ell+2} , \quad L f_\ell = \left(s - \frac{\ell}{2}\right) f_{\ell-2} , \quad \Delta f_\ell = s(1-s)f_\ell.$$

It follows that the irreducible admissible $(g, K)$-modules of $SL(2, \mathbb{R})$ can be realized as subquotients of $H(\varepsilon, s)(K)$ for some $(\varepsilon, s) \in \{0, 1\} \times \mathbb{C}$:

- If $\lambda = s(1-s)$ is not of the form $\frac{k}{2} \left(1 - \frac{k}{2}\right)$ with $k = \varepsilon \mod 2$, then $\mathcal{H}(\varepsilon, s)(K)$ is the unique irreducible admissible $(g, K)$-module on which $\Delta$ acts by $\lambda$. Its set of $K$-types is $2\mathbb{Z} + \varepsilon$. We denote by $\mathcal{P}(\lambda, \varepsilon)$ its isomorphism class and call it principal series representation.

- If $\lambda = \frac{k}{2} \left(1 - \frac{k}{2}\right)$ with $1 < k = \varepsilon \mod 2$, there exists three irreducible subquotients of $\mathcal{H}(\varepsilon, s)(K)$ on which $\Delta$ acts by $\lambda$, with respective sets of $K$-types $\Sigma^+(k)$, $\Sigma^-(k)$ and $\Sigma^0(k)$. The isomorphism classes corresponding to $\Sigma^\pm(k)$ are denoted by $\mathcal{D}^\pm(k)$ and called discrete series representations. The corresponding modules $\mathcal{D}^\pm(1)$ for $k = 1$ are called limits of the discrete series.

References: [B, §2.5].

Week 9

Lecture 24. Unitarizability of the principal series: if $\lambda \geq \frac{1}{4}$, then $\mathcal{P}(\lambda, \varepsilon)$ contains a unitary representative. Conversely, if $(\pi, \mathcal{H})$ is a unitary admissible representation of $SL(2, \mathbb{R})$ on which $\Omega$ acts by $\lambda$, then $\lambda \in \mathbb{R}$. Moreover,

- if $(\pi, \mathcal{H}) \in \mathcal{P}(\lambda, 0)$, then $\lambda > 0$;
- if $(\pi, \mathcal{H}) \in \mathcal{P}(\lambda, 1)$, then $\lambda > \frac{1}{4}$.

This shows that the unitarizable principal series are the $\pi_{\varepsilon, \nu} = \text{Ind}_{MAN}^G \sigma_\varepsilon \otimes \chi_\nu \otimes 1_N$ with $\nu \in i\mathbb{R}$ and possibly $\pi_{0, \nu}$ with $-1 < \nu < 1$. These can be shown to be unitarizable, using intertwining integrals. They are called the complementary series.

Finite dimensional representations: the only finite-dimensional unitary irreducible representations of $GL(n, \mathbb{R})^+$ are one-dimensional, of the form $\det^r$ with $r \in i\mathbb{R}$. As a by-product of the proof, $SL(2, \mathbb{R})$ has no non-trivial finite dimensional unitary irreducible representation.
Unitary irreducible representations that are infinitesimally equivalent, i.e. have isomorphic \((g, K)\)-modules, are unitarily equivalent.

References: [B, §2.6].

**Lecture 25.** Induced representations of finite groups: we consider \(G\) finite group, \(H\) subgroup of \(G\) and \(V\) a representation of \(G\). Restricting \(V\) to a representation of \(H\) gives a functor \(\text{Res}^G_H : \text{Rep}(G) \rightarrow \text{Rep}(H)\).
If \(V \in \text{Rep}(G)\) and \(W \subset V\) is an \(H\)-invariant subspace, then \(W \in \text{Rep}(H)\) and for \(g \in G\), the space \(g \cdot W\) only depends on \(gH\). We say that \(V\) is *induced* by \(W\) if

\[
V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.
\]

Example: the left regular representation of \(G\) is induced by the left regular representation of \(H\). For every \(W \in \text{Rep}(H)\) there exists a unique representation of \(G\) induced by \(W\). We denote it by \(\text{Ind}^G_H W\).

References: Fulton-Harris.

**Lecture 26.** Unitarizability of the discrete series: for \(k \geq 2\), the infinitesimal class \(D^\pm(k)\) admits a unitary representative, namely the space of holomorphic functions \(f\) on \(\mathcal{H}\) such that

\[
\int_{\mathcal{H}} |f(z)|^2 y^k \frac{dx\,dy}{y^2} < \infty
\]
with \(G = \text{SL}(2, \mathbb{R})\) acting by

\[
\pi^\pm(g)f(z) = (\mp bz + d)^{-k}f\left(\frac{az \mp c}{\mp bz + d}\right).
\]
These representations can also be realized as irreducible subrepresentations of the left regular \(L^2(G)\).
Solution of the spectral problem: summary of the correspondence between automorphic forms and unitary irreducible representations of \(\text{SL}(2, \mathbb{R})\). Holomorphic modular forms occur in the discrete series.

References: [B, §2.6, 2.7].

**Lecture 27.** Abstract and concrete \(\mathbb{C}^*-\)algebras, commutative \(\mathbb{C}^*-\)algebras are algebras of continuous functions (Gelfand isomorphism) and all \(\mathbb{C}^*-\)algebras can be seen as algebras of bounded operators on a Hilbert space.
For \(G\) locally compact group, consider the convolution \(*\)-algebra \(C_c(G)\):

\[
f \ast g(s) = \int_G f(t)g(t^{-1}s)\,dt, \quad f^*(t) = \Delta_G(t)^{-1}f(t^{-1})
\]
and equip it with the norm

\[
\|f\|_r = \|\hat{\lambda}_G(f)\|_{\text{op}}
\]
where $\tilde{\lambda}_G(f)$ is the operator of convolution by $f$ on the left, acting on $L^2(G)$. More generally, if $\pi$ is a unitary representation of $G$, define $\tilde{\pi}(f)$ acting on $\mathcal{H}_\pi$ as in Lecture 15 and consider
\[ \|f\|_{\text{max}} = \sup_{\pi} \|\tilde{\pi}(f)\|_{\text{op}}. \]

The completions of $C_c(G)$ with respect to these norms are C*-algebras, respectively denoted by $C^*_r(G)$ and $C^*(G)$. The correspondence $\pi \mapsto \tilde{\pi}$ induced a bijection between unitary (resp. tempered) representations of $G$ and non-degenerate representations of $C^*(G)$ (resp. $C^*_r(G)$). In other words, the study of unitary representations of $G$ is equivalent to the study of Hilbert spaces that are modules over the C*-algebra(s) of $G$.

Hilbert C*-modules and bounded adjointable operators. If $A$ and $B$ are C*-algebras, an $(A, B)$-correspondence is a Hilbert module $E$ over $B$ together with a ∗-morphism $\varphi : A \rightarrow \mathcal{L}_B(E)$.

Given such a bimodule $A E_B$ and a ∗-representation $\mathcal{H}$ of $B$, one can equip the tensor product $E \otimes_B \mathcal{H}$ with the inner product defined by
\[ \langle e_1 \otimes \xi_1, e_2 \otimes \xi_2 \rangle = \langle \xi_1, \langle e_1, e_2 \rangle \xi_2 \rangle. \]

It carries a left action of $A$ via $\varphi$ and the Hilbert completion gives a ∗-representation, denoted $\text{Ind}^A_B \mathcal{H}$.

**Week 10**

**Lecture 28.** Mackey induction for locally compact groups: induces unitary representations to unitary representations. Rieffel’s construction: if $H$ is a closed subgroup of $G$, there exists a C*-correspondence $C^*_c(G)E(G)_{C^*_c(H)}$ such that for every unitary representation $\mathcal{H}$ of $H$, there is a specialization isomorphism $E(G) \otimes_{C^*_c(H)} \mathcal{H} \rightarrow \text{Ind}^G_H \mathcal{H}$ that intertwines the left $C^*(G)$ actions. If $P = L \ltimes N$ is a parabolic subgroup of a real reductive group $G$, there exists a $(C^*_c(G), C^*_c(L))$-correspondence $\mathcal{E}(G/N)$ that realizes parabolic induction: there is a specialization isomorphism of $C^*_c(G)$-modules
\[ \mathcal{E}(G/N) \otimes_{C^*_c(L)} \mathcal{H}_{\sigma \otimes \chi} \sim \rightarrow \text{Ind}^G_P \sigma \otimes \chi \otimes 1_N = \pi_{\sigma, \chi}. \]

Adjoint of the functor $\mathcal{E}(G/N) \otimes_{C^*_c(L)} \cdot$. Case of $p$-adic groups: Frobenius reciprocity and Bernstein’s Second Adjoint Theorem.