Harry Potter and the Linear Extensions of a Naturally Labelled Poset, Enumerated by Descents

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(Tea 3:30 pm Math Lounge)

Abstract

Let "Fred" be a finite partially ordered set, labelled by the numbers 1, 2, 3, up to n so that, whenever an element p is below an element q in the poset, the *label* of p (a natural number) is less than the *label* of q. (In posetese, the permutation $123\dots n$ is a "linear extension" of the poset Fred.) For example, consider the zig-zag-shaped poset with four elements $1, 2, 3, 4$, whose partial ordering is given by $1 < 3 > 2 < 4$.

Look at the linear extensions, that is, the permutations in S_n that respect the partial ordering of Fred, by which we mean the following: If the element labelled i in Fred is below the element labelled j in Fred, then the number i must come before the number j in the permutation. In our example, there are 5 linear extensions: 1234, 2134, 1243, 2143, and 2413. (If Fred were a totally unordered poset, every permutation in S_n would be a linear extension.)

Now take your favourite linear extension and count the number of "descents," that is, the places where a bigger number *immediately* precedes a smaller number. In our example, the number of descents in the linear extensions is given by 0, 1, 1, 2, and 1, respectively. Let h_{42} be the number of linear extensions with 42 descents. The zig-zag has $h_0 = 1$, $h_1 = 3$, and $h_2 = 1$.

According to a theorem of Richard P. Stanley of MIT, if Fred is ranked (that is, if every maximal chain has the same number of elements, r), then the sequence $h_0, h_1, \dots, h_{n-r}$ will be symmetric. In our example, we have the sequence $1, 3, 1$ ($n = 4$ and $r = 2$). If Fred is the 5-element totally unordered poset, we get the "eulerian" numbers $1, 26, 66, 26, 1$ ($n = 5$ and $r = 1$).

At the 1981 Banff Conference on Ordered Sets, Stanley asked for a direct, combinatorial, bijective proof of this theorem. That is, if H_{42} is the *set* of linear extensions with 42 descents, Stanley asked for a bijection between H_0 and H_{n-r}, H_1 and H_{n-r-1}, etc. — in general, between H_k and H_{n-r-k}.

We present a partial solution to the problem, for the cases $k = 0$, 1, and 2. We leave the cases $k = 3$ through ∞ to the reader.

(This talk has nothing to do with the 1975 conjecture of Richard Stanley that the speaker settled in 1999.)

This talk should be accessible to graduate students.

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