On the space of all Lie algebras of a given dimension

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Abstract

The space of all complex Lie algebras of a given dimension n can be naturally identified with the set $\mathcal{L}_n$ of all Lie brackets on $\mathbb{C}^n$. Since the Jacobi identity is determined by polynomial conditions, $\mathcal{L}_n$ is an algebraic subset of the vector space V_n of bilinear forms from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$. The isomorphism class of a Lie algebra $\mu \in \mathcal{L}_n$ is then given by the orbit $\mathrm{GL}(n).\mu$ under the 'change of basis' action of $\mathrm{GL}(n)$ on V_n. This action is very unpleasent from the point of view of invariant theory since any $\mu \in V_n$ is unstable (i.e. $0 \in \overline{\mathrm{GL}(n).\mu}$), which makes very difficult the study of the quotient space $\mathcal{L}_n/\mathrm{GL}(n)$ parameterizing Lie algebras up to isomorphism.

Nevertheless, F. Kirwan and L. Ness have shown that the momentum map for an action can be used to study the orbit space of the null-cone (set of unstable vectors). We consider the momentum map $m : \mathbb{P}V_n \mapsto \mathfrak{iu}(n)$ for the action of $\mathrm{GL}(n)$ on V_n, where $\mathfrak{iu}(n)$ denotes the space of hermitian matrices. The critical points of the functional $F_n = ||m||^2 : \mathbb{P}V_n \mapsto \mathbb{R}$ are studied, in order to understand the stratification of $L_n \subset \mathbb{P}V_n$ defined by the negative gradient flow of F_n, where L_n is the projectivization of $\mathcal{L}_n$. We obtain a description of the critical points which lie in L_n in terms of those which are nilpotent, as well as the minima and maxima of $F_n : L_n \mapsto \mathbb{R}$. A characterization of the critical points modulo isomorphism, as the categorical quotient of a suitable action is considered, and some applications to the study of L_n are given.

This talk should be accessible to graduate students.

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