Nilpotent matrices and the permuation group

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(Tea 3:30 pm Math Lounge)

Abstract

The problem of classifying conjugacy classes of $n \times n$ matrices over a field k has two aspects. One is arithmetic: any degree-n extension field K of k can be embedded in $n \times n$ matrices, in a way that is canonical up to conjugacy. In terms of linear algebra, the corresponding matrices have their eigenvalues in K. For this reason, the arithmetic of k affects conjugacy classes. I will ignore this aspect almost entirely.

The second aspect is independent of the field; one could call it purely "algebraic," if that word is divorced from arithmetic. There are non-zero $n \times n$ matrices all of whose eigenvalues are zero. These are the nilpotent matrices. Any nilpotent $n \times n$ matrix is conjugate to one in Jordan normal form, and in this way conjugacy classes of nilpotent matrices are in bijection with partitions of n.

More than a hundred years ago, Frobenius discovered that exactly the same set—partitions of n—parametrizes the irreducible representations of the symmetric group S_n. Since that time, there has been a tremendous amount of work aimed at using information about S_n and its representations (which are a part of combinatorics and finite mathematics) to study the group $GL(n)$ and its representations (which are a part of algebraic geometry, arithmetic, and analysis).

I'll describe a classical example of this work, due to Green in 1955. The starting point is the analogies

$$\{S_n \text{ acting on } \{1, 2, \ldots, n\}\} \leftrightarrow \{GL(n) \text{ acting on projective space}\}$$

$$\{S_n \text{ on } k\text{-element subsets of } \{1, 2, \ldots, n\}\} \leftrightarrow \{GL(n) \text{ on the Grassmannian of } k\text{-planes}\}.$$

Green developed these analogies in the case of the general linear group over a finite field with q elements. He showed that the corresponding representations of $GL(n)$ are "q-analogues" of symmetric group representations, and that all of these representations decompose in exactly the same way.

I'll explain Green's work with an eye toward later extensions to the case of $GL(n)$ over the real numbers and other infinite fields.

This talk should be entirely accessible to grad students and a lot of it works for undergrads who know about groups and homogeneous spaces.