LECTURE NOTES ON K-THEORY

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Abstract. These notes originated from a series of lectures given at Dartmouth in the winter of 1999. They represent a bit of exploration for all involved including the lecturer.

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Contents

1. Basics

The material in these notes is stolen primarily from [WO93]; however, some bits (and some corrections) were taken from [Mur90] and [Bla86].

One of the significant impediments for the beginner to the subject of noncommutative K-theory is that the theory is considerably more straightforward for the category of unital C^*-algebras with unital homomorphisms. However, it is inevitable that we will want to consider nonunital algebras — at the very least, one often wants to stabilize by tensoring with $\mathcal{K}(\mathcal{H})$, the compact operators on a separable infinite-dimensional Hilbert space. Therefore we will constantly be "adjoining an identity" to C^*-algebras which don't already have one (see [Arv76, Ex 1.1.H]).

Date: 11 July 2007 — 16:38 — Preliminary Version.

We'll follow the treatment and notation of [RW98, §2.3]. Thus $\widetilde{A}$ will denote the subalgebra of $M(A)$ generated by A and the identity. Thus $\widetilde{A} = A$ if $1 \in A$, and $*$-isomorphic to the *vector space* direct sum $A^1 := A + \mathbb{C}$ otherwise. Even if $1 \in A$, the notation A^1 will denote $A + \mathbb{C}$ with the obvious $*$-algebraic structure. If $1 \notin A$, then A^1 has a C^*-norm coming from the isomorphism with $\widetilde{A}$. If $1 \in A$, then $a + \lambda \mapsto (a + \lambda 1, \lambda)$ is a $*$-isomorphism of A^1 onto the *C^*-algebra direct sum*[1] $A \oplus \mathbb{C}$. Again, A^1 inherits a C^*-norm. Therefore we always have a split exact sequence of C^*-algebras

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "j"] & A^1 \arrow[r, "\pi"] & \mathbb{C} \arrow[r] \arrow[l, bend left, dotted, "\iota"] & 0,
\end{tikzcd}
\tag{1.1}
$$

where $j(a) := a + 0$, $\pi(a + \lambda) := \lambda$, and $\iota(\lambda) := 0 + \lambda$. *We will use this notation in conjunction with A^1 throughout these notes.*

The basic building block for K-theory is the projection. Since the sum of projections is only a projection when the projections are orthogonal, we will constantly be putting our projections into matrix algebras where "there is enough room to make them orthogonal." We will also need some notation for matrix algebras over a C^*-algebra A. If $n \geq 1$, then $M_n(A)$ will denote the C^*-algebra of $n \times n$-matrices with entries from A. If $a \in M_n(A)$ and $b \in M_m(A)$, then $a \oplus b$ will denote the block diagonal matrix $\operatorname{diag}(a, b) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in $M_{n+m}(A)$. We shall let φ_n be the inclusion of $M_n(A)$ into $M_{n+1}(A)$ sending a to $a \oplus 0$, and we shall write $M_\infty(A)$ for the *algebraic* direct limit of $\{ M_n(A), \varphi_n) \}$ (Remark A.5). This is mere formalism, and merely allows us to identify a and $a \oplus 0_n$ for all n.[2]

Definition 1.1. We let $P[A] = \{ p \in M_\infty(A) : p = p^* = p^2 \}$. We say that $p \sim q$ if there is a $u \in M_\infty(A)$ such that $p = u^*u$ and $q = uu^*$. (We say that p and q are Murray-von Neumann equivalent.)

Remark 1.2. If $p \sim q$, we can assume that p, q, and u are in $M_n(A)$ for some n. Thus u is a partial isometry (see [Mur90, Theorem 2.3.3]). In particular $u = uu^*u = qu = up$. It is easily checked that $\sim$ is an equivalence relation on $P[A]$. The set of equivalence classes in denoted by $V(A)$, and we write $[p]$ for the equivalence class of p.

Theorem 1.3. *Suppose that $p, q, r, s \in P[A]$.*

(a) *If $p \sim r$ and $q \sim s$, then $p \oplus q \sim r \oplus s$.*
(b) *$p \oplus q \sim q \oplus p$.*
(c) *If $p, q \in M_n(A)$ and $pq = 0$, then $p + q \sim p \oplus q$ in $M_\infty(A)$.*

Proof. Suppose $p = u^*u$ and $r = uu^*$ while $q = v^*v$ and $s = vv^*$. Then $w := \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$ suffices to prove (a).

To prove (b), use $u := \left(\begin{smallmatrix} 0 & q \\ p & 0 \end{smallmatrix}\right)$. To prove (c), let $u := \left(\begin{smallmatrix} p & q \\ 0_n & 0_n \end{smallmatrix}\right)$, then recalling that $pq = 0$, it is easy to check that u implements an equivalence between $(p + q) \oplus 0_n$ and $p \oplus q$. $\square$

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[1] I have written elements of the vector space direct sum as simply $a + \lambda$, and reserved the notation (a, λ) for elements of the algebraic direct sum.

[2] More precisely, if $a \in M_k(A)$, then $a \oplus 0_n = \varphi_{k,k+n}(a)$, and it follows that $\varphi^k(a) = \varphi^{k+n}(a \oplus 0_n$. Since the φ^k are injections in this case, there is no danger in dropping them.

Corollary 1.4. *The binary operation*

$$[p] + [q] := [p \oplus q]$$

makes $V(A)$ into an abelian semigroup with identity equal to the class of the zero projection.

Example 1.5. Since two projections in M_n are equivalent if and only if they have the same rank, we see that $V(\mathbb{C}) \cong (\mathbb{N}, +)$. Since $M_n(M_m) \cong M_{nm}$, we also have $V(M_m) \cong (\mathbb{N}, +)$. In fact, if $\mathcal{H}$ is an infinite-dimensional Hilbert space, then $V\big(\mathcal{K}(\mathcal{H})\big)$ is also isomorphic to $(\mathbb{N}, +)$. On the other hand, since any two infinite rank projections in $B(\mathcal{H})$ are equivalent, $V\big(B(\mathcal{H})\big) \cong (\mathbb{N} \cup \{\infty\}, +)$. However, since all nonzero projections have norm one, and since a projection in $C_0(\mathbb{R}^k, M_n) \cong M_n\big(C_0(\mathbb{R}^k)\big)$ must be a projection-valued function on $\mathbb{R}^k$ vanishing at infinity, it follows that $V\big(C_0(\mathbb{R}^k)\big) = \{0\}$.

Remark 1.6. Notice that $V\big(\mathcal{K}(\mathcal{H})\big)$ is a cancellative semigroup: $x+r = y+r$ implies that $x = y$. On the other hand, $V\big(B(\mathcal{H})\big)$ is not: $x + \infty = y + \infty$ for all x and y.

Definition 1.7. We let $K_{00}(A)$ denote the Grothendieck group $\mathcal{G}\big(V(A)\big)$. As is standard, we abuse notations slightly, and denote the class of $\big([p], [q]\big)$ in $K_{00}(A)$ by $[p] - [q]$. The natural map of $V(A)$ into $K_{00}(A)$ sending $[p]$ to $[p] - 0$ is denoted by ι_A.

Example 1.8. We have $K_{00}\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$. If $\mathcal{H}$ is infinite dimensional, then $K_{00}\big(B(\mathcal{H})\big) = \{0\}$. In particular, the natural map $\iota_A : V(A) \to K_{00}(A)$ need not be injective.

Remark 1.9. Our notation for elements in $K_{00}(A)$ can be misleading. Because $V(A)$ need not be cancellative, $[p] - [q] = 0$ need not imply that $[p] = [q]$ in $V(A)$ (consider $B(\mathcal{H})$). But if $A = \mathbb{C}$, for example, then it is true that $[p] - [q] = 0$ if and only if $p \sim q$.

If $\alpha : A \to B$ is a $*$-homomorphism, then $(a_{ij}) \mapsto \big(\alpha(a_{ij})\big)$ is a $*$-homomorphism from $M_n(A)$ to $M_n(B)$. Although this map is more properly denoted α_n or $\alpha \otimes \mathrm{id}$,[3] notational convenience dictates that we usually denote this homomorphism with the same symbol α. In any event, α induces a well-defined semigroup homomorphism $V(\alpha) : V(A) \to V(B)$ defined by $V(\alpha)\big([p]\big) = [\alpha(p)]$. The induced homomorphism $\mathcal{G}\big(V(\alpha)\big)$ from $K_{00}(A)$ to $K_{00}(B)$ is denoted by $K_{00}(\alpha)$.

Proposition 1.10. *Both V and K_{00} are covariant functors from the category of C^*-algebras (and $*$-homomorphisms) to the category of abelian groups. If $\alpha : A \to B$ is a $*$-homomorphism, then*

$$\begin{tikzcd}
V(A) \arrow[r, "V(\alpha)"] \arrow[d, "\iota_A"'] & V(B) \arrow[d, "\iota_B"] \\
K_{00}(A) \arrow[r, "K_{00}(\alpha)"'] & K_{00}(B)
\end{tikzcd}$$

commutes.

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[3]The latter notation comes from identifying $M_n(A)$ with $A \otimes M_n$.

Remark 1.11. The notations $V(\alpha)$ and $K_{00}(\alpha)$ quickly become burdensome to many, and it is standard to use α_* for both maps, and to hope the meaning is clear from context.

The notation K_{00} indicates that $K_{00}(A)$ is nearly what we want. As we shall see, it is exactly the right thing when $1 \in A$. Unfortunately, simply using $K_{00}(\widetilde{A})$ will not have good functorial properties. The solution is given in the following definition. At least to me, it is a good deal more subtle than first impressions might indicate.

Definition 1.12. Suppose that A is a C^*-algebra. Let $\pi : A^1 \to \mathbb{C}$ be the natural map (see (1.1)), and $K_{00}(\pi) : K_{00}(A^1) \to \mathbb{Z}$ the induced map. Then we define $K_0(A) = \ker K_{00}(\pi)$.

Remark 1.13. In view of Remark 1.9, the elements of $K_0(A)$ are exactly those classes $[p] - [q] \in K_{00}(A^1)$ for which $\pi(p) \sim \pi(q)$.

Proposition 1.14. *K_0 is a covariant functor from the category of C^*-algebras to abelian groups. If $\alpha : A \to B$ is a $*$-homomorphism, then $K_0(\alpha) := K_{00}(\alpha^1)|_{K_0(A)}$, where $\alpha^1 : A^1 \to B^1$ is defined by $\alpha^1(a + \lambda) := \alpha(a) + \lambda$. Of course, $K_0(\alpha)$ is also often written α_*.*

Proof. The only slightly nonstandard thing to check is that $K_0(\alpha)\big(K_0(A)\big) \subset K_0(B) = \ker K_{00}(\pi^B)$. But this follows from the functorality of K_{00} applied to $\pi^B \circ \alpha^1 = \pi^A$. □

Let $j : A \to A^1$ be the obvious map (as in (1.1)), and note that $K_{00}(j)$ maps $K_{00}(A)$ into $K_0(A)$. Viewing this as a map into $K_0(A)$, we get what is called the *natural homomorphism* $j_A : K_{00}(A) \to K_0(A)$. It is natural in that

$$\begin{tikzcd} K_{00}(A) \arrow[r, "K_{00}(\alpha)"] \arrow[d, "j_A"'] & K_{00}(B) \arrow[d, "j_B"] \\ K_0(A) \arrow[r, "K_0(\alpha)"] & K_0(B) \end{tikzcd}$$

commutes for all $*$-homomorphisms $\alpha : A \to B$. Although the proof of the following proposition should be easier, it does at least justify the assertion that our definition of K_0 does not "mess things up" when $1 \in A$.

Proposition 1.15. *If $1 \in A$, then the natural homomorphism $j_A : K_{00}(A) \to K_0(A)$ is an isomorphism.*

Proof. We let ψ denote the compression of A^1 onto the corner determined by $1 \in A$; thus, $\psi(a + \lambda) = a + \lambda 1 + 0$. Since $\psi \circ j = \mathrm{id}_A$, we have $K_{00}(\psi) \circ K_{00}(j) = \mathrm{id}_{K_{00}(A)}$. Formally, if we let β be the restriction of $K_{00}(\psi)$ to $K_0(A)$, then

$$\beta \circ j_A = \mathrm{id}_{K_{00}(A)} .$$

On the other hand, let $\iota : \mathbb{C} \to A^1$ be given by $\iota(\lambda) = 0 + \lambda$, and let $\kappa : \mathbb{C} \to A^1$ be given by $\kappa(\lambda) = \lambda 1 + 0$. Notice that

$$\mathrm{id}_{A^1} = j \circ \psi - \kappa \circ \pi + \iota \circ \pi.$$

In particular, if $[p] - [q] \in K_0(A)$, then $\pi(p) \sim \pi(q)$ (Remark 1.13). Thus $\kappa \circ \pi(p) \sim \kappa \circ \pi(q)$ and $\iota \circ \pi(p) \sim \iota \circ \pi(q)$. Therefore

$$[p] - [q] = [j \circ \psi(p)] - [j \circ \psi(q)].$$

This implies

$$j_A \circ \beta = \mathrm{id}_{K_0(A)}\,.$$

Thus β is a two-sided inverse for j_A and the result follows. $\square$

For future reference, we introduce the first of two additional equivalence relations in $V(A)$. Although we use the multiplier algebra $M(A)$ in the definition, it will suffice here to remark only that A is an essential ideal in the unital C^*-algebra $M(A)$ and that $\widetilde{A} \subset M(A)$, while $M_n(\widetilde{A}) \subset M\big(M_n(A)\big)$.

Definition 1.16. If p and q are projections in A, then we say the p is unitarily equivalent to q, written $p \approx q$, if there is a unitary $w \in M(A)$ such that $q = wpw^*$. We can extend $\approx$ to an equivalence relation on $P[A]$ in the obvious way.

Lemma 1.17. *Suppose that p and q are projections in A. If $p \approx q$, then $p \sim q$. On the other hand, if $p \sim q$, then $p \oplus 0 \approx q \oplus 0$ in $M_2(A)\tilde{}$. In particular, $V(A)$ is also the set of $\approx$ equivalence classes in $P[A]$.*

Proof. Suppose that w is a unitary such that $q = wpw^*$. Then $v = wp$ is a partial isometry satisfying $p = v^*v$ and $q = vv^*$.

Let v be a partial isometry with initial projection p and final projection q as above. If we are happy to find a unitary in $M_2(\widetilde{A})$, then $u = \left(\begin{smallmatrix} v & 1-q \\ 1-p & v^* \end{smallmatrix}\right)$ is a unitary in $M_2(\widetilde{A})$ such that $\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right) = u\left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)u^*$.

To get a unitary in $M_2(A)\tilde{}$, we can follow [Bla86]. Let

$$u := \begin{pmatrix} 1-q & v \\ v^* & 1-p \end{pmatrix}\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}.$$

Then $u \in M_2(A)\tilde{}$. Notice that

$$\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}^2 = \left(1_2 - \begin{pmatrix} p & p \\ p & p \end{pmatrix}\right)^2 = 1_2,$$

while

$$\begin{pmatrix} 1-q*v & \\ v^* & 1-p \end{pmatrix}^2 = \left(1_2 - \begin{pmatrix} q & v \\ v^* & p \end{pmatrix}\right)^2 = 1_2.$$

It follows that $u^*u = uu^* = 1_2$, and u is indeed unitary. Now compute that on the one hand,

$$\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix},$$

and using $vp = v$ and $qv = v$,

$$\begin{pmatrix} 1-q & v \\ v^* & 1-p \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} 1-q & v \\ v^* & 1-p \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $u(p \oplus 0)u^* = q \oplus 0$. $\square$

Remark 1.18. We let p_n represent the class of 1_n in $M_\infty(A^1)$. This is convenient in that we can view p_n as a projection in $M_{n+k}(A)$ without having to put up with $1_n \oplus 0_k$. This notation is unfortunate, but seems to be used frequently in, for example, [Bla86] and [WO93]. Even this notation is abused, and p_n can also be used to denote the corresponding projection in $M_\infty(A)$ when $1 \in A$; thus, we also use p_n to describe the corresponding projection in M_{n+k}.

This brings to what Wegge-Olsen calls the "Portrait of K_0".

Proposition 1.19. *Let A be a C^*-algebra.*

(a) *Every element in $K_0(A)$ has a representative of the form $[p]-[q]$, where for some $r \in \mathbb{Z}^+$, p and q are projections in $M_r(A^1)$ such that $p-q \in M_r(A)$. If $1 \in A$, then p and q can be chosen in $M_r(A)$.*

(b) *One can take $q = p_n$ in part* (a) *for some $n \le r$.*

(c) *If $[p]-[q] = 0$ in $K_0(A)$ and $p, q \in M_r(A^1)$, then there is a $m \in \mathbb{Z}^+$ and a $k \ge m+r$ such that $p \oplus p_m \approx q \oplus p_m$ in $M_k(A^1)$.*

If $1 \in A$, then (b) *and* (c) *hold with p_n in $M_\infty(A)$ (see Remark 1.18).*

We need the following. We use the notation from (1.1) and Remark 1.18.

Lemma 1.20. *If p is a projection in $M_r(A^1)$ and $n = \operatorname{rank} \pi(p)$, then there is a projection $q \in M_r(A^1)$ such that $q \approx p$ and $q - p_n \in M_r(A)$.*

Proof. Choose a unitary $u \in M_r$ such that $u\pi(p)u^* = p_n$. Let $q := \iota(u)p\iota(u)^*$. □

Proof of Proposition 1.19. By definition each $x \in K_0(A)$ is of the form $[p'] - [q']$ with p' and q' projections in $M_k(A^1)$ such that $\pi(p') \sim \pi(q')$. Lemma 1.20 implies there is a $n \le k$ and projections $p \sim p'$ and $q \sim q'$ such that both $p - p_n$ and $q - p_n$ belong to $M_k(A)$. Thus $x = [p] - [q]$, and $p - q \in M_k(A)$. This proves the first part of (a). If $1 \in A$, the assertion follows from the surjectivity of the natural isomorphism j_A (Proposition 1.15).

(b) Now let $x = [r] - [s]$ for $r, s \in P[A^1]$. For large enough n, $s \le p_n$. Thus $p_n - s \in P[A^1]$. Moving far enough down the diagonal, there is $r_1 \in P[A^1]$ such that $r_1 \approx r$ and $r_1 p_n = 0$. Since r_1, $p_n - s$, and s are pairwise orthogonal, Theorem 1.3(c) implies

$$\begin{aligned} [r_1 \oplus (p_n - s)] - [p_n] &= [r_1] + [p_n - s] + [s] - [s] - [p_n] \\ &= [r_1] + [p_n] - [s] - [p_n] \\ &= [r_1] - [s] = [r] - [s] \\ &= x. \end{aligned}$$

Since $x \in K_0(A)$, the projection $\pi(r_1 \oplus p_n - s)$ is equivalent to $\pi(p_n)$, and therefore has rank n. Using Lemma 1.20 again, there is a $p \in P[A^1]$ such that $p \approx r_1 \oplus p_n - s$ and $p - p_n \in P[A]$. Since $x = [p] - [p_n]$, this proves (b).

(c) Now suppose that $[p] - [q] = 0$ in $K_0(A)$. The definition of equivalence in $\mathcal{G}\big(V(A^1)\big)$ implies that there is an $r \in P[A^1]$ such that

$$[p] + [r] = [q] + [r].$$

For some m, $r \le p_m$, and by Theorem 1.3

$$\begin{aligned} p \oplus p_m &= p \oplus (r + p_m - r) \\ &\sim p \oplus r \oplus (p_m - r) \\ &\sim q \oplus r \oplus (p_m - r) \\ &\sim q \oplus p_m. \end{aligned}$$

Now (c) follows from Lemma 1.17.[4]

If $1 \in A$, the arguments in parts (b) and (c) can be repeated with p_n replaced with the obvious projection in $P[A]$. □

[4]We can also put $\equiv$ in (c), and I should say so somewhere.

Remark 1.21. In Murphy's treatment [Mur90], he defines p and q in $V(\widetilde{A})$ to be *stably equivalent* if for some m, $p \oplus p_m \sim q \oplus p_m$. Writing $p \simeq q$ for stable equivalence, Murphy proves that $\simeq$ is an equivalence relation and that $MV(\widetilde{A}) := P[\widetilde{A}]/\simeq$ is a *cancellative* semigroup with identity. He then defines $K_0^M(\widetilde{A})$ to be the Grothendieck group of $MV(\widetilde{A})$. Part (c) of the previous result implies $K_0^M(\widetilde{A}) \cong K_{00}(\widetilde{A})$, which is isomorphic to $K_0(\widetilde{A})$ by Proposition 1.15. Murphy's approach has the advantage of starting from a cancellative semigroup. But this approach is not common elsewhere in the literature.

We will need some rather detailed information about projections in C^*-algebras in order to uncover some basic facts about K_0. For example, we'll want to see that $K_0(A)$ is always countable if A is separable (Remark 2.2).

2. Projections in C^*-algebras

Lemma 2.1. *Suppose that p and q are projections in A such that $\|p - q\| < 1$. Then there is a unitary $u \in \widetilde{A}$ such that $q = upu^*$. We can also arrange that $\|1 - u\| \leq \sqrt{2}\|p - q\|$.*

Proof. Let $v = 1 - p - q + 2qp$, and compute that
$$v^*v = 1 - (p - q)^2 = vv^*.$$
In particular, v is normal. Recall that $|v| := (v^*v)^{\frac{1}{2}}$.

Since $\|p - q\| < 1$, we have $\|(p - q)^2\| < 1$, and v^*v is invertible in $\widetilde{A}$. Since v is normal, this forces v to be invertible too. Thus $u := v|v|^{-1}$ is invertible, and
$$u^*u = |v|^{-1}v^*v|v|^{-1} = 1;$$
and u is a unitary. Furthermore
$$vp = (1 - p - q + 2qp)p = qp \quad \text{while} \quad qv = q(1 - p - q + 2qp) = qp.$$
It follows that
$$vp = qv \quad \text{and} \quad pv^* = v^*q \quad \text{which implies that} \quad pv^*v = v^*qv = v^*vp.$$
Thus p commutes with $|v|^{-1}$. Therefore
$$up = v|v|^{-1}p = vp|v|^{-1} = qv|v|^1 = qu,$$
and $q = upu^*$ as desired.

To get the norm estimate, notice that $\mathrm{Re}(v) := (v + v^*)/2 = 1 - (p - q)^2 = v^*v = |v|^2$. Since v, v^*, and $|v|^{-1}$ all commute, it follows that $\mathrm{Re}(u) = \mathrm{Re}(v)|v|^{-1} = |v|$. Now compute
$$\begin{aligned} \|1 - u\|^2 &= \|(1 - u^*)(1 - u)\| = \|2 \cdot 1 - u - u^*\| && (2.1)\\ &= 2\|1 - \mathrm{Re}(u)\| \\ &= 2\|1 - |v|\| && (2.2) \end{aligned}$$
But $v^*v = 1 - (p - q)^2$ is a positive operator of norm at most one, so the same is true of $|v|$. Since $(1 - t) \leq 1 - t^2$ on $[0, 1]$, the functional calculus implies that (2.2) is less than or equal to
$$2\|1 - |v|^2\| = 2\|(p - q)^2\| = 2\|p - q\|^2.$$
It follows that (2.1) is bounded by $2\|p - q\|^2$ as required. □

Corollary 2.2. *If A is separable, then $V(A)$, $K_{00}(A)$, and $K_0(A)$ are countable.*

We can also use the functional calculus to prove that a self-adjoint element which is nearly idempotent, is close to a projection.

Lemma 2.3. *Suppose $a = a^*$ in A and that $\|a^2 - a\| < \frac{1}{4}$. Then there is a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$.*

Proof. We may replace A with the subalgebra $C^*(\{\, a\,\})$ generated by a. Recall that the spectral theorem implies that

$$C^*(\{\, a\,\}) \cong C_0\big(\sigma(a)\big) := \{\, f \in C\big(\sigma(a)\big) : f(0) = 0\,\}.$$

The norm condition on a implies that if $\lambda \in \sigma(a)$, then $\lambda \neq \frac{1}{2}$, and $|\lambda| < \frac{3}{2}$. Thus

$$S := \{\, \lambda \in \sigma(a) : |\lambda| > \frac{1}{2}\,\} = \{\, \lambda \in \sigma(a) : |\lambda| \geq \frac{1}{2}\,\}$$

is clopen in $\sigma(a)$. Therefore characteristic function $\mathbb{I}_S$ is a projection in $C_0\big(\sigma(a)\big)$, and

$$|\lambda - \mathbb{I}_S(\lambda)| < \frac{1}{2} \quad \text{for all } \lambda \in \sigma(a).$$

Thus $\|\operatorname{id} - \mathbb{I}_S\|_\infty < \frac{1}{2}$ and there is a projection $p \in A$ such that $\|a - p\| < \frac{1}{2}$. □

The next lemma says that two projections that are approximately equivalent are actually equivalent provided the element implementing the equivalent looks enough like an partial isometry. Despite this awkward hypothesis, the lemma will play a crucial rôle in Theorem 2.5.

Lemma 2.4. *Suppose that p and q are projections in A. Let $u \in A$ be such that*

$$\|p - u^*u\| < 1, \quad \|q - uu^*\| < 1, \quad \textit{and} \quad u = qup.$$

Then $p \sim q$.

Proof. Using $u = qup$, we see that u^*u belongs to the unital C^*-algebra pAp. The first equation implies that u^*u is invertible in pAp. Similarly, uu^* in invertible in qAq. Let $z = |u|^{-1}$ in pAp, and set $w := uz$. Then $w^*w = zu^*uz = p$.

On the other hand, $ww^* \in qAq$, and

$$\begin{aligned} uu^*ww^* &= uu^*uz^2u^* = u|u|^2z^2u^* \\ &= upu^* = qup^3u^*q \\ &= uu^*. \end{aligned}$$

Since uu^* is invertible in qAq, we have $ww^* = q$. □

Now we want to look see how projections in direct limits of C^*-algebras behave. For the basics and notations for direct limits, see Appendix A.4.

Theorem 2.5. *Suppose that $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$.*

(a) *If p is a projection in A, then there is a $n \in \mathbb{Z}^+$ and a projection $q \in A_n$ such that $p \approx \varphi^n(q)$.*
(b) *If p and q are projections in A_n, and if $\varphi^n(p) \sim \varphi^n(q)$ in A, then there is a $m \geq n$ such that $\varphi_{nm}(p) \sim \varphi_{nm}(q)$ in A_m.*

Proof. Fix a projection $p \in A$. We can find $a_k \in A_{n_k}$ such that $\varphi^{n_k}(a_k) \to p$. Since $p = p^*$, we can replace a_k by $(a_k + a_k^*)/2$ and assume that $a_k = a_k^*$. Since $p = p^2$, we also have $\varphi^{n_k}(a_k^2) \to p$. Thus, $\varphi^{n_k}(a_k^2 - a_k) \to 0$. Thus we can choose m and $a \in A_m$ such that $a = a^*$ and

$$\|\varphi^m(a) - p\| < \frac{1}{2} \quad \text{and} \quad \|\varphi^m(a) - \varphi^m(a^2)\| < \frac{1}{4}.$$

By Theorem A.3, there is a $n \geq m$ such that

$$\|\varphi_{mn}(a) - \varphi_{mn}(a^2)\| < \frac{1}{4}.$$

Using the functional calculus (Lemma 2.3), there is a projection $q \in A_n$ such that $\|\varphi_{mn}(a) - q\| < \frac{1}{2}$. Therefore

$$\begin{aligned}
\|p - \varphi^n(q)\| &\leq \|p - \varphi^m(a)\| + \|\varphi^m(a) - \varphi^n(q)\| \\
&= \|p - \varphi^m(a)\| + \|\varphi^n(\varphi_{mn}(a) - q)\| \\
&< \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

More functional calculus (Lemma 2.1) implies that $p \approx \varphi^n(q)$; this proves (a).

Now suppose that $\varphi^n(p) \sim \varphi^n(q)$ in A. Let u be a partial isometry such that $\varphi^n(p) = u^*u$ and $\varphi^n(q) = uu^*$. Notice that

$$u = uu^*u = u\varphi^n(p) = \varphi^n(q)u.$$

Let $v_k \in A_{n_k}$ be such that $\varphi^{n_k}(v_k) \to u$. We can replace v_k with $\varphi_{nn_k}(q)v_k\varphi_{nn_k}(p)$, so that we can assume $\varphi^{n_k}(v_k) \to u$ and $\varphi_{nn_k}(q)v_k\varphi_{nn_k}(p) = v_k$. Since we have

$$\varphi^n(p) = \lim_k \varphi^{n_k}(v_k^*v_k) \quad \text{and} \quad \varphi^n(q) = \lim_k \varphi^{n_k}(v_kv_k^*),$$

we can find $k \geq n$ and $v \in A_k$ such that

$$\begin{gathered}
v = \varphi_{nk}(q)v\varphi_{nk}(p), \\
\|\varphi^n(p) - \varphi^k(v^*v)\| = \|\varphi^k(\varphi_{nk}(p) - v^*v)\| < 1, \quad \text{and} \\
\|\varphi^n(q) - \varphi^k(vv^*)\| = \|\varphi^k(\varphi_{nk}(q) - vv^*)\| < 1.
\end{gathered}$$

Since

$$\|\varphi^k(\varphi_{nk}(p) - v^*v)\| = \lim_r \|\varphi_{kr}(\varphi_{nk}(p) - v^*v)\| = \lim_r \|\varphi_{nr}(p) - \varphi_{kr}(v)^*\varphi_{kr}(v)\|,$$

we can find a $m \geq k$ and $w = \varphi_{km}(v)$ such that

$$\|\varphi_{nm}(p) - w^*w\| < 1 \quad \text{and} \quad \|\varphi_{nm}(q) - ww^*\| < 1,$$

and

$$w = \varphi_{km}(v) = \varphi_{km}\big(\varphi_{nk}(q)v\varphi_{nk}(p)\big) = \varphi_{nm}(q)w\varphi_{nm}(p).$$

Now the clever computations in Lemma 2.4 imply that $\varphi_{nm}(p) \sim \varphi_{nm}(q)$ in A_m. This completes the proof. $\square$

3. The continuity of K_0

One of the most important functorial properties of K_0 is that it respects direct limits of C^*-algebras — this is what is meant by the phrase "K_0 is continuous". To make this precise we will need to define both direct limits of sequences of semigroups and groups. A direct limit in any category is defined as follows. Let $\{ G_n \}_{n=1}^{\infty}$ be a sequence of objects, and $\tau_n : G_n \to G_{n+1}$ morphisms. A family of morphisms $h_n : G_n \to H$ is called *compatible* if

$$\begin{tikzcd} G_n \arrow[rr, "\tau_n"] \arrow[dr, "h_n"'] & & G_{n+1} \arrow[dl, "h_{n+1}"] \\ & H & \end{tikzcd} \tag{3.1}$$

commutes for all n. Notice that if, for $m \geq n$, we define $\tau_{nm} = \tau_{m-1} \circ \cdots \circ \tau_{n+1} \circ \tau_n$ and $\tau_{nn} = \mathrm{id}_{G_n}$, then (3.1) is equivalent to $h_m \circ \tau_{nm} = h_n$ for all $m \geq n$. An object G together with compatible homomorphisms $\tau^n : G_n \to G$ is called a direct limit of $\{ G_n, \tau_n \}$ if whenever $h_n : G_n \to H$ are a compatible family of morphisms, then there is a unique morphism $h : G \to H$ such that

$$\begin{tikzcd} & G \arrow[dd, "h"] \\ G_n \arrow[ur, "\tau^n"] \arrow[dr, "h_n"'] & \\ & H \end{tikzcd}$$

commutes. A direct limit, if it exists, is unique up to isomorphism.

In the category of sets, where the morphisms are just functions, we can form a direct limit as follows. Let

$$\coprod_{n=1}^{\infty} G_n = \{ (n, x) : x \in G_n \}$$

be the disjoint union. Then there is a smallest equivalence relation on $\coprod G_n$ such that $(n, x) \sim (n + 1, \tau_n(x))$. It is not hard to see that

$$(n, x) \sim (m, y) \Longleftrightarrow \text{there is a } k \geq \max\{ m, n \} \text{ such that } \tau_{nk}(x) = \tau_{mk}(y).$$

Let G be the quotient space $\coprod G_n/\sim$, and let $[n, x]$ be the class of (n, x) in G, then we can define $\tau^n : G_n \to G$ by $\tau^n(x) = [n, x]$. To see that (G, τ^n) is an inductive limit, let $h_n : G_n \to Y$ be a compatible family of functions. Notice that if $n \leq m$, then $\tau^n(x) = \tau^m(x')$ if and only if $x' = \tau_{nm}(x)$. Thus we can define $h : G \to Y$ by $h\bigl(\tau^n(x)\bigr) = h_n(x)$. Thus inductive limits exist in the category of sets and maps.[5]

Lemma 3.1. *Every direct system $\{ G_n, \tau_n \}$ of abelian groups (semigroups) has a direct limit $(G, \tau^n) := \varinjlim(G_n, \tau_n)$, which is unique up to isomorphism.*

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[5]Inductive limits also exist in the category of topological spaces and continuous maps. We simply equip G with the topology where V is open in G if and only if $(\tau^n)^{-1}(V)$ is open in G_n for all n. Then the maps constructed above are all continuous.

Sketch of Proof. Let (G, τ^n) the direct limit of $\{ G_n, \tau_n \}$ as sets. If each G_n is a semigroup, then we can define an associative operation on G as follows:

$$[n, x] + [m, y] := [k, \tau_{nk}(x) + \tau_{mk}(y)],$$

where $k \geq \max\{ n, m \}$. The maps τ^n are clearly additive. If each G_n is a group, then G is a group with identity equal to $[n, 0]$ and $-[n, x] = [n, -x]$. That G has the appropriate universal property is checked as above. $\square$

In the category of groups, there is another proof.

Sketch of Alternate Proof. Let

$$G_1 = \{ (x_i) \in \prod G_i : \tau_n(x_n) = x_{n+1} \quad \text{for all sufficiently large } n \}.$$

Let $F = \{ (x_i) \in G_1 : x_n = 0 \quad \text{for all sufficiently large } n \}$. Then we can set $G = G_1/F$ with $\tau^n : G_n \to G$ defined by

$$\tau^n(x)(m) = \begin{cases} \tau_{nm}(x) & \text{if } m \geq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The rest is routine. $\square$

Remark 3.2. Note that we have $G = \bigcup_n \tau^n(G_n)$ and that $\tau^n(a) = \tau^m(b)$ if and only if there is a $k \geq \max\{ n, m \}$ such that $\tau_{nk}(a) = \tau_{mk}(b)$.

Example 3.3. Let $G_n = \mathbb{Z}$ for all n, and let $\tau_n(m) := s(n)m$, where $s : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a given function. Let $s!(n) := s(1)s(2) \cdots s(n)$, and define

$$\begin{aligned} \mathbb{Z}(s) &:= \{ m/s!(n) : m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}^+ \} \\ &= \{ m/p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} : m \in \mathbb{Z} \text{ and } p_j \text{ a prime} \\ &\qquad\qquad\qquad\qquad \text{such that } p_j^{n_j} \mid s!(n) \text{ for some } n \}. \end{aligned}$$

If $\tau^n(m) := m/s!(n)$, then $(\mathbb{Z}(s), \tau^n)$ equal to $\varinjlim(G_n, \tau_n)$.

Proof. We just have to see that $\mathbb{Z}(s)$ has the right universal property! So, let $h_n : \mathbb{Z} \to H$ be a family of compatible homomorphisms. Thus if $k \geq n$ we have

$$h_n(m) = h_k\big(\tau_{nk}(m)\big) = h_k\Big(\frac{s!(k)}{s!(n)} m\Big).$$

In other words, if $m/s!(n) = m'/s!(k)$, then $h_n(m) = h_k(m')$, Thus we can define $h : G \to H$ by

$$h\Big(\frac{m}{s!(n)}\Big) = h_n(m). \quad \square$$

The universal property approach can be quite helpful.

Example 3.4. Suppose that $(G, \tau^n) = \varinjlim(G_n, \tau_n)$. Suppose that $h_n : G_n \to H$ is a family of compatible homomorphisms, and that $h : G \to H$ is the corresponding homomorphism. Notice that $\tau_n(\ker h_n) \subset \ker h_{n+1}$, and $\tau^n(\ker h_n) \subset \ker h$. Then

$$(\ker h, \tau^n) = \varinjlim(\ker h_n, \tau_n).$$

Proof. Suppose that $\rho_n : \ker h_n \to K$ are compatible homomorphisms. Note that $\ker h = \bigcup \tau^n(\ker h_n)$. If $m \leq n$ and $\tau^n(a) = \tau^m(b)$, then Remark 3.2 implies there is a $k \geq n$ such that $\tau_{nk}(a) = \tau_{mk}(b)$. Thus, $\rho_n(a) = \rho_m(b)$, and we can define $\rho : \ker h \to K$ by $\rho\big(\tau^n(a)\big) = \rho_n(a)$. $\square$

Example 3.5. Suppose that $\{ V_n, \tau_n \}$ is direct sequence of semigroups with identities. Let $(V, \tau^n) = \varinjlim (V_n, \tau_n)$. Then

$$\big(\mathcal{G}(V), \mathcal{G}(\tau^n)\big) = \varinjlim \big(\mathcal{G}(V_n), \mathcal{G}(\tau_n)\big).$$

Proof. Suppose that we have compatible group homomorphisms $h_n : \mathcal{G}(V_n) \to H$. Define $h_n' : h_n \circ \iota_{V_n}$. Then the h_n' are compatible with the τ_n:

$$\begin{tikzcd}
V_n \arrow[dr, dashed, "\iota_{V_n}"] \arrow[ddrr, bend left, "h_n'"] \arrow[dddd, "\tau_n"'] & & \\
& \mathcal{G}(V_n) \arrow[dr, dashed, "h_n"] \arrow[dd, dashed, "\mathcal{G}(\tau_n)"] & \\
& & H \\
& \mathcal{G}(V_{n+1}) \arrow[ur, dashed, "h_{n+1}"'] & \\
V_{n+1} \arrow[ur, dashed, "h_{n+1}"'] \arrow[uurr, bend right, "h_{n+1}'"'] & &
\end{tikzcd}$$

Therefore, there is a semigroup homomorphism $h : V \to H$ such that $h \circ \tau^n = h_n'$ for all n. I claim that $\mathcal{G}(h) : \mathcal{G}(V) \to H$ has the right property: namely, $\mathcal{G}(h) \circ \mathcal{G}(\tau^n) = h_n$. However, this follows from chasing around the following diagram:[6]

$$\begin{tikzcd}
\mathcal{G}(G) \arrow[rr, "\mathcal{G}(h)"] & & H \\
& V \arrow[ul, dashed, "\iota_V"] \arrow[ur, dashed, "h"'] & \\
& V_n \arrow[u, dashed, "\tau^n"] \arrow[uur, dashed, "h_n'"'] \arrow[dl, dashed, "\iota_{V_n}"'] & \\
\mathcal{G}(V_n) \arrow[uuu, "\mathcal{G}(\tau^n)"] \arrow[uuurr, bend right, "h_n"'] & &
\end{tikzcd}$$

□

These observations about direct limits, now allow us to prove the following result which is what is meant by the continuity of K_0.

Theorem 3.6. *Suppose that (A, φ^n) is the direct limit of $\{ A_n, \varphi_n \}$. Then*

$$\big(K_0(A), K_0(\varphi^n)\big) = \varinjlim \big(K_0(A_n), K_0(\varphi_n)\big). \tag{3.2}$$

[6] Of course, it is also a direct consequence of the fact that $\mathcal{G}$ is a functor from semigroups to groups and that $\mathcal{G}(h_n') = h_n$.

The same is true for V and K_{00}:

$$\big(V(A),V(\varphi^n)\big)=\varinjlim\big(V(A_n),V(\varphi_n)\big) \tag{3.3}$$

$$\big(K_{00}(A),K_{00}(\varphi^n)\big)=\varinjlim\big(K_{00}(A_n),K_{00}(\varphi_n)\big). \tag{3.4}$$

We need a preliminary result before we give the proof of the "of course it's true" variety. Recall that if $\varphi : A \to B$ is a $*$-homomorphism, then $\varphi^1 : A^1 \to B^1$ is defined by $\varphi^1(a+\lambda) := \varphi(a)+\lambda$.

Lemma 3.7. *Suppose that (A,φ^n) is the C^*-direct limit of $\{\, A_n,\varphi_n \,\}$. Then $(A^1,(\varphi^n)^1)$ is the direct limit of $\{\, A^1_n,\varphi^1_n \,\}$.*

Proof. We just have to show that $(A^1,(\varphi^n)^1)$ has the right universal property. So, suppose that $\alpha_n : A^1_n \to B$ are $*$-homomorphisms which are compatible with the φ^1_n's: that is, $\alpha_n = \alpha_{n+1}\circ\varphi^1_n$ for all n. It follows that the α_n are compatible with the φ_n's; thus the universal property of A implies that there is a $*$-homomorphism $\varphi : A \to B$ such that $\alpha_n = \varphi\circ\varphi^n$. Since $\alpha_n(0+1)=\alpha_m(0+1)$ for all n and m, we can denote the common value by b; note that b is a projection. Furthermore, b acts as the identity on $\bigcup_n \alpha_n(A_n) = \varphi\big(\bigcup_n \varphi^n(A_n)\big)$. It follows that b acts as the identity on $\varphi(A)$. Thus, we can define a $*$-homomorphism $\tilde\varphi : A^1 \to B$ by $\tilde\varphi(a+\lambda) := \varphi(a)+\lambda b$. Thus $\tilde\varphi\circ(\varphi^n)^1(a+\lambda) = \varphi\big((\varphi^n(a)\big)+\lambda b = \alpha_n(a)+\lambda b = \alpha_n(a+\lambda)$, as required. Note that $\tilde\varphi$ is uniquely determined since $\bigcup\varphi^n(A_n)$ dense in A, which forces $\bigcup(\varphi^n)^1(A^1_n)$ to be dense in A^1. □

Proof of Theorem 3.6. I claim it will suffice to prove (3.3). If (3.3) holds, then Example 3.5 implies that K_{00} is continuous (i.e., (3.4) holds). In particular, we can combine this with Lemma 3.7 to conclude that

$$\big(K_{00}(A^1),K_{00}((\varphi^n)^1)\big)=\varinjlim\big(K_{00}(A^1_n),K_{00}(\varphi^1_n)\big).$$

Let $\pi_n : A^1_n \to \mathbb{C}$ be the natural map. The maps $K_{00}(\pi_n)$ are compatible with the $K_{00}(\varphi^1_n)$, and so there is a unique map $\tau : K_{00}(A^1)\to\mathbb{Z}$ such that $\tau\circ K_{00}((\varphi^n)^1) = K_{00}(\pi_n)$. If $\pi : A^1 \to \mathbb{C}$ is the natural map, then $\pi\circ(\varphi^n)^1 = \pi_n$; thus, we have $\tau = K_{00}(\pi)$. Since $K_0(A) = \ker K_{00}(\pi)$ and $K_0(A_n) = \ker K_{00}(\pi_n)$, the sufficiency of the claim is given by Example 3.4.

To prove (3.3), let (H,ψ_i) be the direct limit semigroup of $\{\, V(A_i),V(\varphi_i)\,\}$. The functorality of V implies that $V(\varphi^i) : V(A_i)\to V(A)$ are compatible with the $V(\varphi_i)$. The universal property of H implies that there is a semigroup homomorphism $\Lambda : H \to V(A)$ such that

$$\begin{tikzcd} & H \arrow[dd, "\Lambda"] \\ V(A_i) \arrow[ur, "\psi_i"] \arrow[dr, "V(\varphi^i)"'] & \\ & V(A) \end{tikzcd} \tag{3.5}$$

commutes. It suffices to prove that Λ is a bijection. To do this, we will apply Theorem 2.5 to $M_n(A) = \varinjlim_k M_n(A_k)$.

Let p be a projection in $M_n(A)$. Theorem 2.5 allows us to find a projection $q \in M_n(A_k)$ such that $p \sim \varphi^k(q)$. Thus,

$$[p] = [\varphi^k(q)] = V(\varphi^k)\big([q]\big) = \Lambda\big(\psi_k([q])\big),$$

and it follows that Λ is surjective.

Now suppose that $x, y \in H$ satisfy $\Lambda(x) = \Lambda(y)$. We can assume that there are projections p and q in $M_n(A_j)$ such that

$$x = \psi_j\big([p]\big) \quad \text{and} \quad y = \psi_j\big([q]\big).$$

The commutativity of (3.5) implies that $\varphi^j(p)$ and $\varphi^j(q)$ define the same class in $V(A)$. Increasing n if necessary, we can assume that $\varphi^j(p) \sim \varphi^j(q)$ in $M_n(A)$. Thus, Theorem 2.5 implies that there exists $m \geq j$ such that

$$\varphi_{jm}(p) \sim \varphi_{jm}(q) \quad \text{in } M_n(A_m).$$

Thus $[\varphi_{jm}(p)] = [\varphi_{jm}(q)]$ in $V(A_m)$. Therefore,

$$\begin{aligned} x = \psi_j\big([p]\big) &= \psi_m \circ V(\varphi_{jm})\big([p]\big) = \psi_m\big([\varphi_{jm}(p)]\big) \\ &= \psi_m\big([\varphi_{jm}(q)]\big) = \psi_j\big([q]\big) \\ &= y. \end{aligned}$$

Thus Λ is injective. This completes the proof. □

Now we can give one of the "basic" examples in the subject. I would be interested to find a proof of this corollary which did not require the overhead of Theorem 3.6.

Corollary 3.8. *$K_0\big(\mathcal{K}(\mathcal{H})\big) \cong \mathbb{Z}$.*

First Proof. The only content is when $\dim \mathcal{H} = \infty$. Let $\varphi_n : M_n \to M_{n+1}$ be given by $\varphi_n(a) := a \oplus 0$. Then $\mathcal{K}(\mathcal{H}) = \varinjlim(M_n, \varphi_n)$. Since $V(\varphi_n)\big([e_{11}^n]\big) = [e_{11}^{n+1}]$, it follows that $K_{00}(\varphi_n) = \mathrm{id}_{\mathbb{Z}}$. Now the result is immediate from Theorem 3.6. □

Since $K_{00}(A)$ is much more tractable than $K_0(A)$, the following corollary of Theorem 3.6 is particularly nice.[7] Notice that if A is the inductive limit of unital C^*-algebras, for example an AF-algebra, then A need not be unital itself. But A will have an approximate identity $\{\, q_n \,\}_{n=1}^{\infty}$ consisting of projections. To see this, just let $q_n = \varphi^n(1_n)$ where 1_n is the identity in A_n. If $\{\, q_n \,\}_{n=1}^{\infty}$ is such an approximate identity, then the definition of approximate identity requires that $q_n \leq q_{n+1}$ as positive elements, but in fact it is an exercise to see that $q_n \leq q_{n+1}$ as projections; that is, $q_{n+1}q_n = q_n$.[8]

Corollary 3.9. *Suppose that A has a countable approximate identity $\{\, q_n \,\}$ consisting of projections. Then the natural map $j_A : K_{00}(A) \to K_0(A)$ is an isomorphism.*

Proof. We have $A = \overline{\bigcup q_n A q_n}$ and $q_{n+1} \geq q_n$ implies that $q_n A q_n \subset q_{n+1} A q_{n+1}$. Thus $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$, where the φ_n and φ^n are the appropriate inclusion maps, and $A_n := q_n A q_n$. Since each $q_n A q_n$ is unital, the natural maps j_{A_n} :

[7] Since Davidson [Dav96] uses K_{00} in place of K_0, this is particularly important for his treatment.

[8] Suppose $p - q \geq 0$ in A. We may as well assume that A is a subalgebra of $B(\mathcal{H})$. Then for any vector $h \in H$, we have $\|qh\|^2 = (qh \mid qh) = (qh \mid h) \leq (ph \mid h) = \|ph\|^2 \leq \|p\|$. It follows that $qh = h$ implies that $ph = h$; this means $pq = q$.

$K_{00}(A_n) \to K_0(A_n)$ are isomorphisms. Thus, we have the following commutative (infinite) diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & K_{00}(A_n) \arrow[r, "K_{00}(\varphi_n)"] \arrow[d, "j_{A_n}"'] & K_{00}(A_{n+1}) \arrow[r] \arrow[d, "j_{A_{n+1}}"] \arrow[rr, bend left, "K_{00}(\varphi^{n+1})"] & \cdots & K_{00}(A) \arrow[d, dotted, "j_A"] \\
\cdots \arrow[r] & K_0(A_n) \arrow[r, "K_0(\varphi_n)"] & K_0(A_{n+1}) \arrow[r] \arrow[rr, bend right, "K_0(\varphi^{n+1})"'] & \cdots & K_0(A).
\end{tikzcd}$$

Now general nonsense implies that $K_{00}(\varphi^n)(x) \mapsto K_0(\varphi^n)\big(j_{A_n}(x)\big)$ defines an isomorphism of $K_{00}(A)$ onto $K_0(A)$ (with inverse $K_0(\varphi^n)(x) \mapsto K_{00}(\varphi^n)\big(j_{A_n}^{-1}(x)\big)$). Since $K_{00}(\varphi^n)\big(j_{A_n}(x)\big) = j_A\big(K_{00}(\varphi^n)(x)\big)$, we conclude this isomorphism is $j_A : K_{00}(A) \to K_0(A)$, and j_A is an isomorphism as claimed. $\square$

Second proof of Corollary 3.8. Since $\mathcal{K}(\mathcal{H})$ has an approximate identity of projections, Corollary 3.9 applies. But $K_{00}(\mathcal{K}(\mathcal{H}))$ is clearly isomorphic to $\mathbb{Z}$. $\square$

Example 3.10. Let $s : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a given function. For convenience, define $s(0) = 1$, and let $s!(n) := s(1)s(2)\ldots s(n)$. Define $\varphi_n : M_{s!(n-1)} \to M_{s!(n)}$ by

$$\varphi_n(m) := \underbrace{m \oplus m \oplus \cdots \oplus m}_{s(n)\text{-times}}.$$

We let M_s be the inductive limit of $\{\, A_n, \varphi_n \,\}$, where $A_1 = \mathbb{C}$ and $A_n = M_{s!(n-1)}$:

$$\mathbb{C} \xrightarrow{\varphi_1} M_{s!(1)} \xrightarrow{\varphi_2} M_{s!(2)} \xrightarrow{\varphi_3} \cdots.$$

Then $K_0(\varphi_n)\big([e_{11}^n]\big) = s(n) \cdot [e_{11}^{n+1}]$, and $K_0(M_s)$ is the direct limit

$$\mathbb{Z} \xrightarrow{\times s(1)} \mathbb{Z} \xrightarrow{\times s(2)} \mathbb{Z} \longrightarrow \cdots,$$

which is $\mathbb{Z}(s)$ as in Example 3.3.

4. Stability

In the this section, we want to consider the C^*-tensor product of a C^*-algebra A with the compact operators $\mathcal{K}$ on a separable infinite dimensional Hilbert space. For the full story, consult Appendix B of [RW98]. The basic idea is as follows. We may as well assume that A is a C^*-subalgebra of $B(\mathcal{V})$ for some Hilbert space $\mathcal{V}$, and that $\mathcal{K}$ is the algebra of compact operators on a Hilbert space $\mathcal{H}$ with orthonormal basis $\{\, h_i \,\}_{i=1}^\infty$. If $T \in B(\mathcal{V})$ and $S \in B(\mathcal{H})$, then

$$T \otimes S(v \otimes h) := Tv \otimes Sh$$

defines a bounded operator $T \otimes S$ in on the Hilbert space tensor product $\mathcal{V} \otimes \mathcal{H}$. It can be shown that $\|T \otimes S\| = \|T\|\|S\|$ [RW98, Lemma B.2]. Furthermore, the map $(T, S) \mapsto T \otimes S$ induces an injective map of the algebraic tensor product $B(\mathcal{V}) \odot B(\mathcal{H})$ into $B(\mathcal{V} \otimes \mathcal{H})$ [RW98, Lemma B.3]. If $e_{ij} := h_j \otimes \bar{h}_i$ is the usual matrix unit in $B(\mathcal{H})$ (i.e., the rank-one operator from the span of h_j to the span of h_i), then $(a_{ij}) \mapsto \sum a_{ij} \otimes e_{ij}$ defines a $*$-isomorphism of $M_n(A)$ onto a subalgebra of $B(\mathcal{V} \otimes \mathcal{H})$ which, for the purposes of this discussion, we define to be $A \otimes M_n$. Note that $A \otimes M_n \subset A \otimes M_{n+1}$. We define

$$A \otimes \mathcal{K} = \overline{\bigcup A \otimes M_n};$$

thus

$$(A \otimes \mathcal{K}, \varphi^n) = \varinjlim (A_n \otimes M_n, \varphi_n),$$

where the φ^n and the φ_n are just the inclusion maps.[9] Let $\alpha^n : A \to A \otimes M_n$ be defined by $\alpha^n(a) := a \otimes e_{11}$. We get a map[10] $\alpha : A \to A \otimes \mathcal{K}$ defined by $\alpha(a) = a \otimes e_{11}$.

Theorem 4.1. *The map $\alpha : A \to A \otimes \mathcal{K}$ sending a to $a \otimes e_{11}$ induces a natural isomorphism $K_0(\alpha) : K_0(A) \to K_0(A \otimes \mathcal{K})$*

The proof relies on the following "finite dimensional version". This comment merely reflects the facts that M_r is the compact operators on a r-dimensional space, and that $M_r(A)$ is isomorphic to $A \otimes M_r$, and the natural isomorphism intertwines α^r and α_r, where α_r is defined below.

Proposition 4.2. *Let $r \geq 2$. Define $\alpha_r : A \to M_r(A)$ by $\alpha_r(a) := a \oplus 0_{r-1}$. Then $K_0(\alpha_r) : K_0(A) \to K_0\big(M_r(A)\big)$ is an isomorphism.*

Proof. Fix $n \in \mathbb{Z}^+$ and let $\zeta : M_n\big(M_r(A^1)\big) \to M_{nr}(A^1)$ be the "obvious" isomorphism ([RW98, Example B.19]). Let $\alpha_r^1 : A^1 \to M_r(A)^1$ be the natural extension and note that if p is a projection in $M_n(A^1)$, then[11]

$$\zeta\big(\alpha_r^1(p)\big) \approx p \oplus (r-1) \cdot \iota\big(\pi(p)\big), \tag{4.1}$$

where the unitary implementing the equivalence is the permutation matrix associated to

$$\begin{pmatrix} 1 & 2 & \dots & r & r+1 & r+2 & \dots & 2r & (n-1)r+1 & (n-1)r+2 & \dots & nr \\ 1 & r+1 & \dots & (n-1)r+1 & 2 & r+2 & \dots & (n-1)r+2 & r & 2r & \dots & nr \end{pmatrix}.$$

Now we want to see that $K_0(\alpha_r)$ is bijective. (Keep in mind that $K_0(\alpha_r)$ is the restriction of $K_{00}(\alpha_r^1)$.) Suppose that $K_0(\alpha_r)\big([p]-[q]\big) = 0$ in $K_0\big(M_r(A)\big)$. Then we may assume (Proposition 1.19) that $p, q \in M_n(A^1)$ and that there exists m such that

$$\alpha_r^1(p \oplus 1_m) = \alpha_r^1(p) \oplus 1_m \sim \alpha_r^1(q) \oplus 1_m = \alpha_r^1(q \oplus 1_m).$$

Since ζ is a isomorphism, (4.1) implies that

$$p \oplus 1_m \oplus (r-1) \cdot \iota\big(\pi(p) \oplus 1_m\big) \sim q \oplus 1_m \oplus (r-1) \cdot \iota\big(\pi(q) \oplus 1_m\big). \tag{4.2}$$

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[9]It is not obvious that our construction of $A \otimes \mathcal{K}$ is independent of how we represent A as operators on Hilbert space. None the less, it is independent, and this follows from [RW98, Theorem B.9]. In our case, it is not so difficult to see that $A \otimes \mathcal{K}$ as defined above is the direct limit of the $M_n(A)$, and so is uniquely defined.

[10]This map is not "natural": it depends on the choice of h_1.

[11]To see where (4.1) comes from consider that case $n = r = 2$. Then α_2^1 maps

$$\begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix} \mapsto \begin{pmatrix} a+\alpha & 0 & b+\beta & 0 \\ 0 & \alpha & 0 & \beta \\ c+\gamma & 0 & d+\delta & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \approx \begin{pmatrix} a+\alpha & b+\beta & 0 & 0 \\ c+\gamma & d+\delta & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{pmatrix}$$

Since $[p]-[q] \in K_0(A)$, we have $\pi(p) \sim \pi(q)$, and it follows from (4.2) that $[p]-[q] = 0$ in $K_0(A)$. Thus $K_0(\alpha_r)$ is injective.

But surjectivity is easy. If p is a projection in $M_n\big(M_r(A)\big)$, then

$$\zeta\big(\alpha_r^1(\zeta(p))\big) \approx \zeta(p) \oplus (r-1) \cdot \iota\big(\pi(\zeta(p))\big).$$

Then, since $\zeta^{-1}\big(\iota\big(\pi(\zeta(p))\big)\big) = \iota\big(\pi(p)\big)$,

$$\begin{aligned} \alpha_r^1\big(\zeta(p)\big) &\approx p \oplus (r-1) \cdot \zeta^{-1}\big(\iota\big(\pi(z(p))\big)\big) \\ &\approx p \oplus (r-1) \cdot \iota\big(\pi(p)\big) \end{aligned}$$

Consequently, if $[p]-[q] \in K_0\big(M_r(A)\big)$, then $\pi(p) \sim \pi(q)$, and it follows that

$$K_0(\alpha_r)\big([\zeta(p)] - [\zeta(q)]\big) = [\alpha_r^1(\zeta(p))] - [\alpha_r^1(\zeta(q))] = [p] - [q] \quad \square$$

Proof of Theorem 4.1. Since the diagram

$$\begin{tikzcd} {} \arrow[r] & A \otimes M_n \arrow[r, "\varphi_n"] & A \otimes M_{n+1} \arrow[r] \arrow[rr, bend left, "\varphi^{n+1}"] & \cdots & A \otimes \mathcal{K} \\ {} \arrow[r] & A \arrow[u, "\alpha_n"] \arrow[r, "\mathrm{id}"] & A \arrow[u, "\alpha_{n+1}"] \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \cdots & A \arrow[u, "\alpha"] \end{tikzcd}$$

commutes, the continuity of K_0 (Theorem 3.6) and the isomorphism of $M_r(A)$ and $A \otimes M_r$ implies that we obtain a commutative diagram

$$\begin{tikzcd} {} \arrow[r] & K_0(A \otimes M_n) \arrow[r, "K_0(\varphi_n)"] & K_0(A \otimes M_{n+1}) \arrow[r] \arrow[rr, bend left, "K_0(\varphi^{n+1})"] & \cdots & K_0(A \otimes \mathcal{K}) \\ {} \arrow[r] & K_0(A) \arrow[u, "K_0(\alpha_n)"] \arrow[r, "\mathrm{id}"] & K_0(A) \arrow[u, "K_0(\alpha_{n+1})"] \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \cdots & K_0(A) \arrow[u, "K_0(\alpha)"] \end{tikzcd}$$

Since each $K_0(\alpha_n)$ is an isomorphism by Proposition 4.2, it follows that $K_0(\alpha)$ is too.

If $\varphi : A \to B$ is a homomorphism, then the maps $\varphi : M_n(A) \to M_n(B)$ induce a homomorphism $\varphi \otimes \mathrm{id} : A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ A \otimes \mathcal{K} \arrow[r, "\varphi\otimes\mathrm{id}"] & B \otimes \mathcal{K} \end{tikzcd}$$

commutes. The naturality of $K_0(\alpha)$ then follows by functorality:

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(\varphi)"] \arrow[d, "K_0(\alpha_A)"'] & K_0(B) \arrow[d, "K_0(\alpha_B)"] \\ K_0(A \otimes \mathcal{K}) \arrow[r, "K_0(\varphi\otimes\mathrm{id})"] & K_0(B \otimes \mathcal{K}) \end{tikzcd} \quad \square$$

5. Homotopy

Definition 5.1. Two projections p and q in a C^*-algebra A are homotopic if there is a continuous projection-valued function $r : [0,1] \to A$ such that $r_0 = r$ and $r_1 = q$. In this case we write $p \equiv q$.

Our first task is to see that homotopic projections are necessarily unitary equivalent. This requires a bit of overhead which will be of use latter on.

Lemma 5.2. *Suppose that p is a projection in a C^*-algebra A. Let*

$$\mathcal{N} := \{\, q \in A : q = q^* = q^2 \text{ and } \|q - p\| < 1 \,\}.$$

Then there is a continuous map $q \mapsto u_q$ from $\mathcal{N}$ to the unitaries in $\widetilde{A}$ such that

$$u_p = 1 \quad \text{and} \quad q = u_q p u_q^*. \tag{5.1}$$

If $A = M_n(B^1)$ and $\pi(p) = p_m$ for some $m \le n$, then we can assume $\pi(u_q) = 1_n$ whenever $\pi(q) = p_m$.

Proof. For each $q \in \mathcal{N}$, let $v_q := 2q - 1$ and $z_q := v_q v_p + 1$. Note that v_q is a self-adjoint unitary (aka a *symmetry*). A straightforward calculation reveals that

$$q z_q = z_q p. \tag{5.2}$$

And

$$\begin{aligned} \|z_q - 2\| &= \|v_q v_p - 1\| = \|v_q(v_p - v_q)\| \\ &\le \|v_p - v_q\| \\ &= 2\|q - p\|. \end{aligned}$$

Since $\|q - p\| < 1$ if $q \in \mathcal{N}$, this implies that z_q is invertible in $\widetilde{A}$, and (5.2) implies that $q = z_q p z_q^{-1}$. Then $u_q := z_q |z_q|^{-1}$ is a unitary, and it is not hard to check that $q = u_q p u_q^*$ [WO93, Lemma 5.2.4].

Notice that $\|z_q - z_r\| = \|v_q v_p - v_r v_p\| \le \|v_q - v_r\| = 2\|q - r\|$. Thus $q \mapsto z_q$ is continuous. Since $z \mapsto z|z|^{-1}$ is continuous[12] on on the invertible elements of $\widetilde{A}$, (5.1) follows.

To prove the final assertion, notice that $\pi(v_q) = 1_m \oplus -1_{n-m}$. Thus, $\pi(z_q) = 2\cdot 1_n$ and $\pi(u_q) = 1_n$ as required. $\square$

Corollary 5.3. *Suppose that $t \mapsto r_t$ is a continuous projection-valued function from $[0,1]$ to A. Then there is a continuous unitary-valued function u from $[0,1]$ to $\widetilde{A}$ such that $u_0 = 1$ and $r_t = u_t r_0 u_t^*$. If $A = M_n(B^1)$ and $\pi(r_t) = p_m$ for some $m \le n$ and all t, then we can assume that $\pi(u_t) = 1_n$ for all t.*

Proof. If $\|r_0 - r_t\| < 1$ for all t, then the result follows immediately from Lemma 5.2. Otherwise, we can choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\|r_{t_{i-1}} - r_t\| < 1$ if $t \in [t_{i-1}, t_i]$ and such that there is a continuous unitary-valued function u^i from $[t_{i-1}, t_i]$ to $\widetilde{A}$ such that $u^i_{t_{i-1}} = 1$ and $r_t = u^i_t r_{t_{i-1}} (u^i_t)^*$ for $t \in [t_{i-1}, t_i]$. The result follows by gluing together the u^i:

$$u_t := u^i_t u^{i-1}_{t_{i-1}} \cdots u^1_{t_1} \quad \text{if } t \in [t_{i-1}, t_i]. \quad \square$$

---

[12] If $f \in C_0(\mathbb{R})$, then the map $a \mapsto f(a)$ is easily seen to be continuous from the self-adjoint elements in A to A.

Proposition 5.4. *Suppose that p and q are projections in A. If $p \equiv q$, then $p \approx q$. On the other hand, if $p \approx q$, then $p \oplus 0 \equiv q \oplus 0$ in $M_2(A)$. In particular, $V(A)$ is also the set of $\equiv$ equivalence classes in $P[A]$.*

Since $r_0 \equiv r_1$ in Corollary 5.3, the first assertion in Proposition 5.4 follows. The second assertion requires some results about unitaries which will also be quite useful when we turn to the definition of K_1.

First we need some notation from [WO93]. We let $\mathrm{U}(A)$ be the group of unitary elements in $\widetilde{A}$, and $\mathrm{GL}(A)$ the group of invertible elements in $\widetilde{A}$. Then we also use

$$\begin{aligned}
\mathrm{GL}_n(A) &:= \mathrm{GL}\big(M_n(\widetilde{A})\big)\\
\mathrm{U}_n(A) &:= \mathrm{U}\big(M_n(\widetilde{A})\big)\\
M_n^1(A) &:= \{\, a \in M_n(A^1) : \pi(a) = 1_n \,\}\\
\mathrm{GL}_n^1(A) &:= \{\, a \in \mathrm{GL}_n(A^1) : \pi(a) = 1_n \,\}\\
\mathrm{U}_n^1(A) &:= \{\, u \in \mathrm{U}_n(A^1) : \pi(u) = 1_n \,\}.
\end{aligned}$$

In particular, $\mathrm{U}_1(A) = \mathrm{U}(A)$. Elements of $M_n^1(A)$, $\mathrm{GL}_n^1(A)$, and $\mathrm{U}_n^1(A)$ are called *normalized*. Notice that if $n \geq 2$, then $M_n(A^1) \supsetneq M_n(A)^1$. However, we do always have $\mathrm{GL}_n^1(A) = \mathrm{GL}_1^1\big(M_n(A)\big)$ and $\mathrm{U}_n^1(A) = \mathrm{U}_1^1\big(M_n(A)\big)$.

Notice that $M_n^1(A)$ is not an algebra — or even a vector space! It is a semigroup and both $\mathrm{GL}_n^1(A)$ and $\mathrm{U}_n^1(A)$ are sub*groups*.

Lemma 5.5. *Suppose that A is a C^*-algebra.*
 (a) *$\mathrm{GL}_n^1(A)$ is open in $M_n^1(A)$, as well as locally convex and locally path connected.*
 (b) *$\mathrm{U}_n^1(A)$ is a deformation retract of $\mathrm{GL}_n^1(A)$.*
 (c) *$\mathrm{U}_n^1(A)$ is locally path connected.*

Similar statements hold for $\mathrm{GL}_n(A)$ and $\mathrm{U}_n(A)$.

Remark 5.6. It follows that the connected components and path components of $\mathrm{U}_n(A)$ ($U_n^1(A)$, $\mathrm{GL}_n(A)$, or $\mathrm{GL}_n^1(A)$) coincide. The notation $\mathrm{U}_n(A)_0$ ($U_n^1(A)_0$, $\mathrm{GL}_n(A)_0$, or $\mathrm{GL}_n^1(A)_0$) is used to denote the connected component of the identity.

Proof. Let B be a unital C^*-algebra — I have in mind $M_n(A^1)$. Suppose that $x \in \mathrm{GL}(B)$ and $\|a\| < \|x^{-1}\|^{-1}$. Then

$$\|1_n - (1_n - x^{-1}a)\| < 1;$$

thus $1_n - x^{-1}a \in \mathrm{GL}(B)$. Therefore $x - a \in \mathrm{GL}(B)$. It follows that if $\|x - y\| < \|x^{-1}\|^{-1}$ and $t \in [0,1]$, then we have $x - t(y - x) \in \mathrm{GL}(B)$. If $x, y \in \mathrm{GL}_n^1(A)$, then so is $x - t(y - x)$. This proves (a).

We now want to define $F : \mathrm{GL}(B) \times [0,1] \to \mathrm{GL}(B)$ such that

$$\begin{aligned}
F(z,1) = z &\quad \text{for all } z \in \mathrm{GL}(B),\\
F(z,0) \in \mathrm{U}(B) &\quad \text{for all } z \in \mathrm{GL}(B), \text{ and}\\
F(u,t) = u &\quad \text{for all } u \in \mathrm{U}(B) \text{ and } t \in [0,1].
\end{aligned}$$

If $z \in \mathrm{GL}(B)$, then so is z^*z and $z \mapsto |z|^{-1}$ is continuous. Furthermore, $u = z|z|^{-1}$ is unitary: it is certainly invertible and

$$u^{-1} = |z|z^{-1} = |z|^{-1}|z|^2z^{-1} = |z|^{-1}z^*zz^{-1} = |z|^{-1}z^* = u^*.$$

Since $|z| \geq 0$ and $0 \notin \sigma(|z|)$, we can define

$$F(z,t) = z|z|^{-1}\exp\bigl(t\log|z|\bigr)$$

If $B = M_n(A^1)$ and $\pi(z) = 1_n$, then $\pi\bigl(F(z,t)\bigr) = \pi(z)|\pi(z)|^{-1}\exp\bigl(t\log|\pi(z)|\bigr) = 1_n$.

This proves (b), and (c) follows from (b). $\square$

Remark 5.7. One can improve on (c) above. If $u, v \in \mathrm{U}(A)$ satisfy $\|u - v\| < 2$, then u and v are homotopic in $\mathrm{U}(A)$.

Proof. Since $uv^* \in \mathrm{U}(A)$, $\sigma(uv^*) \subset \mathbb{T}$. Since $\|u-v\| = \|uv^*-1\| < 2$, $-1 \notin \sigma(uv^*)$. Therefore $\log uv^*$ is well-defined in $\widetilde{A}$, and we can define

$$u_t := \exp\bigl(t\log uv^*\bigr)v.$$

Since $1 = \overline{\exp(t\log z)}\exp(t\log z)$ if $z \in \mathbb{T}$, it follows that each u_t is unitary. This suffices as $u_0 = v$ and $u_1 = u$. Furthermore, if u and v are normalized, so is u_t. $\square$

Remark 5.8. Thus two unitaries can fail to be homotopic only when they are maximally far apart: $\|u - v\| = 2$. If $\widetilde{A}$ is closed under Borel functional calculus, then the above proof shows that $\mathrm{U}(A)$ is path connected. This means, as we shall see, that $K_1(A) \cong \{\,0\,\}$ for all von Neumann algebras.

Remark 5.9. If u and v are unitaries in $\widetilde{A}$ which are homotopic in $\mathrm{GL}(A)$, then they are homotopic in $\mathrm{U}(A)$.

Proof. Apply the retraction to the homotopy. $\square$

Recall that an *elementary row operation* on a matrix $M \in M_n(A)$ consists of one of the following:

(a) Multiply a row by any element in $\mathrm{GL}(A)_0$.
(b) Add a multiple of one row of M to a different row of M.
(c) Interchange two rows of M.

Of course, there is a corresponding notion of *elementary column operations*. A matrix $E \in M_n(A)$ is called an *elementary matrix* if it is obtained from the identity 1_n via one of the elementary row operations above. A crucial observation is that if E is an elementary matrix, then EM is the matrix obtained from M via the same elementary row operation as that which defines E.

Theorem 5.10. *Suppose that x and y are elements of $\mathrm{GL}_n(A)$, and that y can be obtained from x via a finite sequence of elementary row and column operations. Then x and y are homotopic in $\mathrm{GL}_n(A)$. In particular, if u and v are invertible (resp., unitary) in $\widetilde{A}$, then the 2×2-matrices*

$$\begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

are mutually homotopic in $\mathrm{GL}_2(A)$ (resp., $\mathrm{U}_2(A)$). If u and v are normalized, then these matrices are mutually homotopic in $\mathrm{GL}_2^1(A)$ (resp., $\mathrm{U}_2^1(A)$).

Proof. It suffices to see that any elementary matrix is homotopic to the identity. This is straightforward. For example, to see that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is homotopic to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, consider

$$u_t := \begin{pmatrix} \cos\bigl(\frac{\pi}{2}t\bigr) & \sin\bigl(\frac{\pi}{2}t\bigr) \\ \sin\bigl(\frac{\pi}{2}t\bigr) & \cos\bigl(\frac{\pi}{2}t\bigr) \end{pmatrix}$$

To establish the statement about normalized homotopies, we have to write down explicit ones. To do this, let $w_t := (u\oplus 1)\cdot u_t\cdot(v\oplus 1)\cdot u_t^*$ and $z_t := u_t\cdot(u\oplus v)\cdot u_t^*$. Then w is a homotopy between $\left(\begin{smallmatrix} u & 0\\ 0 & v\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} uv & 0\\ 0 & 1\end{smallmatrix}\right)$. Similarly, z is a homotopy between $\left(\begin{smallmatrix} v & 0\\ 0 & u\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} u & 0\\ 0 & v\end{smallmatrix}\right)$. If u and v are normalized, then $\pi(w_t) = \pi(u_t u_t^*) = 1_2 = \pi(z_t)$. $\square$

We will invoke the above result mostly via the following corollary.

Corollary 5.11. *If $u \in \mathrm{U}(A)$, then $\begin{pmatrix} u & 0\\ 0 & u^*\end{pmatrix}$ is homotopic to the identity in $\mathrm{U}_2(A)$. If u is normalized, then the homotopy can be taken in $\mathrm{U}_2^1(A)$.*

Proof of Proposition 5.4. We only have to verify the last statement. So, suppose $u \in \mathrm{U}(A)$ satisfies $q = upu^*$. By Corollary 5.11, we can choose a unitary homotopy w_t from $\left(\begin{smallmatrix} u & 0\\ 0 & u^*\end{smallmatrix}\right)$ to 1_2. Then $p_t := w_t(p\oplus 0)w_t^*$ is a continuous path of projections in $M_2(A)$ connecting $p\oplus 0$ and $q \oplus 0$. $\square$

If A and B are C^*-algebras and $\gamma : A \to C\big([0,1], B\big)$ is a homomorphism, then we'll write $\epsilon_t : C\big([0,1], B\big) \to B$ for the evaluation map and γ_t for the composition $\epsilon_t \circ \gamma$.

Definition 5.12. We say that two homomorphisms $\alpha : A \to B$ and $\beta : A \to B$ are homotopic if there is a homomorphism $\gamma : A \to C\big([0,1], B\big)$ such that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. In this case we write $\alpha \sim_h \gamma$. We say that $\alpha : A \to B$ is an equivalence if there exists a homomorphism $\beta : B \to A$ such that $\alpha\circ\beta \sim_h \mathrm{id}_B$ and $\beta\circ\alpha \sim_h \mathrm{id}_A$. We say that α is a deformation if there exists β such that $\beta\circ\alpha \sim_h \mathrm{id}_A$ and $\alpha\circ\beta = \mathrm{id}_B$. In this event, we say that B is a deformation retract of A. Finally, we say that A is contractible if $\mathrm{id}_A \sim_h 0$.

Example 5.13. Note that $\mathbb{C}$ is *not* contractible as a C^*-algebra; this will follow from Corollary 5.16. If X is a *compact* contractible space, then D is a deformation retract of $C(X, D)$. In particular, $\mathbb{C}$ is a deformation retract of $C(X)$.

Proof. Fix $x_0 \in X$ and let $\varphi : [0,1]\times X \to X$ be a continuous function such that $\varphi(0,x) = x$ and $\varphi(1,x) = x_0$ for all $x \in X$. Define $\alpha : C(X,D) \to D$ by $\alpha(f) := f(x_0)$ and $\beta : D \to C(X,D)$ by sending d to the constant function $x \mapsto d$. Clearly, $\alpha\circ\beta = \mathrm{id}_D$. If $\gamma : C(X,D) \to C\big([0,1], C(X,D)\big)$ is defined by

$$\gamma_t(f)(x) = f\big(\varphi(t,x)\big),$$

then $\gamma_0 = \mathrm{id}_{C(X,D)}$ while $\gamma_1 = \beta\circ\alpha$. $\square$

Remark 5.14. If X is a locally compact contractible space which is not compact, then β will not map into $C_0(X,D)$. In fact, $C_0(X,D)$ will, in general, not retract onto D. As we shall see in due course, $\mathbb{C}$ can not be a deformation retract of $C_0(\mathbb{R}) = C_0(R,\mathbb{C}) \cong C_0\big((0,1)\big)$.[13]

Theorem 5.15. *Suppose that α and β are homomorphisms from A to B. If $\alpha \sim_h \beta$, then $K_0(\alpha) = K_0(\beta)$.*

Proof. Suppose that $x = [p] - [q]$ in $K_0(A)$, and let γ_t be a homotopy from α to β. Then γ_1^1 is a homotopy from α^1 to β^1. In particular, $\alpha^1(p) \equiv \beta^1(p)$ and

[13]Ref needed

$\alpha^1(q) \equiv \beta^1(q)$. Thus

$$\begin{aligned} K_0(\alpha)(x) &= [\alpha^1(p)] - [\alpha^1(q)] = [\beta^1(p)] - [\beta^1(q)] \\ &= K_0(\beta)(x). \quad \square \end{aligned}$$

Corollary 5.16. *If $\alpha : A \to B$ is a homotopy equivalence, then $K_0(\alpha)$ is an isomorphism of $K_0(A)$ onto $K_0(B)$. In particular, $K_0(A) = \{\, 0 \,\}$ if A is contractible.*

Proof. The zero map clearly induces the zero map on K-theory. $\square$

Before concluding this section, I want to recall the connections between invertible elements in A and *exponentials* in A. If $a \in A$, then even if a is not normal (so that the usual functional calculus does not apply),

$$\exp(a) := 1 + a + \frac{a^2}{2!} + \cdots = \sum_{n=o}^{\infty} \frac{a^n}{n!}$$

converges in $\widetilde{A}$ to an element in $\mathrm{GL}(A)$. Of course, if a is normal, then this is the same element defined by the functional calculus, so there is no harm in using the same notation. Then $\exp(A)$ is defined to be the subgroup of $\mathrm{GL}(A)$ generated be $\{\, \exp(a) : a \in A \,\}$.[14] It isn't obvious that this group is closed in $\mathrm{GL}(A)$, but in fact it is open.[15] Proving this requires the holomorphic functional calculus (see, for example, [Rud73]). Let $z_0 = \exp(a_1) \cdots \exp(a_n)$ and suppose that

$$\|z - z_0\| < \|z_0^{-1}\|^{-1}.$$

Let $z' = zz_0^{-1}$. Then $\|z' - 1\| \leq \|z - z_0\| \|z_0^{-1}\| < 1$, so $\sigma(z' - 1) \subset B_1(0)$ and $\sigma(z') \subset \{\, \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0 \,\}$. In particular, there is an $a_0 := \log(z')$ such that $\exp(a_0) = z'$ [Rud73, Theorem 10.30]. Thus,

$$z = z' z_0 = \exp(a_0) \exp(a_1) \cdots \exp(a_n) \in \exp(A).$$

Lemma 5.17. *For any C^*-algebra,* $\exp(A) = \mathrm{GL}(A)_0$.

Proof. Since $\exp(A)$ is both closed and open, it suffices to see that $\exp(A) \subset \mathrm{GL}(A)_0$. But if $z = \exp(a_1) \cdots \exp(a_n)$, then $z_t := \exp(ta_1) \cdots \exp(ta_n)$ is a homotopy connecting z to 1 in $\mathrm{GL}(A)$. $\square$

Corollary 5.18. *Suppose that $\alpha : A \to B$ is a surjective, unital, $*$-homomorphism. If $x \in \mathrm{GL}(B)_0$ (resp., $\mathrm{U}(B)_0$), then there is an $x' \in \mathrm{GL}(A)_0$ (resp., $U(A)_0$) such that $\alpha(x') = x$.*

Proof. Let $x = \exp(b_1) \cdots \exp(b_n)$. Choose a_i such that $\alpha(a_i) = b_i$ for all i. If $x' := \exp(a_1) \cdots \exp(a_n)$, then $x' \in \mathrm{GL}(A)_0$ and maps onto x. If x is unitary, then we can replace x' by $u' := x'|x'|^{-1}$. Then u' is connected to x' as in the proof of Lemma 5.5, and therefore $u' \in \mathrm{GL}(A)_0$. $\square$

Corollary 5.19. *Suppose that J is an ideal in A and that $u \in \mathrm{U}(\widetilde{A}/J)$. Then there is a unitary $w \in \mathrm{U}_2(A)_0$ such that $\pi_J(w) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$.*

Proof. Since $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right) \in U_2(\widetilde{A}/J)_0$, the previous corollary applies. $\square$

[14] Notice that in general, $\exp(a)\exp(b) \neq \exp(a+b)$ unless a and b commute. Thus, the set of exponentials is not closed under multiplication.

[15] An open subgroup of a topological group is necessarily closed: $H = G \setminus \bigcup_{g \notin H} gH$.

Remark 5.20. The holomorphic functional calculus is not to be sneezed at. Even when $A = M_n$, it allows us to conclude that $\mathrm{GL}_n(\mathbb{C})_0 = \mathrm{GL}_n(\mathbb{C})$ — since the spectrum of a matrix is always finite, it can't separate 0 and ∞. (Notice that one can show $\mathrm{U}_n(\mathbb{C})_0 = \mathrm{U}_n(\mathbb{C})$ using ordinary spectral theory from Linear Algebra.)

6. Half Exactness

We can now prove that an exact sequence of C^*-algebras induces what is called a half-exact sequence of the corresponding K-groups. In this section, we will start to adopt the standard notation α_* for the induced group homomorphism $K_0(\alpha)$ corresponding to a $*$-homomorphism α.

Theorem 6.1. *Suppose that*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{6.1}$$

is an exact sequence of C^*-algebras. Then

$$K_0(A) \xrightarrow{\alpha_*} K_0(B) \xrightarrow{\beta_*} K_0(C) \tag{6.2}$$

is exact.

It definitely is not the case that α_* need be injective, or that β_* need be surjective. We will give specific examples below, but the idea is that there are more partial isometries to implement equivalences in B than in A (which corresponds to an ideal in B), and projections in C may not lift to projections in B.

Proof of Theorem 6.1. Recall that $\alpha_* = K_0(\alpha)$ is the restriction of $K_{00}(\alpha^1)$ to $K_0(A)$. If $x \in K_0(A)$, then Proposition 1.19 implies that we can assume $x = [p] - [p_n]$ for $p \in M_k(A^1)$ and $p - p_n \in M_k(A)$ for some $k \geq n$. Since $\beta \circ \alpha = 0$, we must have $\beta^1 \circ \alpha^1(p) = p_n$. Thus

$$\beta_* \circ \alpha_*(x) = [\beta^1 \circ \alpha^1(p)] - [\beta^1 \circ \alpha^1(p_n)] = [p_n] - [p_n] = 0.$$

Thus $\operatorname{im} \alpha_* \subset \ker \beta_*$

Now suppose that $y \in \ker \beta_* \subset K_0(B)$. As above, we can assume $y = [q] - [p_n]$ with $q \in M_k(B^1)$, $q - p_n \in M_k(B)$, and $k \geq n$. Since

$$\beta_*(y) = [\beta^1(q)] - [\beta^1(p_n)] = [\beta^1(q)] - [p_n],$$

Proposition 1.19 implies that there is a v and $m \geq k + v$ such that

$$\beta^1(q) \oplus p_v \approx p_n \oplus p_v \quad \text{in } M_m(C^1).$$

Let $u \in \mathrm{U}_m(C^1)$ be such that $u\big(\beta^1(q) \oplus p_v\big)u^* = p_n \oplus p_v$. Corollary 5.19 implies that there is a $w \in \mathrm{U}_{2m}(B^1)$ such that $\beta^1(w) = u \oplus u^*$. Let $r := w(q \oplus p_v \oplus 0_n)w^*$ in $M_{2m}(B^1)$. Then r is a projection and

$$\beta^1(r) = (u \oplus u^*)\big(\beta^1(q) \oplus p_v \oplus 0_n\big)(u^* \oplus u) = p_n \oplus p_v \oplus 0_n.$$

The exactness of (6.1) implies there is a $s \in M_{2m}(A^1)$ such that $r = \alpha^1(s)$. Replacing s by $(s + s^*)/2$, we can assume that $s = s^*$. Since α^1 is injective, we can assume that $s = s^2$, and hence, that s is projection. Since $[r] = [q \oplus p_v]$,

$$\begin{aligned} y = [q] - [p_n] &= [q \oplus p_v] - [p_n \oplus p_v] \\ &= [r] - [p_{n+v}] \\ &= \alpha_*([s] - [p_{n+v}]). \end{aligned}$$

Therefore $\ker \beta_* \subset \operatorname{im} \alpha_*$. This completes the proof. $\square$

Example 6.2. Consider the exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H}) \to 0.$$

Then we get the exact sequence

$$\mathbb{Z} \xrightarrow{\alpha_*} \{0\} \xrightarrow{\beta^*} K_0\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big),$$

and α_* is certainly not injective.[16] To see that β_* need not be surjective, consider $B = C\big([0,1]\big)$, $A = C_0\big((0,1)\big)$, and $C = \mathbb{C} \oplus \mathbb{C}$. Then, once[17] we establish that $K_0\big(C_0(\mathbb{R})\big) = \{0\}$, the exact sequence

$$0 \xrightarrow{\alpha_*} \mathbb{Z} \xrightarrow{\beta_*} \mathbb{Z} \oplus \mathbb{Z}$$

forces β_* not to be surjective.

Remark 6.3. Note that Theorem 6.1 fails with K_{00} in place of K_0. For example, take $A = C_0(\mathbb{R}^2)$, $B = C(S^2)$ and $C = \mathbb{C}$. The K_{00} groups are 0, $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$. (Here we have used that nontrivial facts that $K_{00}(C(S^2)) = K_0(C(S^2)) \cong K^0(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.)

7. Definition of K_1

It should be clear by now that unitaries play a critical rôle in the theory and computation of K_0. In this chapter we'll get a closer glimpse of why.

Just as $K_0(A)$ is defined in terms of equivalence classes of projections in matrix algebras over A, we want to define $K_1(A)$ in terms of homotopy classes of unitaries in matrix algebras. We have injections of $\mathrm{GL}^1_n(A)$ into $\mathrm{GL}^1_{n+1}(A)$ and $\mathrm{U}^1_n(A)$ into $\mathrm{U}^1_{n+1}(A)$ given by $x \mapsto x \oplus 1$. We define

$$\mathrm{GL}^1_\infty(A) := \varinjlim \mathrm{GL}^1_n(A) \quad \text{and} \quad \mathrm{U}^1_\infty(A) := \varinjlim \mathrm{U}^1_n(A).$$

We notice that the connecting maps take the connected components $\mathrm{GL}^1_n(A)_0$ and $\mathrm{U}^1_n(A)_0$ into $\mathrm{GL}^1_{n+1}(A)_0$ and $\mathrm{U}^1_{n+1}(A)_0$, respectively. In particular, $x \mapsto x \oplus 1$ induces a homomorphism of the quotient $\mathrm{U}^1_n(A)/\,\mathrm{U}^1_n(A)_0$ into $\mathrm{U}^1_{n+1}(A)/\,\mathrm{U}^1_{n+1}(A)_0$ and we can define $K_1(A)$ as follows.

Definition 7.1. If A is a C^*-algebra, then

$$K_1(A) := \varinjlim \mathrm{U}^1_n(A)/\,\mathrm{U}^1_n(A)_0.$$

If we define $\mathrm{U}^1_\infty(A)_0 = \varinjlim \mathrm{U}^1_n(A)_0$, then it is not hard to check that

$$K_1(A) = \mathrm{U}^1_\infty(A)/\,\mathrm{U}^1_\infty(A)_0. \tag{7.1}$$

Remark 7.2. If $u \in \mathrm{U}^1_n(A)$, then $[u]$ will denote the class of u in $K_1(A)$. Every class in $K_1(A)$ has such a representative, and $[u] = [v]$ if and only if there are $m, n, k \in \mathbb{Z}^+$ such that $u \oplus 1_m \sim_h v \oplus 1_n$ in $\mathrm{U}^1_k(A)$.

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[16] In fact, $K_0\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big) = \{0\}$, but this is, I think, not so easy to see.

[17] Well, given that $K_0(C(\mathbb{T})) \cong \mathbb{Z}$, then $K_{00}(C_0(\mathbb{R})^1) = K_0(C(\mathbb{T})) \cong \mathbb{Z}$, and it is not hard to see that $K_{00}(A^1) \cong K_0(A) \oplus \mathbb{Z}$.

Lemma 7.3. *Let u and v be in $\mathrm{U}_n^1(A)$. If $y = \iota(y')$, where y' is a unitary matrix in M_n, then $yuy^* \in \mathrm{U}_n^1(A)$ and is homotopic to u in $\mathrm{U}_n^1(A)$. In particular, $[u] = [yuy^*]$ in $K_1(A)$. Furthermore, if u and v are homotopic in $\mathrm{U}_n(A^1)$ then u and v are homotopic in $\mathrm{U}_n^1(A)$. In particular, $[u] = [v]$ in $K_1(A)$*

Proof. That $yuy^* \in \mathrm{U}_n^1(A)$ is straightforward. Remark 5.20 implies that $y \sim_h 1_n$ in $\mathrm{U}_n(\mathbb{C})$. The first assertion follows. Now suppose that $t \mapsto a_t$ is a homotopy from u to v in $\mathrm{U}_n(A^1)$. Let $y_t = \iota\bigl(\pi(a_t^*)\bigr)$. Then $y_0 = y_1 = 1_n$, and $t \mapsto y_t a_t y_t^*$ is a homotopy from u to v in $\mathrm{U}_n^1(A)$. □

Remark 7.4. We still need to see that the groups $\mathrm{GL}_n^1(A)/\mathrm{GL}_n^1(A)_0$ and $\mathrm{U}_n^1(A)/\mathrm{U}_n^1(A)_0$ are isomorphic for $n = 1, 2, \dots, \infty$. This allows us to replace U with GL in the above discussions.

Notice that there is no reason to suspect that $\mathrm{U}_n^1(A)/\mathrm{U}_n^1(A)_0$ is abelian. Therefore the next lemma gives one excuse for passing to the direct limit.

Lemma 7.5. *$K_1(A)$ is an abelian group with respect to the operation coming from the direct limit. The identity is the class of 1 and the inverse of $[u]$ is $[u^*]$.*

Proof. If $m \le n$, $u \in \mathrm{U}_m^1(A)$, and $v \in \mathrm{U}_n^1(A)$, then $[u][v] := [(u \oplus 1_{n-m})v]$ is the group operation on the inductive limit K_1. Since Theorem 5.10 implies that

$$[(u \oplus 1_{n-m})v] = [(u \oplus 1_{n-m}) \oplus v] = [v \oplus (u \oplus 1_{n-m})] = [v][u],$$

the operation is commutative. □

The object of this section is to prove Theorem 7.6 which states that $K_1(A)$ is naturally isomorphic to $K_0(SA)$ where $SA := A \otimes C_0(\mathbb{R})$.[18] To explain what natural means in this context, it is helpful to think of passing to the suspension as a functor. If $\alpha : A \to B$ is a homomorphism, then $S\alpha : SA \to SB$ is given by the restriction of $\mathrm{id} \otimes \alpha$ to SA. That is, $S\alpha(f)(z) := \alpha\bigl(f(z)\bigr)$, where we have identified

$$SA = \{\, f \in C(\mathbb{T}, A) : f(1) = 0 \,\}.$$

Then the word "natural" above simply means that given $\alpha : A \to B$, then the diagram

$$\begin{tikzcd} K_1(A) \arrow[r, "\Theta_A"] \arrow[d, "\alpha_*"'] & K_0(SA) \arrow[d, "S\alpha_*"] \\ K_1(B) \arrow[r, "\Theta_B"'] & K_0(SB) \end{tikzcd} \tag{7.2}$$

commutes. Then it follows that K_1 enjoys many of the same functorial properties as K_0. For example, given an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0\ ,$$

of C^*-algebras, then it is easy to see that

$$0 \longrightarrow SA \xrightarrow{S\alpha} SB \xrightarrow{S\beta} SC \longrightarrow 0$$

[18] In some abbreviated treatments, $K_1(A)$ is actually defined to be $K_0(SA)$. The naturality of the isomorphism of Theorem 7.6 means there is little harm in this.

is exact. Since K_0 is half-exact, we can use (7.2) to prove that K_1 is half-exact. With a bit more work, one can prove that K_1 preserves direct limits, etc.

It will be helpful to keep in mind that homotopies with values in SA^1 are in one-to-one correspondence with continuous functions f from $[0,1]\times\mathbb{T}\to A^1$ which are scalar valued on $[0,1]\times\{\,1\,\}$ and for which $z\mapsto\pi\bigl(f(t,z)\bigr)$ is constant for all $t\in[0,1]$. Given such an f, we can define γ from $C\bigl([0,1],SA^1\bigr)\subset C\bigl([0,1]\times\mathbb{T},A^1\bigr)$ to SA^1 by $\gamma_t(z)=f(t,z)$, and conversely. (The point is, that elements of SA^1 are of the form $f+\lambda$ where $f\in SA$, and $SA^1\neq\{\,f:\mathbb{T}\to A^1: f(1)\in\mathbb{C}1\,\}$.)

Theorem 7.6. *There is an isomorphism $\Theta_A: K_1(A)\to K_0(SA)$ which is natural as in* (7.2) *above.*

Proof. Let u be a normalized unitary in $\mathrm{U}_n^1(A)$. We can apply Corollary 5.11 to produce a homotopy $t\mapsto w_t$ in $\mathrm{U}_{2n}^1(A)$ connecting $w_0=1_{2n}$ to $u\oplus u^*$. Using the notation p_n for the class of 1_n in $V(A)$, we can define a path of projections by

$$t\mapsto q_t:=w_tp_nw_t^*\quad(t\in[0,1]). \tag{7.3}$$

Since $q_0=p_n=q_1$ and $\pi(q_t)=p_n$ for all t, (7.3) defines a projection $q^{u,w}$ in $M_{2n}(SA^1)$.[19] Then we can define a class $[q^{u,w}]-[p_n]$ in $K_0(SA)$. We want to show that this class depends only on the class $[u]$ of u in $K_1(A)$. To do this, we have to show that $\theta(u):=[q^{u,w}]-[p_n]$ does not depend on w, that $\theta(u)=\theta(u\oplus 1_m)$, and that $u\sim_h v$ implies $\theta(u)=\theta(v)$.

To do this, suppose that $t\mapsto a_t$ is a homotopy from u to v in $\mathrm{U}_n^1(A)$, and let $q^{u,w}$ and $q^{v,z}$ be projections constructed as above. But let x be the map $t\mapsto w_t\cdot(u^*a_t\oplus ua_t^*)\cdot z_t^*$ from $[0,1]$ to $\mathrm{U}_{2n}(A^1)$. It is easy to see that $x_0=1_{2n}=x_1$, and that $\pi(x_t)=1_{2n}$ for all t. Thus x defines an element of $\mathrm{U}_{2n}^1(SA)$. Since $p_n=1_n\oplus 0_n$,

$$\begin{aligned}xq^{v,z}x^*=xzp_nz^*x^*&=w\cdot(u^*a\oplus ua^*)\cdot p_n\cdot(a^*u\oplus au^*)\cdot w^*\\&=wp_nw^*\\&=q^{u,w}.\end{aligned}$$

Thus the class of $\theta(u)=q^{u,w}$ does not depend on w or the class of u in $\mathrm{U}_n^1(A)/\,\mathrm{U}_n^1(A)_0$.

Now we need to consider $\theta(u\oplus 1_m)$. Let $t\mapsto w_t$ be the path in $\mathrm{U}_{2n}^1(A)$ connecting 1_{2n} and $u\oplus u^*$ as above; thus, $\theta(u)=[q^{u,w}]-[p_n]$. Choose a scalar permutation matrix y such that

$$y\cdot(u\oplus u^*\oplus 1_m\oplus 1_m)\cdot y^*=u\oplus 1_m\oplus u^*\oplus 1_m.$$

Define $z_t:=y\cdot(w_t\oplus 1_{2m})\cdot y^*$, and check that $z_t\in\mathrm{U}_{2n+2m}(A)$. Furthermore, $z_0=1_{2n+2m}$ and $z_1=u\oplus 1_m\oplus u^*\oplus 1_m$. Thus, $\theta(u\oplus 1_m)=[q^{u\oplus 1_m,z}]-[p_{n+m}]$, where $q^{u\oplus 1_m,z}=zp_{n+m}z^*$. But

$$\begin{aligned}zp_{n+m}z^*&\approx(w\oplus 1_{2m})\cdot y^*p_{n+m}y\cdot(w^*\oplus 1_{2m})\\&=((w\oplus 1_{2m})\cdot(1_n\oplus 0_n\oplus 1_m\oplus 0_m)\cdot(w^*\oplus 1_{2m})\\&=wp_nw^*\oplus p_m\\&=q^{u,w}\oplus p_m.\end{aligned} \tag{7.4}$$

[19]Some care is called for here; $q^{u,w}$ is a matrix of functions on $\mathbb{T}$. The off diagonal entries are in SA proper as are n of the diagonal entries. Thus $q^{u,w}$ is a matrix of elements in SA^1. The other n diagonal entries are of the form $f+1$ with $f\in SA$. It should be noted that, since $w_0\neq w_1$, w does not define a unitary in $U_{2n}(SA^1)$ and $q^{u,w}$ is not necessarily trivial in $K_0(SA)$.

Thus $\theta(u \oplus 1_m) = [q^{u\oplus 1_m, z}] - [p_{n+m}] = \theta(u)$, and we obtained a well-defined map $\Theta_A : K_1(A) \to K_0(SA)$.

To see that Θ_A is a homomorphism, consider $u, v \in \mathrm{U}_n^1(A)$. Let w and z be as above so that $\Theta_A([u]) = [q^{u,w}] - [p_n]$ and $\Theta_A([v]) = [q^{v,z}] - [p_n]$. Note that $yp_{2n}y^* \approx p_n \oplus p_n$ via a scalar permutation matrix y. Therefore,

$$\Theta_A\big([uv]\big) = \Theta\big([u \oplus v]\big) = [q^{u\oplus v, w\oplus z}] - [p_{2n}],$$

where

$$\begin{aligned} q^{u\oplus v, w\oplus z} &= y^*(w \oplus z)yp_{2n}y^*(w^* \oplus z^*)y \\ &\approx (w \oplus z)(p_n \oplus p_n)(w^* \oplus z^*) \\ &= q^{u,w} \oplus q^{v,z}. \end{aligned}$$

Thus $\Theta_A\big([uv]\big) = \Theta_A([u]) + \Theta_A([v])$.

Naturality is proved similarly. Suppose $\alpha : A \to B$ is a homomorphism, and that $u \in \mathrm{U}_n^1(A)$. Let $\Theta_A\big([u]\big) = [q^{u,w}] - [p_n]$. Note that $\alpha_*([u]) = [\alpha^1(u)]$, and that $t \mapsto \alpha^1(w_t)$ is a path of unitaries connecting 1_{2n} and $\alpha^1(u) \oplus \alpha^1(u)^*$. Thus

$$\Theta_B\big(\alpha_*([u])\big) = [q^{\alpha^1(u),\alpha^1(w)}] - [p_n],$$

where $q^{\alpha^1(u),\alpha^1(w)}$ is given by $t \mapsto \alpha^1(w)p_n\alpha^1(w_t)^*$. Thus

$$[q^{\alpha^1(u),\alpha^1(w)}] = S\alpha_*\big([wp_nw^*]\big) = S\alpha_*\big([q^{u,w}]\big).$$

Thus $\Theta_B \circ \alpha_* = S\alpha_* \circ \Theta_A$.

To prove that Θ_A is injective, we suppose that $u, v \in \mathrm{U}_n^1(A)$ and that

$$\Theta_A\big([u]\big) = \Theta_A\big([v]\big). \tag{7.5}$$

Using the notation above, we let $q^{u,w'}$ be the projection in $M_{2n}(SA^1)$ defined by $w'p_nw'^*$, where $t \mapsto w'_t$ is a homotopy in $\mathrm{U}_{2n}^1(A)$ from 1_{2n} to $u \oplus u^*$. It follows from (7.5) that for suitable w' and z', we have $[q^{u,w'}] - [q^{v,z'}] = 0$ in $K_0(SA)$. Proposition 1.19 implies that there is a $m \in \mathbb{Z}^+$ and $k \geq 2n + 2m$ such that

$$q^{u,w'} \oplus p_m \approx q^{v,z} \oplus p_m \quad (\text{in } M_k(SA^1)). \tag{7.6}$$

On the other hand, (7.4) implies that for suitable w and z we have

$$q^{u\oplus 1_m, w} \approx q^{u,w'} \oplus p_m \quad \text{and} \quad q^{v\oplus 1_m, z} \approx q^{v,z'} \oplus p_m \quad (\text{in } M_{2n+2m}(SA^1)). \tag{7.7}$$

(It should be noted that the p_m's which appear in (7.6) and (7.7) are not, technically, the same. One denotes $1_m \oplus 0_{k-2n-m}$ and the other $1_m \oplus 0_m$. But this isn't a serious issue.) Together, (7.6) and (7.7) imply that there is a unitary $a \in \mathrm{U}_k(SA^1)$ such that $q^{v\oplus 1_m, z} \oplus 0_d = a(q^{u\oplus 1_m, w} + 0_d)a^*$, where $d = k - 2n - m$. That is, $zp_{n+m}z^* \oplus 0_d = a(wp_{n+m}w^* \oplus 0_d)a^*$. Thus if we replace p_{n+m} with $p_{n+m} \oplus 0_d$, we have

$$(z \oplus 1_d)p_{n+m}(z^* \oplus 1_d) = a(w \oplus 1_d)p_{n+m}(w \oplus 1_d)a^*. \tag{7.8}$$

Now we view elements of SA^1 as function from $[0,1]$ into A^1 which attain the same *scalar* value at 0 and 1. Then we can define $x_t := (w_t^* \oplus 1_d)a_t^*(z_t \oplus 1_d) \in \mathrm{U}_k(A^1)$. Using (7.8), it is not hard to see that x_t commutes with p_{n+m}. It follows from the rules for matrix multiplication, that $x_t = b_t \oplus c_t$ for $b_t \in \mathrm{U}_{n+m}(A^1)$ and $c_t \in \mathrm{U}_{k-n-m}(A^1)$. Since $x_0 = b_0 \oplus c_0 = a_0^*$, there are unitary matrices $y \in M_{n+m}$ and $y' \in M_{k-n-m}$ such that $a_0^* = y \oplus y' = a_1^*$. Since

$$x_1 = (u^* \oplus 1_m \oplus u \oplus 1_m \oplus 1_d)(y \oplus y')(v \oplus 1_m \oplus v^* \oplus 1_m \oplus 1_d),$$

It follows that $t \mapsto y^* b_t$ is a homotopy in $\mathrm{U}_{n+m}(A^1)$ from 1_{n+m} to $y^*(u^* \oplus 1_m)y(v \oplus 1_m)$. It follows from Lemma 7.3 that

$$[1] = [y^*(u^* \oplus 1_m)y(v \oplus 1_m)] = [u^*v] = [u]^{-1}[v],$$

and $[u] = [v]$ in $K_1(A)$ as required.

It remains to show that Θ_A is surjective. So let $x \in K_0(SA)$. By Proposition 1.19, we can assume $x = [f] - [p_n]$ where f is a projection in $M_k(SA^1)$ such that $f - p_n \in M_k(SA)$. We can assume that $k - 2n = 2m$ for $m \in \mathbb{N}$. Since $t \mapsto f_t$ is a path of projections in $M_k(A^1)$ such that $f_t - p_n \in M_k(A)$ and $f_0 = f_1 = p_n$, Corollary 5.3 implies there is a path $t \mapsto w_t$ in $\mathrm{U}^1_k(A)$ such that $w_0 = 1_k$ and $f_t = w_t p_n w_t^*$. Since $w_1 p_n w_1^* = p_n$, we have $w_1 = u \oplus \tilde{v}$ for $u \in \mathrm{U}^1_n(A)$ and $\tilde{v} \in \mathrm{U}^1_{k-n}(A)$. We'd like to construct a homotopy from $\tilde{v}$ to $u^* \oplus 1_{2m}$; however, I don't see any way to do this. But we can replace k by $2k$ and w_t by $w_t \oplus u^* \oplus v^*$. (We certainly have $2k - 2n$ even, so this causes no harm.) Now $v = \tilde{v} \oplus u^* \oplus \tilde{v}^*$, and there is a homotopy in $\mathrm{U}^1_{n-k}(A)$ from v to $u^* \oplus 1_{n-2k}$ by Corollary 5.11. Multiplying by v^* we get a homotopy $t \mapsto d_t$ in $\mathrm{U}^1_{n-k}(A)$ from 1_{k-n} to $v^*(u^* \oplus 1_{2m})$. Let $t \mapsto z_t$ be a homotopy in $\mathrm{U}^1_k(A)$ from 1_k to $u \oplus u^* \oplus 1_{2m}$ of the form $z_t = z_t' \oplus 1_{2m}$. Then if $e_t := z_t p_n z_t = z_t' p_n z_t'$, it follows that $\Theta_A([u]) = [e] - [p_m]$. Now let $x_t := w_t(1_n \oplus d_t)z_t^* \in \mathrm{U}^1_k(A)$. Certainly, $x_0 = 1_k$, and

$$\begin{aligned} x_1 &= (u \oplus v)(1_n \oplus v^*(u^* \oplus 1_{2m}))(u^* \oplus u \oplus 1_{2m}) \\ &= uu^* \oplus (vv^*(u^* \oplus 1_{2m})(u \oplus 1_{2m})) \\ &= 1_k. \end{aligned}$$

Thus $x \in \mathrm{U}^1_k(SA)$, and

$$\begin{aligned} xex^* &= w(1_n \oplus d)z^*(zp_nz^*)z(1 \oplus d^*)w^* \\ &= wp_nw^* \\ &= f. \end{aligned}$$

Therefore $[e] = [f]$, and $\Theta_A([u]) = [f] - [p_m] = x$ as required. □

8. The Long Exact Sequence in K-Theory

For motivation, we recall some basic material about Fredholm operators on Hilbert space. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Then quotient $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the *Calkin Algebra*. An operator $T \in B(\mathcal{H})$ is called *Fredholm* if its image $q(T)$ is invertible in the Calkin Algebra. The theory, as presented in Douglas's book [Dou98] for example, tells us that

$$j(T) := \dim\ker T - \dim\ker T^*$$

is a (finite) integer which depends only on the class of $q(T)$ in

$$\mathrm{GL}\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big)/\,\mathrm{GL}\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big)_0 \cong \mathrm{U}\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big)/\,\mathrm{U}\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big)_0 \cong \mathbb{Z}$$

[Dou98, Theorem 5.36]. Because $B(\mathcal{H})$ admits polar decompositions, a unitary in the Calkin algebra lifts to a partial isometry in $B(\mathcal{H})$.[20] But if we identify $K_0(\mathbb{C})$

[20]Let T be any lift of a unitary u in $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then we can write $T = V|T|$ for a partial isometry V. Since $q(|T|) = q(T^*T)^{\frac{1}{2}} = 1$, $q(V) = u$ as required.

with $\mathbb{Z}$, then we have $j(V) = [1 - V^*V] - [1 - VV^*]$. Note that

$$W := \begin{pmatrix} V & 1 - VV^* \\ 1 - V^*V & V^* \end{pmatrix}$$

is a unitary in $\mathrm{U}_2\big(B(\mathcal{H})\big)$ covering $u \oplus u^*$ in $\mathrm{U}_2\big(B(\mathcal{H})/\mathcal{K}(\mathcal{H})\big)$. And a little calculation reveals that

$$\begin{aligned}[W(1 \oplus 0)W^*] - [1 \oplus 0] &= [VV^* \oplus 1 - V^*V] - [1 \oplus 0] \\ &= [1 - V^*V] - [1 - VV^*] \\ &= j(V).\end{aligned} \tag{8.1}$$

If

$$0 \longrightarrow J \xrightarrow{\ i\ } A \xrightarrow{\ q\ } A/J \longrightarrow 0$$

is an exact sequence of C^*-algebras with J an ideal in A, then we want to mimic the index map from Fredholm theory. Unfortunately, a unitary in $\mathrm{U}_n^1(A/J)$ need not lift to a partial isometry in $M_n(A^1)$. But Corollaries 5.11 and 5.18 we can always find a unitary $w \in \mathrm{U}_{2n}^1(A)$ such that $q^1(w) = u \oplus u^*$. Then $wp_nw^* - p_n \in M_{2n}(A)$ and belongs to the kernel of q. Therefore

$$\tilde{\partial}(u, w) := [wp_nw^*] - [p_n] \tag{8.2}$$

defines a class in $K_0(J)$.

Theorem 8.1. *The class of* (8.2) *depends only on the class* $[u]$ *in* $K_1(A/J)$ *and defines a homomorphism* $\partial : K_1(A/J) \to K_0(J)$ *such that*

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{q_*} K_1(A/J) \xrightarrow{\partial} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/J)$$

is exact.

Remark 8.2. Since ∂ coincides with the Fredholm index map when $A = B(\mathcal{H})$ and $J = \mathcal{K}(\mathcal{H})$, ∂ is called the *index map* in K-theory.

Proof. The first order of business is to show that the right-hand side of (8.2) depends only on the class of u in $K_1(A/J)$. Suppose that $w_1 \in \mathrm{U}_{2n}^1(A)$ is another lift of $u \oplus u^*$. Then $z := w_1^*w \in \mathrm{U}_{2n}^1(J)$, and implements an equivalence between wp_nw^* and $w_1p_nw_1^*$ in $V(J)$. Therefore the right-hand side of (8.2) is independent of the choice of lift w. Now suppose that $u \sim_h v$ in $\mathrm{U}_n^1(A/J)$. Then $x := u^*v$ and ux^*u^* both belong to $\mathrm{U}_n^1(A/J)_0$. Corollary 5.18 implies there are $a, b \in \mathrm{U}_n^1(A)_0$ such that $q^1(a) = x$ and $q^1(b) = ux^*u^*$. But

$$v \oplus v^* = ux \oplus x^*u^* = (u \oplus u^*)(x \oplus ux^*u^*),$$

and this lifts to $w(a \oplus b)$. Since $a \oplus b$ commutes with p_n,

$$w(a \oplus b)p_n(a^* \oplus b^*)w^* = wp_nw^*,$$

and the right-hand side of (8.2) is independent of the homotopy class of u. To see that the right-hand side of (8.2) depends only on $[u]$, we have to see what happens when we replace u by $u \oplus 1_m$ for some $m \in \mathbb{Z}^+$. Let w be a lift for $u \oplus u^*$ and y a scalar matrix[21] such that

$$q^1(y)(u \oplus u^* \oplus 1_{2m})q^1(y) = u \oplus 1_m \oplus u^* \oplus 1_m.$$

[21] The term *scalar matrix* refers to a matrix of the form $\iota(y')$ for some $y \in M_k$ for appropriate k. Note that $q^1(\iota(y'))$ is again a scalar matrix for the same y'.

Thus $y(w\oplus 1_{2m})y^*$ is a lift of $u\oplus 1_m\oplus u^*\oplus 1_m$. Furthermore,

$$\begin{aligned} y(w\oplus 1_{2m})y^*p_{n+m}y(w^*\oplus 1_{2m})y^* &\approx (w\oplus 1_{2m})(p_n\oplus p_m)(w^*\oplus 1_{2m}) \\ &= wp_nw^*\oplus p_m. \end{aligned} \tag{8.3}$$

But $[wp_nw^*\oplus p_m]-[p_{n+m}]$ equals the left-hand side of (8.2) in $K_0(J)$. Thus $\partial\bigl([u]\bigr)$ is well-defined.

To see that ∂ is a homomorphism, it suffices to consider $u,v\in \mathrm{U}_n^1(A/J)$. Let w and z be lifts of $u\oplus u^*$ and $v\oplus v^*$, respectively. As above, we can find a scalar matrix y such that $y(w\oplus z)y^*$ is a lift of $u\oplus v\oplus u^*\oplus v^*$ and $y^*p_{2n}y=p_n\oplus p_n$. Therefore

$$\begin{aligned} \partial\bigl([u][v]\bigr) &= \partial\bigl([uv]\bigr) \\ &= \partial\bigl([u\oplus v]\bigr) \\ &= [y(w\oplus z)y^*p_{2n}y(w^*\oplus z^*)y^*] \\ &= [(w\oplus z)(p_n\oplus p_n)(w^*\oplus z^*)] \\ &= [wp_nw^*\oplus zp_nz^*]-[p_n\oplus p_n] \\ &= \partial\bigl([u]\bigr)+\partial\bigl([v]\bigr). \end{aligned}$$

Thus ∂ is a homomorphism.

Lemma 8.3. $\ker\partial=\operatorname{im}q_*$.

Proof. The image of q_* consists of those classes in $K_1(A/J)$ of the form $[q^1(v)]$ for some $v\in\mathrm{U}_n^1(A)$. But then $w=v\oplus v^*$ is a lift of $q^1(v)\oplus q^1(v)^*$, and then $wp_nw^*=p_n$. Therefore $\partial\bigl([q^1(v)]\bigr)=0$, and $\operatorname{im}q_*\subset\ker\partial$.

Now suppose that $[u]\in\ker\partial$ for some $u\in\mathrm{U}_n^1(A/J)$. Let w be a lift of $u\oplus u^*$. Since $[wp_nw^*]-[p_n]=0$ in $K_0(J)$, Proposition 1.19 implies that there is a $m\in\mathbb{Z}^+$ and $k\geq 2n+m$ such that

$$wp_nw^*\oplus p_m\approx p_n\oplus p_m \tag{8.4}$$

via a unitary in $\mathrm{U}_k(J^1)$. There is no harm in insisting that $k\geq 2n+2m$. Let y be a scalar matrix as in (8.3), and let $z:=y(w\oplus 1_{2m})y^*\oplus 1_d$, where $d=k-2n-2m$. Then

$$wp_nw^*\oplus p_m\approx zp_{n+m}z^* \tag{8.5}$$

via a scalar unitary matrix in $\mathrm{U}_k(J^1)$. Combining (8.4) and (8.5), there is a unitary matrix $x\in\mathrm{U}_k(J^1)$ such that

$$x(zp_{n+m}z^*)x^*=p_{n+m}.$$

The scalar matrix $a:=\iota\bigl(\pi(x)\bigr)\in\mathrm{U}_k(J^1)$ commutes with p_{n+m} (since z is normalized). Note that $q^1(a^*x)=1_k=\pi(a^*x)$. Since J^1 is an ideal in A^1, $a^*xz\in\mathrm{U}_k^1(J)$, and $q^1(a^*xz)=q^1(z)=u\oplus 1_m\oplus u^*\oplus 1_m\oplus 1_d$. Since a^*xz commutes with p_{n+m}, it must be of the form $b\oplus c$ for $b\in\mathrm{U}_{n+m}^1(J)$ and $c\in\mathrm{U}_{k-n-m}^1(J)$. In particular, $q^1(b)=u\oplus 1_m$; thus $[u]=[u\oplus 1_m]=[q^1(b)]\in\operatorname{im}q_*$. This completes the proof of the lemma. $\square$

Lemma 8.4. $\operatorname{im}\partial=\ker i_*$.

Proof. Since $[wp_nw^*] = [p_n]$ in $V(A)$, it is immediate that $\operatorname{im}\partial \subset \ker i_*$. Let $x \in \ker i_*$. Then we can write $x = [e] - [p_n]$ for some projection $e \in M_k(J^1)$ with $e - p_n \in M_k(J)$ and $k \geq n$. Since $[e] - [p_n] = 0$ in $K_0(A)$, we can, by Lemma 1.17 and Proposition 1.19, replace e by $e \oplus p_m$ and n by $n+m$ and suppose that there is a unitary $w_1 \in \mathrm{U}_{2n}(A^1)$ such that $w_1p_nw_1^* = p_n$. Since $\pi(e) = p_n$, $\pi(w_1)$ commutes with p_n, and we can replace w_1 with $\iota\big(\pi(w_1)^*\big)w_1$ and assume that w_1 is normalized. Since $w := w_1 \oplus w_1^* \in \mathrm{U}^1_{4n}(A)_0$, we can assume that $e = wp_nw^*$ in $M_{4n}(A^1)$ with $w \in \mathrm{U}^1_{4n}(A)_0$. Since $q^1(e) = p_n$, $q^1(w)$ commutes with p_n, and $q^1(w) = u_1 \oplus u_2$ with $u_1 \in \mathrm{U}^1_n(A/J)$ and $u_2 \in \mathrm{U}^1_{3n}(A/J)$. Using $u_1 \oplus u_2 \in \mathrm{U}^1_{4n}(A/J)_0$, it follows that $[u_1 \oplus 1_{2n}][u_2] = 1$ in $K_1(A/J)$. Thus for some $m \in \mathbb{Z}^+$, $u_1^* \oplus 1_{2n+m}$ and $u_2 \oplus 1_m$ are homotopic in $\mathrm{U}^1_{3n+m}(A/J)$. In particular, $(u_1^* \oplus 1_{2n+m})(u_2^* \oplus 1_m) \in \mathrm{U}_{3n+m}(A/J)_0$ has a lift (Corollary 5.18) $v \in \mathrm{U}^1_{3n+m}(A)_0$. Let

$$z := (1_n \oplus v)(w \oplus 1_m).$$

Note that

$$q^1(z) = \big(1_n \oplus (u_1^* \oplus 1_{2n+m})(u_2^* \oplus 1_m)\big)(u_1 \oplus u_2 \oplus 1_m) = u_1 \oplus u_2 \oplus 1_{2n+m}.$$

Furthermore,

$$\begin{aligned} zp_nz^* &= (1_n \oplus v)(w \oplus 1_m)p_n(w^* \oplus 1_m)(1_n \oplus v^*) \\ &= (1_n \oplus v)(e + 0_{3n+m})(1_n \oplus v^*) \\ &= e. \end{aligned}$$

Therefore $x = [e] - [p_n] = [zp_nz^*] - [p_n]$. But if z_1 is a lift of $u_1 \oplus u_1^*$, then $z_1 \oplus 1_{2n+m}$ is a lift of $u_1 \oplus u_1^* \oplus 1_{2n+m}$ and $(z_1 \oplus 1_{2n+m})z^* \in \mathrm{U}^1_{4n+m}(J)$. In particular, $[zp_nz^*] = [z_1p_nz_1^*]$ in $V(J)$. Thus

$$\begin{aligned} x &= [e] - [p_n] \\ &= [zp_nz^*] - [p_n] \\ &= [z_1p_nz_1^*] - [p_n] \\ &= \partial\big([u_1]\big). \quad \square \end{aligned}$$

Proof of Theorem 8.1 continued. Lemmas 8.3 and 8.4 give us exactness at $K_1(A/J)$ and $K_0(J)$. Since K_0 is half-exact by Theorem 6.1 and the half-exactness of K_1 follows from that of K_0 by Theorem 7.6, the result is proved. $\square$

9. C^*-Algebras with Identity

The goal of this section is to establish a number of realizations of $K_0(A)$ which closely parallel some classic ones for $K^0(X) \cong K_0\big(C(X)\big)$. In particular, we want to show that if A has an identity, then $K_0(A)$ is isomorphic to the Grothendieck group of the semigroup of isomorphism classes of finitely generated projective A-modules. We also want to present a result I found in Higson's [Hig90, Theorem 3.31] which generalizes the result [Ati89, Theorem A1][22] which states that $K_0\big(C(X)\big)$ is isomorphic to the set of homotopy classes of maps from X into the set of Fredholm operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$. Specifically, we want to show that $K_0(A)$ is isomorphic to the set of homotopy classes of generalized Fredholm operators on Hilbert A-modules. (Higson [Hig90] looks at operators

[22]Higson [Hig90] attributes this to Atiyah and Janich

on *all* countably generated A-modules, while Wegge-Olsen [WO93, Chap. 17] considers only operators on Kasparov's universal Hilbert module H_A.) For most of this section, *we will assume that A has an identity.*

9.1. **Projective modules.** Projective modules arise naturally in the theory. As you know, $K_0(A)$ is generated by projections in matrix algebras over A. Since a idempotent in a C^*-algebra must be similar to a projection, it should not be surprising that we could also build $K_0(A)$ out of idempotents in matrix algebras over A. So before proceeding with the formal definitions, I'll briefly outline how finitely generated projective A-modules are related to such idempotents. First, lets recall that if A is a ring with identity, then a right A-module[23] X is *finitely generated* if it has a finite spanning set $\{x_1, \dots, x_n\}$. Such a module is called *free* if it has a basis — that is, a spanning set which is also linearly independent in the obvious sense. This is equivalent to saying that $\mathsf{X} \cong A^n$ for some n. An A-module P is called *projective* if every homomorphism out of P into a quotient module lifts. Thus given a surjective homomorphism $q : \mathsf{E} \to \mathsf{F}$ and a homomorphism $f : \mathsf{P} \to \mathsf{F}$, there is a homomorphism $h : \mathsf{P} \to \mathsf{E}$ such that

$$\begin{tikzcd} & \mathsf{E} \arrow[d, "q"] \\ \mathsf{P} \arrow[ur, dashed, "h"] \arrow[r, "f"'] & \mathsf{F} \end{tikzcd}$$

commutes. It is pretty easy to see that free modules are projective.

Now suppose that P is projective with generators $\{x_1, \dots, x_n\}$. Then we have a surjection $q : A^n \to \mathsf{P}$ and a map c such that

(9.1)
$$\begin{tikzcd} & A^n \arrow[d, "q"] \\ \mathsf{P} \arrow[ur, dashed, "c"] \arrow[r, "\mathrm{id}"'] & \mathsf{P} \end{tikzcd}$$

commutes. In particular, c is injective, and we can identify P with $c(\mathsf{P})$ so that $A^n \cong \mathsf{P} \oplus \ker q$. This P is (isomorphic to) a direct summand of a free module. Note that in this case, we can view q as a module map from A^n to itself and that $q^2 = q$. Since A has an identity, module maps from A^n to itself are in natural one-to-one correspondence with matrices in $M_n(A)$, and we can view q as an idempotent in the matrix ring $M_n(A)$. Finally, notice that given an idempotent q, the A-module $\mathsf{P} := q(A^n)$ is projective. To see this, note that A^n is also projective, given a surjection g and a map f as below, we get a map h such that

$$\begin{tikzcd} & & \mathsf{E} \arrow[d, "g"] \\ A^n \arrow[urr, dashed, "h"] \arrow[r, "q"'] & q(A^n) \arrow[ur, dotted, "h'"] \arrow[r, "f"'] & \mathsf{F} \end{tikzcd}$$

commutes. But then $h' := h|_{q(A^n)}$ satisfies

$$g\big(h'\big(q(a)\big)\big) = f\big(q^2(a)\big) = f\big(q(a)\big),$$

[23]We should say *unitary* module as we are assuming that $x \cdot 1_R = x$ for all $x \in \mathsf{X}$. Note that if $1 \in A$, then any Hilbert A-module is unitary (because $x \cdot 1 - x$ has length zero).

and $q(A^n)$ is projective. Thus finitely generated projective modules correspond naturally to idempotents in matrix rings over A.

With that algebra safely behind us, we now assume that A is a C^*-algebra with identity 1_A. One point of confusion best dealt with early is the following. A (right) Hilbert A-module X is *countably generated* if there is a countable subset D such that

$$\mathsf{X} = \overline{D \cdot A} := \overline{\operatorname{span}}\{\, a \cdot d : d \in D \text{ and } a \in A \,\}. \tag{9.2}$$

Naturally then, a Hilbert module should be *finitely generated* when we can take D finite above. Thus we will have to say that X is *algebraically* finitely generated to indicate that X is finitely generated as a module over the ring A. For definiteness, we'll say that X is topologically generated when we want to refer to (9.2).

A remarkable theorem of Kasparov [RW98, Theorem 5.49] implies that all the modules we want to consider are complemented submodules (as Hilbert modules[24]) of a universal Hilbert A-module H_A. We recall from [RW98, Proposition 2.15] that

$$\mathsf{H}_A := \{\, (a_i) \in \prod_{i=1}^{\infty} A : \sum a_i^* a_i \text{ converges in } A \,\}.$$

Alternatively, H_A can be realize as the external tensor product $A \otimes \ell^2$ [RW98, Lemma 3.43]. (Of course, ℓ^2 could be replaced by any separable infinite-dimensional Hilbert space $\mathcal{H}$.) Then Kasparov's theorem implies that any *countably generated* Hilbert A-module X is isomorphic to a Hilbert module direct summand of H_A; alternatively, $\mathsf{X} \oplus \mathsf{H}_A \cong \mathsf{H}_A$.

It should be kept in mind that, although Hilbert modules are natural and straightforward generalizations of Hilbert spaces, there are a number of properties which we take for granted in Hilbert space that often do not hold for Hilbert modules. For example, a Hilbert module will often fail to be *self-dual*: we say that X is self-dual if given a *bounded* A-linear map $\varphi : \mathsf{X} \to A_A$, then there exists $y \in \mathsf{X}$ such that $\varphi(x) = \langle y \, , x \rangle_A$ for all $x \in \mathsf{X}$. But if $1 \in A$, then A_A, and therefore A^n are easily seen to be self-dual.[25] Note also, that if X is self-dual, then the usual Hilbert space proof shows that the set of bounded A-linear operators $B_A(\mathsf{X})$ on X coincides with $\mathcal{L}(\mathsf{X})$. Our interest in self-dual modules is due to the following lemma which will be crucial in establishing the uniqueness assertion in Theorem 9.8.

Lemma 9.1. *Suppose that* E *is an orthogonal direct summand of* A^n *and that* F *is a Hilbert A-module which is isomorphic to* E *as an A-module. Then* E *and* F *are isomorphic as Hilbert A-modules.*[26]

For the proof of the lemma, we'll need to grapple with yet another defect[27] of Hilbert modules: operators need not have polar decompositions. We pause here to

[24] I am emphasizing that "direct summand" is taken in the category of Hilbert modules. This is logically distinct from being simply an A-module direct summand. In Hilbert space, this is the distinction between being an orthogonal direct summand, and simply a vector space or *skew* summand. Consequently, the terms "orthogonal summand" or "orthogonal direct sum" will also be used to make this distinction.

[25] A result of Frank [Fra90] shows that H_A can be self-dual only when A is finite dimensional.

[26] Two Hilbert A-modules E and F are isomorphic as A-modules if there is a merely a bijective module map $u : \mathsf{E} \to \mathsf{F}$. If they in addition u preserves inner-products, then we say that E and F are isomorphic as Hilbert A-modules or that E and F are *unitarily* equivalent. Note that u^{-1} is an adjoint for u so that u is necessarily a unitary in $\mathcal{L}(\mathsf{E}, \mathsf{F})$.

[27] In computer science, this would be a "feature".

give a rather longer discussion than required to prove Lemma 9.1 since we'll need polar decompositions in Section 9.3.

Definition 9.2. Let X and Y be Hilbert A-modules. An operator $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ has a polar decomposition if there is a partial isometry $V \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ such that $T = V|T|$ and

$$\ker V = \ker T \qquad \ker V^* = \ker T^*$$
$$\operatorname{Range} V = \overline{T(\mathsf{X})} \qquad \operatorname{Range} V^* = \overline{|T|(\mathsf{X})}.$$

Recall that if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator between Hilbert *spaces* $\mathcal{H}_i$, then the business of polar decomposition goes fairly easily. Then map sending $|T|h \in \mathcal{H}_1$ to $Th \in \mathcal{H}_2$ is both well-defined and isometric. This gives a isometry u of $\overline{|T|(\mathcal{H}_1)}$ onto $\overline{T(\mathcal{H}_1)}$ in $\mathcal{H}_2$. We get an operator $v := u \oplus 0 : \mathcal{H}_1 \to \mathcal{H}_2$, by simply setting v to be zero on $|T|(\mathcal{H}_1)^\perp$. Then $T = v|T|$ is our *polar decomposition.* This proof breaks down for $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ because $\overline{|T|(\mathsf{X})}$ may not be complemented.[28] It can also break down because v may not be adjointable as an operator from Y to X.

Example 9.3 (Noncomplemented submodule). Let $\mathsf{X} = C\big([0,1]\big)$ viewed a right Hilbert module over itself. Let $\mathsf{Y} = \{\, f \in C\big([0,1]\big) : f(0) = 0 \,\}$. Then Y is a closed submodule of X and it is easily checked that $\mathsf{Y}^\perp = \{\, 0 \,\}$. Thus Y is not complemented in X.

Lemma 9.4. *Suppose that* X *and* Y *are Hilbert A-modules and that* $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$. *Then T has a polar decomposition if and only if* $\overline{T(\mathsf{X})}$ *is complementable in* Y *and* $\overline{|T|(\mathsf{X})}$ *is complementable in* X.

Proof. If $T = V|T|$ is a polar decomposition for T, then $I_{\mathsf{X}} = (I_{\mathsf{X}} - V^*V) \oplus V^*V$ and $I_{\mathsf{Y}} = (I_{\mathsf{Y}} - VV^*) \oplus VV^*$ give the required decompositions of X and Y.

Since $\ker T = \ker |T|$, elementary considerations show that $|T|(\mathsf{X})^\perp = \ker T$ and $T(\mathsf{X})^\perp = \ker T^*$. Thus if the ranges of T and $|T|$ are complementable, then

$$\mathsf{Y} = \overline{T(\mathsf{X})} \oplus \ker T^*$$
$$\mathsf{X} = \overline{|T|(\mathsf{X})} \oplus \ker T.$$

Thus, as in the Hilbert space case, we can define $V = U \oplus 0 : \mathsf{X} \to \mathsf{Y}$ such that $T = V|T|$. Then $W' := U^{-1} \oplus 0$ (relative to $\overline{T(\mathsf{X})} \oplus \ker T^*$) is easily seen to be an adjoint for V. The rest is straightforward. $\square$

Example 9.5 (Operator without Polar Decomposition). Let $\mathsf{X} := C\big([0,1]\big)$ and Y be as in Example 9.3. Define $g \in \mathsf{X}$, by $g(t) = t$ for all $t \in [0,1]$. Define $T : \mathsf{X} \to \mathsf{X}$ by $T(f) = gf$. Then it is easily checked that $T \in \mathcal{L}(\mathsf{X})$ with $T^* = T$. Furthermore, $\ker T = \{\, 0 \,\}$. Thus if T had a polar decomposition, then $T(\mathsf{X})$ would have to be dense. But $T(\mathsf{X}) \subset \mathsf{Y}$ is certainly not dense.

Fortunately, the following will be enough for our purposes.

---

[28]We say that a closed submodule E of X is complemented if $\mathsf{X} = \mathsf{E} \oplus \mathsf{E}^\perp$. Since $\mathsf{E}^{\perp\perp}$ always contains E, E fails to be complementable when $\mathsf{E}^{\perp\perp}$ is strictly larger than E. For example, in a Hilbert module, one can have $\mathsf{E}^\perp = \{\, 0 \,\}$ without E being dense.

Theorem 9.6 ([WO93, Theorem 15.3.8])**.** *Suppose that* X *and* Y *are Hilbert A-modules, and that* $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ *has* closed range. *Then both* T^* *and* $|T|$ *also have closed range. In fact, the ranges of* T*,* $|T|$*, and* T^* *are complementable and we have*

$$\mathsf{X} = \ker |T| \oplus |T|(\mathsf{X}) = \ker T \oplus T^*(\mathsf{Y}) \quad \textit{and} \quad \mathsf{Y} = \ker T^* \oplus T(\mathsf{X}).$$

In particular, T *has a polar decomposition* $= V|T|$*.*

Proof. Suppose that $T(\mathsf{X})$ is closed. Then $U : |T|(\mathsf{X}) \to T(\mathsf{X})$ is isometric and surjective. Since $T(X)$ is complete, so is $|T|(\mathsf{X})$; hence $|T|(\mathsf{X})$ is closed. Since we can approximate $|T|^{\frac{1}{2}}$ by polynomials in $|T|$ without constant term,

$$|T|(\mathsf{X}) = |T|^{\frac{1}{2}}|T|^{\frac{1}{2}}(\mathsf{X}) \subset |T|^{\frac{1}{2}}(\mathsf{X}) \subset \overline{|T|(\mathsf{X})} = |T|(\mathsf{X}), \quad \text{and}$$
$$\ker |T| \subset \ker |T|^{\frac{1}{2}} \subset \ker |T|^{\frac{1}{2}}|T|^{\frac{1}{2}} = \ker |T|.$$

Thus $|T|(\mathsf{X}) = |T|^{\frac{1}{2}}(\mathsf{X})$ and $\ker |T|^{\frac{1}{2}} = \ker |T|$. Now if $x \in \mathsf{X}$, then $|T|^{\frac{1}{2}}x = |T|y$ for some $y \in \mathsf{X}$ and $x = (x - |T|^{\frac{1}{2}}y) + |T|^{\frac{1}{2}}y$. That is, $x \in \ker |T|^{\frac{1}{2}} \oplus |T|^{\frac{1}{2}}(\mathsf{X})$. We have shown that

$$\mathsf{X} = \ker |T|^{\frac{1}{2}} \oplus |T|^{\frac{1}{2}}(\mathsf{X})$$
$$= \ker |T| \oplus |T|(\mathsf{X}) \tag{9.3}$$
$$= \ker T \oplus |T|(\mathsf{X}). \tag{9.4}$$

Therefore $|T|(\mathsf{X})$ is complemented.

To show that $T(X)$ is complemented, we'll show that $T^*(\mathsf{Y})$ is closed and complemented. Then we can repeat the proof with T^* in place of T and conclude that $T(\mathsf{X})$ is complemented. First, note that $T^*T(\mathsf{X}) = |T|^2(\mathsf{X}) \subset |T|(\mathsf{X})$. If $x \in |T|(\mathsf{X})$, then $x = |T|y$ for some $y \in \mathsf{X}$. By (9.3), we can write $y = y_1 + y_2$ with $y_1 \in \ker |T|$ and $y_2 = |T|z$ for some $z \in \mathsf{X}$. Then $x = |T|y = |T|(y_1 + y_2) = |T|^2 z = T^*Tz \in T^*(\mathsf{Y})$. This shows that $|T|(\mathsf{X}) = T^*T(\mathsf{X})$. By (9.4),

$$\mathsf{X} = \ker T \oplus T^*T(\mathsf{X}). \tag{9.5}$$

Since

$$T^*T(\mathsf{X}) \subset T^*(\mathsf{Y}) \subset (\ker T)^\perp = T^*T(\mathsf{X}),$$

we see that $T^*(\mathsf{Y}) = T^*T(\mathsf{X}) = (\ker T)^\perp$ is closed and

$$\mathsf{X} = \ker T \oplus T^*(\mathsf{Y})$$

as required. □

Proof of Lemma 9.1. Let $T : \mathsf{E} \to \mathsf{F}$ be an A-module isomorphism. Since A^n is self-dual, it is not hard to see that E is also self-dual in its inherited structure. Furthermore, a sequence $\{ a_k := (a_i^k) \}$ in E converges to $a := (a_i)$ if and only if $a_i^k \to a_i$ in A for each $i = 1, 2, \ldots, n$. But if P is the projection of A^n onto E and $\{ e_i \}_{i=1}^n$ is the usual basis for A^n, then

$$T\big((a_i)\big) = T\big(P\big((a_i)\big)\big)$$
$$= \sum_{i=1}^n T\big(P(e_i)\big) \cdot a_i.$$

In particular, T is continuous and therefore bounded. Since E is self-dual, T is adjointable. Since T is surjective (and therefore certainly has closed range), Theorem 9.6 implies that $|T|$ is also surjective.[29] Therefore $|T|x \mapsto Tx$ is a unitary transformation U of E onto F as required. □

Example 9.7 (Nonclosed finitely generated submodule). Consider the operator $T \in \mathcal{L}(\mathsf{X})$ defined in Example 9.5. The range of T is the submodule of X generated by the identity function g. Since T doesn't have a polar decomposition, Theorem 9.6 implies T can't have closed range. Therefore $T(\mathsf{X})$ is a finitely generated submodule of X which is not closed.

We'll write $\{\, \mathbf{e}_i \,\}$ for the usual basis for H_A and identify A^n with the subspace spanned by $\{\, \mathbf{e}_1, \dots, \mathbf{e}_n \,\}$. We'll also write p_n for the projection of H_A onto A^n.

Theorem 9.8 ([WO93, Theorem 15.4.2])**.** *Let A be a C^*-algebra with identity and suppose that E is a right A-module. Then the following conditions are equivalent.*

(a) *E is a finitely generated projective A-module.*
(b) *E is isomorphic to an algebraic direct summand in A^n for some $n \in \mathbb{Z}^+$.*
(c) *E is isomorphic to a Hilbert module direct summand in A^n for some $n \in \mathbb{Z}^+$.*
(d) *E is isomorphic to a closed finitely generated submodule of H_A.*
(e) *There is a $q \in \mathcal{K}(\mathsf{H}_A)$ such that E is isomorphic to $q(\mathsf{H}_A)$.*
(f) *There is a $q \in \mathcal{K}(\mathsf{H}_A)$ with $q \le p_n$ such that E is isomorphic to $q(\mathsf{H}_A)$.*

In particular, every finitely generated projective A-module admits a Hilbert A-module structure which is self-dual and unique up to unitary isomorphism. With respect to this structure, the isomorphisms in (c)–(f) *can be taken to be unitary.*

Note that the hypothesis in part (d) is not redundant in view of Example 9.7. For the proof, we'll want the following lemma which follows almost immediately from lemmas 2.3 and 2.1.

Lemma 9.9. *Let A be a C^*-algebra with identity. If $q \in \mathcal{K}(\mathsf{H}_A)$ is a projection, then for sufficiently large $n \in \mathbb{Z}^+$, there is a unitary $u \in \mathcal{L}(\mathsf{H}_A)$ such that $uqu^* \le p_n$.*

Proof. It is straightforward to check that $\{\, p_n \,\}$ is an approximate unit for $\mathcal{K}(\mathsf{H}_A)$. Then for large n, we can assume that $\|q - p_n q p_n\| < \frac{1}{12}$. If $q_n := p_n q p_n$, then q_n is self-adjoint and $\|q_n\| \le 1$. Furthermore,

$$\|q_n^2 - q_n\| \le \|q_n(q_n - q)\| + \|(q_n - q)q\| + \|q - q_n\| < \frac{1}{4}.$$

Since q_n is self-adjoint, a little functional calculus (Lemma 2.3) implies there is a projection $q' \in \mathcal{K}(\mathsf{H}_A)$ such that $\|q' - q_n\| < \frac{1}{2}$. Then $\|q - q'\| < 1$, and q and q' are unitarily equivalent in $\mathcal{L}(\mathsf{H}_A)$ by Lemma 2.1.

Thus, it will suffice to see that q' is majorized by p_n. A quick look at the proof of Lemma 2.3 reveals that q' is constructed via the functional calculus $q' = f(q_n)$ for a function f vanishing at 0. This means we can approximate f with polynomials in q_n without constant terms. It follows that $p_n q' = q' p_n = q'$; that is $q' \le p_n$. □

Proof of Theorem 9.8. (*a*) $\iff$ (*b*): follows almost by definition (cf., (9.1) and following discussion).

(*b*) $\implies$ (*d*): Let $A^n = \mathsf{F} \boxplus \mathsf{E}$. (Here, $\boxplus$ denotes an algebraic or skew direct sum.) Let P be the skew projection of A^n onto E. Then P is an A-module map and is

[29] We don't really need the full power of Theorem 9.6; we only need (9.4).

given by a matrix in $M_n(A)$. In particular, P is continuous and $\mathsf{E} = P(A^n)$ is closed in A^n. Since we have identified A^n with its image in H_A, this suffices.

$(d) \implies (e)$: Let $\{ f_1, \dots, f_k \} \subset \mathsf{E} \subset \mathsf{H}_A$ be a set of generators for E. Then $\theta_{f_i,\mathbf{e}_i}$ belongs to $\mathcal{K}(\mathsf{H}_A)$ as does

$$K := \sum_{i=1}^{k} \theta_{f_i,\mathbf{e}_i}.$$

We clearly have $K(\mathsf{H}_A) \subset \mathsf{E}$. But if $\{ a_1, \dots, a_k \} \subset A$, then $a = \sum_{i=1}^k \mathbf{e}_i \cdot a_i \in \mathsf{H}_A$, and $K(a) = \sum f_i \cdot a_i$, so $K(\mathsf{H}_A) = \mathsf{E} = K(A^k)$. Thus K has closed range and we can apply Theorem 9.6 to conclude that $K = V|K|$, where V is a partial isometry with support projection V^*V equal to the orthogonal complement of $\ker V$ or $\operatorname{Range} K^*$. Since $K^* = \sum_i \theta_{\mathbf{e}_i,f_i}$, this implies that $V^*V \le p_k$. It follows that $V^*V = V^*Vp_k$ is compact. Since $V = VV^*V$, it follows that V is compact, and so is the range projection $Q := VV^*$. Again by Theorem 9.6,

$$\operatorname{Range} Q = \operatorname{Range} V = \operatorname{Range} K = \mathsf{E}.$$

$(e) \implies (f)$: In view of Lemma 9.9, we can assume there is a compact projection q' such that $q' \le p_n$ and that there is a unitary $u \in \mathcal{L}(\mathsf{H}_A)$ such that $uq = q'u$. Then $u : \mathsf{H}_A \to \mathsf{H}_A$ is an inner-product preserving A-module map taking $q(\mathsf{H}_A)$ into $q'(\mathsf{H}_A)$ with inverse $u^* = u^{-1}$.

$(f) \implies (c)$: Viewing A^n as the closed submodule $p_n(\mathsf{H}_A)$, we can decompose A^n as

$$q(\mathsf{H}_A) \oplus (p_n - q)(\mathsf{H}_A).$$

And $(c) \implies (b)$ is trivial.

Thus if E is a finitely generated projective module, then (c) implies that E inherits a self-dual Hilbert A-module structure as an orthogonal summand in A^n. Now the uniqueness and other assertions follow from Lemma 9.1. □

If we start with a Hilbert module, then we can add two more characterizations to the list in Theorem 9.8.

Corollary 9.10. *Suppose that $1 \in A$ and that E is a algebraically finitely generated* Hilbert *A-module. Then E is projective.*

Proof. The Kasparov Stablization Theorem allows us to assume that E is a closed submodule of H_A. (The projection Q of H_A onto E is an A-linear operator which is self-adjoint and therefore bounded.) Thus the result follows immediately from Theorem 9.8. □

Corollary 9.11. *Suppose that E is a Hilbert A-module. Then E is finitely generated and projective if and only if $1_E \in \mathcal{K}(\mathsf{E})$. Moreover, if $1_{\mathsf{E}} \in \mathcal{K}(\mathsf{E})$, then there is a $n \in \mathbb{Z}^+$ and elements $\{ x_i, y_i \}_{i=1}^n \subset \mathsf{E}$ such that*

$$1_{\mathsf{E}} = \sum_{i=1}^{n} \theta_{x_i,y_i}.$$

Proof. If E is finitely generated and projective, then Theorem 9.8 implies that we may as well assume that $\mathsf{E} = q(\mathsf{H}_A)$ for some compact projection q. But then q is an identity for $\mathcal{K}\big(q(\mathsf{H}_A)\big)$, and if $\sum_i \theta_{x_i,y_i}$ is close to q, then so is

$$q\Big(\sum_i \theta_{x_i,y_i}\Big)q = \sum_i \theta_{qx_i,qy_i} \in \mathcal{K}\big(q(\mathsf{H}_A)\big).$$

Thus, $q \in \mathcal{K}\big(q(\mathsf{H}_A)\big)$.

Now assume that $\mathcal{K}(\mathsf{E})$ has an identity. Then [RW98, Proposition 5.50] implies that E is countably generated. Thus we can assume (by Kasparov's result) that E is a orthogonal direct summand of H_A. Since $1 \in \mathcal{K}(\mathsf{E})$, it is not hard to see that the projection Q onto E is the norm limit of operators $\theta_{x,y}$ with $x, y \in \mathsf{E}$. Thus, $Q \in \mathcal{K}(\mathsf{H}_A)$, and E is finitely generated and projective by Theorem 9.8.

The ideal of "finite-rank" operators $\operatorname{span}\{\, \theta_{x,y} : x, y \in \mathsf{E}\,\}$ is dense in $\mathcal{K}(\mathsf{E})$. If $1_{\mathsf{E}} \in \mathcal{K}(\mathsf{E})$, then the invertibles are open in $\mathcal{K}(\mathsf{E})$, and 1_{E} must be finite-rank. $\square$

9.2. **Alternate realizations of $K_0(A)$.** Now recall from Lemma 1.17 that we can view $V(A)$ as the semigroup of *either* Murray-von Neumann or unitary equivalence classes of projections $P(A)$ in $M_\infty(A)$ where $[p] + [q] := [p \oplus q]$. Let $V_1(A)$ be the semigroup of A-module isomorphism classes of algebraically finitely generated projective A-modules where $[\mathsf{E}] + [\mathsf{F}] := [\mathsf{E} \oplus \mathsf{F}]$. Here, we have some flexibility in the underlying set for $V_1(A)$ (thanks to Theorem 9.8). In particular, we can restrict ourselves to algebraically finitely generated Hilbert A-modules (so that $\mathsf{E} \oplus \mathsf{F}$ means orthogonal direct sum), or even to algebraically finitely generated closed submodules of H_A. Also, we define $V_2(A)$ to be the collection of Murray-von Neumann equivalence classes[30] of projections in $\mathcal{K}(\mathsf{H}_A)$. With a little work, you can verify that $V_2(A)$ is a semigroup with respect to the operation

$$[p] + [q] := [p' + q'] \quad \text{where } [p] = [p'],\ [q] = [q'], \text{ and } p' \perp q'.$$

(The existence of such p' and q' can be deduced from Lemma 9.9 which allows us to assume that $p, q \le p_n$; now simply "shift" q so that $p \perp q$.)

Theorem 9.12 ([WO93, Exercise 15.K])**.** *Let A be a C^*-algebra with identity. The map sending a projection $p \in M_n(A)$ to the A-module $p(A^n)$ induces a semigroup isomorphism $\varphi : V(A) \to V_1(A)$. Similarly, the map sending a compact projection $q \in \mathcal{K}(\mathsf{H}_A)$ to the A-module $q(\mathsf{H}_A)$ induces a semigroup isomorphism $\psi : V_2(A) \to V_1(A)$. Thus the Grothendieck groups $\mathcal{G}\big(V_1(A)\big)$ and $\mathcal{G}\big(V_2(A)\big)$ are both naturally isomorphic to $K_0(A)$.*

Proof. Given φ and ψ, $\mathcal{G}(\varphi)$ and $\mathcal{G}(\varphi) \circ \mathcal{G}(\psi)$ provide isomorphisms of both $\mathcal{G}\big(V_1(A)\big)$ and $\mathcal{G}\big(V_2(A)\big)$ with $\mathcal{G}\big(V(A)\big) := K_{00}(A)$, which, since $1 \in A$, is naturally isomorphic to $K_0(A)$ by Proposition 1.15. Thus, we just have to verify the assertions about φ and ψ.

Since $p(A^n)$ and $(p + 0_k)(A^{n+k})$ are clearly isomorphic, we get a well-defined map $\varphi_0 : P(A) \to V_1(A)$. But if $[p] = [q]$ in $V(A)$, then we can assume that $p, q \in M_n(A)$ and that there exists $u \in \mathrm{U}_n(A)$ such that $qu = up$. Then $u : A^n \to A^n$ is a module isomorphism mapping $p(A^n)$ into $q(A^n)$. Furthermore $u^*|_{q(A^n)}$ is an inverse to $u|_{p(A^n)}$. Thus $\varphi_0(p) = \varphi_0(q)$, and we obtain a map $\varphi : V(A) \to V_1(A)$ such that $\varphi\big([p]\big) = [p(A^n)]$. It is not hard to check that φ is a semigroup homomorphism:

$$\begin{aligned} \varphi\big([p] + [q]\big) &= \varphi\big([p \oplus q]\big) \\ &= [(p \oplus q)A^{n+m}] \\ &= [p(A^n)] + [q(A^m)] = \varphi\big([p]\big) + \varphi\big([q]\big). \end{aligned}$$

[30] The partial isometry implementing an equivalence between two compact projections, must itself be compact. Thus we can view this as equivalence in $\mathcal{L}(\mathsf{H}_A)$ or in $\mathcal{K}(\mathsf{H}_A)$.

Note that φ is surjective by Theorem 9.8. To see that φ is injective, suppose that $\varphi\bigl([p]\bigr)=\varphi\bigl([q]\bigr)$. Then we can assume $p,q\in M_n(A)$, and that there is an A-module isomorphism $v:p(A^n)\to q(A^n)$. Lemma 9.1 allows us to assume that v is unitary. Let $w:=v\oplus 0$ (relative to $A^n=p(A^n)\oplus(1-p)(A^n)$). It is not hard to see that w is adjointable with adjoint $w^*=v^*\oplus 0$ (relative to $A^n=q(A^n)\oplus(1-q)(A^n)$). Since $w^*w=p$ and $ww^*=q$, it follows that $[p]=[q]$ and φ is injective. This shows that φ is an isomorphism as claimed.

To define ψ, let p and q be projections in $\mathcal{K}(\mathsf{H}_A)$ with $w\in\mathcal{L}(\mathsf{H}_A)$ such that $p=w^*w$ and $q=ww^*$. (Note that any such w must lie in $\mathcal{K}(\mathsf{H}_A)$.) Then $w|_{p(\mathsf{H}_A)}$ is an A-module isomorphism of $p(\mathsf{H}_A)$ onto $q(\mathsf{H}_A)$. Thus we can define $\psi\bigl([p]\bigr)=[p(\mathsf{H}_A)]$. This map is surjective by Theorem 9.8 and a homomorphism because

$$\begin{aligned}\psi\bigl([p]+[q]\bigr)&=[(p'+q')(\mathsf{H}_A)]\\&=[p'(\mathsf{H}_A)\oplus q'(\mathsf{H}_A)]\\&=\psi\bigl([p]\bigr)+\psi\bigl([q]\bigr).\end{aligned}$$

The proof that ψ is injective is similar to that for φ. □

9.3. **Generalized Fredholm Operators.** If X is a Hilbert module, then the finitely generated closed submodules of X are a reasonable analogue of the finite dimensional subspaces of a Hilbert space. (As Example 9.7 shows, the word closed is not redundant here as it would be for finitely generated subspaces of Hilbert spaces.) If π is the quotient map of $\mathcal{L}(\mathsf{X})$ onto $\mathcal{Q}(\mathsf{X}):=\mathcal{L}(\mathsf{X})/\mathcal{K}(\mathsf{X})$, then the subsets

$$\begin{aligned}\mathcal{F}(\mathsf{X})&=\{\,T\in\mathcal{L}(\mathsf{X}):\pi(T)\in\mathrm{GL}\bigl(\mathcal{Q}(\mathsf{X})\bigr)\,\}\\\mathcal{F}''(\mathsf{X})&=\{\,T\in\mathcal{L}(\mathsf{X}):T(\mathsf{X})\text{ is closed and both}\\&\qquad\qquad\ker T\text{ and }\ker T^*\text{ are algebraically finitely generated}\,\}\\\mathcal{F}'(\mathsf{X})&=\{\,T\in\mathcal{L}(X):T+K\in\mathcal{F}''(\mathsf{X})\text{ for some }K\in\mathcal{K}(\mathsf{X})\,\}\end{aligned}$$

are all reasonable candidates for the set of Fredholm operators on X, and Atkinson's Theorem [Ped89, Proposition 3.3.11] implies these sets coincide when $A=\mathbb{C}$. For general Hilbert modules, the relationship is unclear. When $\mathsf{X}=\mathsf{H}_A$, then we shall see that that

$$\mathcal{F}''(\mathsf{H}_A)\subsetneq\mathcal{F}'(\mathsf{H}_A)=\mathcal{F}(\mathsf{H}_A).\tag{9.6}$$

This result is part of Mingo's thesis [Min87], which is where much of the material from [WO93, Chap. 17] comes from.[31] In the interests of time and space, the treatment here will diverge from [WO93] (a.k.a. Mingo), and instead focus on Higson's treatment in [Hig90, §3]. In particular, we make the following definition.

Definition 9.13. If X and Y are Hilbert A-modules, then $T\in\mathcal{L}(\mathsf{X},\mathsf{Y})$ is a (generalized) Fredholm operator if there is a $S\in\mathcal{L}(\mathsf{Y},\mathsf{X})$ such that $1_{\mathsf{X}}-ST\in\mathcal{K}(\mathsf{X})$ and $1_{\mathsf{Y}}-TS\in\mathcal{K}(\mathsf{Y})$. One calls S a parametrix for F. The set of Fredholm operators from X to Y is denoted $\mathcal{F}(\mathsf{X},\mathsf{Y})$.

[31]Some of Mingo's results rely quite heavily on earlier work of Kasparov, Miščenko and Fromenko, and Pimsner, Popa, and Voiculescu. Some of the arguments have simplified over time, but Mingo's treatment is still quite nice and readable.

Of course, when $\mathsf{X} = \mathsf{Y}$, we simply write $\mathcal{F}(\mathsf{X})$ and obtain the same class of operators as above. In the classical case — that is, $\mathsf{X} = \mathsf{Y} \cong \ell^2$, every Fredholm operator T has closed range and one can choose the parametrix S so that $1 - ST$ is the projection onto $\ker T$ while $1 - TS$ is the projection onto $\ker T^*$ [Ped89, Proposition 3.3.11]. In particular, both $1 - ST$ and $1 - TS$ are finte-rank. Unfortunately, not every generalized Fredholm operator need have closed range: the operator $T : \mathsf{X} \to \mathsf{X}$ from Example 9.5 is Fredholm, since $\mathcal{K}(\mathsf{X}) = \mathcal{L}(\mathsf{X})$. However, as Exel points out in [Exe93], but does not prove, one can still demand that the parametrix is an inverse up to generalized finite-rank operators; that is, operators in $\operatorname{span}\{\, \theta_{x,y} : x \in \mathsf{X} \text{ and } y \in \mathsf{Y} \,\}$.

Lemma 9.14. *Suppose that $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ is Fredholm. Then there is a parametrix S such that $1_{\mathsf{X}} - ST$ and $1_{\mathsf{Y}} - TS$ are finite-rank operators.*

Proof. Let $\Theta(\mathsf{X})$ denote the finite-rank operators from X to X. If S is a parametrix for T, then there is a finite-rank $F \in \Theta(\mathsf{X})$ such that $\|1 - (ST + F)\| < 1$. In particular, $ST + F$ is invertible. Let $S' := (ST + F)^{-1} S$. Then

$$\begin{aligned} S'T &= (ST + F)^{-1} ST = (ST + F)^{-1}(ST + F) + (ST + F)^{-1} F \\ &= 1_X + (ST + F)^{-1} F. \end{aligned}$$

Since $\Theta(\mathsf{X})$ is a (not necessarily closed) two-sided ideal in $\mathcal{L}(\mathsf{X})$, $1_{\mathsf{X}} - S'T$ is finite rank.

Similarly, we can construct S'' such that $1_{\mathsf{Y}} - TS''$ is finite rank. But then modulo the *ideal* of finite-rank operators,

$$S' \equiv S'(TS'') \equiv (S'T)S'' \equiv S''.$$

Therefore, $1_{\mathsf{Y}} - TS' = (1_{\mathsf{Y}} - TS'') + T(S'' - S') \in \Theta(\mathsf{Y})$. □

If T has closed range, then we can invoke Theorem 9.6 to recover a more complete analogue of the classical theory.

Lemma 9.15. *Let X and Y be Hilbert A-modules. Suppose that $T \in \mathcal{F}(\mathsf{X}, \mathsf{Y})$ has closed range. Then there is a parametrix S for T such that*

$$T = TST \quad \text{and} \quad S = STS. \tag{9.7}$$

In fact, we can choose S such that $1_{\mathsf{Y}} - TS$ is the (orthogonal) projection onto $\ker T^*$ and $1_{\mathsf{X}} - ST$ is the (orthogonal) projection onto $\ker T$. It follows that $\ker T$ and $\ker T^*$ are algebraically finitely generated closed (hence projective) submodules of X and Y, respectively.

Proof. Since the range of T is closed, Theorem 9.6 implies that $T^*(\mathsf{Y})$ is also closed and that we have

$$\mathsf{X} = T^*(\mathsf{Y}) \oplus \ker T \tag{9.8}$$

and

$$\mathsf{Y} = T(\mathsf{X}) \oplus \ker T^*. \tag{9.9}$$

Then $T|_{T^*(\mathsf{Y})}$ is an injective bounded operator from $T^*(\mathsf{Y})$ onto $T(\mathsf{X})$ and must have a bounded inverse S'. Let $S := S' \oplus 0$ relative to (9.8). Similarly, we can define R' to be the inverse of $T^*|_{T(\mathsf{X})}$ and get a bounded operator $R = R' \oplus 0$ relative to

(9.9). Routine computations[32] show that R is an adjoint for S and thus $S \in \mathcal{L}(\mathsf{Y}, \mathsf{X})$ as required. Similar calculations show that S satisfies (9.7). It is immediate that $1_{\mathsf{X}} - ST$ is an idempotent with range $\ker T$. The definition of S implies that ST is the identity on $T^*(\mathsf{Y})$. Since $\mathsf{X} = \ker T \oplus T^*(\mathsf{Y})$ by Theorem 9.6, $1_{\mathsf{X}} - ST$ is the orthogonal projection as claimed. Similarly, $1_{\mathsf{Y}} - TS$ is the projection onto $\ker T^*$ (because TS is the projection onto the range of T). Using Lemma 9.14, we can find a parametrix S'' such that $1_{\mathsf{X}} - S''T$ and $1_{\mathsf{Y}} - TS''$ are finite-rank operators. Since

$$1_{\mathsf{X}} - ST = (1_{\mathsf{X}} - S''T)(1_{\mathsf{X}} - ST),$$

it follows that $1_{\mathsf{X}} - ST$ is finite-rank: say, $1_{\mathsf{X}} - ST = \sum_{i=1}^n \theta_{x_i,y_i}$. Since $1_{\mathsf{X}} - ST$ is the identity on $\ker T$, it follows that $\ker T$ is finitely generated with generators $\{ x_i \}$. In particular, $\ker T$ is finitely generated projective by Corollary 9.10. A similar argument applies to $1_{\mathsf{Y}} - TS$ and $\ker T^*$.[33] □

Remark 9.16. An operator $S \in \mathcal{L}(\mathsf{Y}, \mathsf{X})$ satisfying (9.7) is called a *pseudo-inverse* for T. Since (9.7) implies that TS is an idempotent with Range equal to Range T, it follows that any operator with a pseudo-inverse has closed range.

Lemma 9.17. *Suppose that $T \in \mathcal{L}(\mathsf{H}_A)$ has closed range and that both* $\ker T$ *and* $\ker T^*$ *are finitely generated (hence projective). Then $T \in \mathcal{F}(\mathsf{H}_A)$.*

Proof. Theorem 9.6 implies that T has a polar decomposition $V|T|$. Notice that $1-V^*V = P$ where P is the projection onto $\ker T$. Then $P \in \mathcal{K}(\mathsf{H}_A)$ by Theorem 9.8 as is $1 - VV^*$. This means $V \in \mathcal{F}(\mathsf{H}_A)$, and it will suffice to see that $|T| + P$ is invertible in $\mathcal{L}(\mathsf{H}_A)$. But Theorem 9.6 implies that $\mathsf{H}_A = \ker |T| \oplus \operatorname{Range} |T|$; therefore, it is not hard to see that $|T| + P$ is bijective. Thus, $|T| + P$ is invertible by the Open Mapping Theorem. □

Definition 9.18. Let X and Y be countably generated Hilbert modules over a unital C^*-algebra A. Suppose that $T \in \mathcal{L}(\mathsf{X}, \mathsf{Y})$ is Fredholm with closed range. Then the index of T is the element $\operatorname{ind} T$ in $K_0(A)$ which is the image of $[\ker T] - [\ker T^*]$ under the natural isomorphism of $\mathcal{G}\big(V_1(A)\big)$ onto $K_0(A)$ given by Theorem 9.12.

Remark 9.19. In the sequel, if E and F are finitely generated projective modules, we'll view $[E] - [F]$ as an element of $K_0(A)$ and suppress any mention of the isomorphism from Theorem 9.12.

Next we want to extend ind to arbitrary Fredholm operators. There are two paths. The first is to work with compact perturbations of operators on H_A. This is Mingo's approach and is treated in detail in [WO93, Chap. 17]. The second follows Higson [Hig90] and Exel [Exe93]. We will sketch the first and look at the second in more detail.

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[32] Let $x \in \mathsf{X}$ and $y \in \mathsf{Y}$. Then (9.8) allows us to write

$$x = T^*(y_0) + x'$$

with $x' \in \ker T = T^*(\mathsf{Y})^\perp$ and $y_0 \in (\ker T^*)^\perp = T(\mathsf{X})$. Similarly,

$$y = T(x_0) + y'$$

with $y' \in \ker T^* = T(\mathsf{X})^\perp$ and $x_0 \in (\ker T)^\perp = T^*(\mathsf{Y})$. Then

$$\langle Sy \, , x\rangle_A = \langle x_0 \, , T^*y_0\rangle_A = \langle Tx_0 \, , y_0\rangle_A = \langle Tx_0 \, , Rx\rangle_A = \langle y \, , Rx\rangle_A .$$

[33] I have not appealed directly to Theorem 9.8 just to avoid having to assume X and Y are countably generated. I'm not sure why I bothered.

Lemma 9.20. *Suppose that u is a unitary in $\mathcal{Q}(\mathsf{H}_A) := \mathcal{L}(\mathsf{H}_A)/\mathcal{K}(\mathsf{H}_A)$. Then there is a partial isometry $V \in \mathcal{L}(\mathsf{H}_A)$ such that $\pi(V) = u$.*

Remark 9.21. If $A = \mathbb{C}$, then $\mathsf{H}_\mathbb{C} = \ell^2$ and $\mathcal{Q}(\mathsf{H}_\mathbb{C})$ is the *Calkin algebra*. In this case, the above result is a straightforward consequence of polar decomposition: if $\pi(Z) = u$ and $Z = V|Z|$, then

$$\begin{aligned} u = \pi(Z) &= \pi(V)\pi(|Z|) \\ &= \pi(V)\big(\pi(Z^*)\pi(Z)\big)^{\frac{1}{2}} \\ &= \pi(V). \end{aligned}$$

Remark 9.22. Lemma 9.20 is valid when H_A is replaced by any X for which $\mathcal{K}(\mathsf{X})$ has a approximate identity consisting of *projections* — note that $\{\, p_n \,\}$ is an approximate unit of projections for $\mathcal{K}(\mathsf{H}_A)$. This will be the only property of H_A used in the proof. (I believe the word "projection" was omitted from Remark 17.1.3 of [WO93].)

Proof of Lemma 9.20. Choose Z in $\mathcal{L}(\mathsf{H}_A)$ with $\pi(Z) = u$. Since $1 - Z^*Z$ is in $\mathcal{K}(\mathsf{H}_A)$, we can find a projection $P \in \mathcal{K}(\mathsf{H}_A)$ such that

$$\|(1-P)(1-Z^*Z)(1-P)\| = \|(1-P) - (1-P)Z^*Z(1-P)\| < 1.$$

This implies that $(1-P)Z^*Z(1-P)$ is invertible in $(1-P)\mathcal{L}(\mathsf{H}_A)(1-P)$ with inverse

$$(1-P) + \sum_{k=1}^{\infty}\big((1-P)Z^*Z(1-P)\big)^k.$$

Since P is invertible in $P\mathcal{L}(\mathsf{H}_A)P$ with inverse P, we see that

$$T := P + (1-P)Z^*Z(1-P)$$

is invertible in $\mathcal{L}(\mathsf{H}_A)$ with

$$T^{-1} = I + \sum_{k=1}^{\infty}\big((1-P)Z^*Z(1-P)\big)^k \geq 0.$$

Since T^{-1} is positive, $T^{-\frac{1}{2}}$ is defined and is the norm limit of polynomials in T^{-1} without constant term. Thus, $PT^{-\frac{1}{2}} = P = T^{-\frac{1}{2}}P$.

Now let $V := Z(1-P)T^{-\frac{1}{2}}$. Since P is compact and $\pi(T) = \pi(Z^*Z) = 1$,

$$\begin{aligned} \pi(V) &= \pi(ZT^{-\frac{1}{2}}) - \pi(ZPT^{-\frac{1}{2}}) \\ &= \pi(Z) - 0 \\ &= u. \end{aligned}$$

All that remains to be shown is that V is a partial isometry. But

$$\begin{aligned} V^*V &= T^{-\frac{1}{2}}(1-P)Z^*Z(1-P)T^{-\frac{1}{2}} \\ &= T^{-\frac{1}{2}}(T-P)T^{-\frac{1}{2}} \\ &= I - P. \quad \square \end{aligned}$$

Lemma 9.23. *If $F \in \mathcal{F}(\mathsf{H}_A)$, then there is a $G \in \mathcal{F}(\mathsf{H}_A)$ with closed range such that $F - G \in \mathcal{K}(\mathsf{H}_A)$.*

Proof. Since $F \in \mathcal{F}(\mathsf{H}_A)$, $\pi(F)\pi(|F|)^{-1}$ us a unitary in $\mathcal{Q}(\mathsf{H}_A)$, Lemma 9.20 implies that there is a partial isometry W in $\mathcal{L}(\mathsf{H}_A)$ such that $\pi(W) = \pi(F)\pi(|F|)^{-1}$. Since $|F|$ is positive, $\log \pi(|F|)$ is defined and we can choose $H \in \mathcal{L}(\mathsf{H}_A)$ such that $\pi(H) = \log \pi(|F|)$. Now we let $G := W \exp(H)$. Since $\exp(H)$ is invertible,

$$\operatorname{Range} G = \operatorname{Range} W;$$

in particular, the range of G is closed. Since

$$\begin{aligned}\pi(G) &= \pi(W)\pi\bigl(\exp(H)\bigr)\\ &= \pi(F)\pi(|F|)^{-1}\exp\bigl(\pi(H)\bigr)\\ &= \pi(F)\pi(|F|)^{-1}\exp\bigl(\log\pi(|F|)\bigr)\\ &= \pi(F),\end{aligned}$$

$F - G$ is compact and $G \in \mathcal{F}(\mathsf{H}_A)$. □

Remark 9.24. Lemmas 9.15 and 9.17 implies that we always have

$$\mathcal{F}''(\mathsf{X}) \subset \mathcal{F}'(\mathsf{X}) \subset \mathcal{F}(\mathsf{X}).,\tag{9.6$'$}$$

at least when $\mathsf{X} = \mathsf{H}_A$, and it is not hard to see how to modify the proof of Lemma 9.17 for general X. The equality in (9.6) follows from Lemma 9.23; unfortunately, I do not see how to extend the argument to general X.

Remark 9.25 (Mingo's Approach). From here on we'll switch to Higson's approach. Mingo's treatment [Min87] is treated in detail in [WO93, Chap. 17], but I'll give a brief sketch here. Using Lemma 9.23, it is natural to attempt to extend the definition of ind to all Fredholm operators $T \in \mathcal{F}(\mathsf{H}_A)$ by setting $\operatorname{ind} T$ equal to $\operatorname{ind} G$, where G is any compact perturbation of T which has closed range. To do this, we would need to see that $\operatorname{ind} S = \operatorname{ind} G$ whenever S and G are Fredholm with closed range satisfying $S - G$ compact. Fortunately this is true, but the proof due to Kasparov [Kas80] is rather complex. A reasonable exposition of Kasparov's argument is given in [WO93] (see [WO93, Corollary 17.2.5]).[34] With Kasparov's result in hand, we get a well-defined map ind : $\mathcal{F}(\mathsf{H}_A) \to K_0(A)$. More hard work is required to show that $\operatorname{ind} T = \operatorname{ind} S$ if and only if T and S are connected by a continuous path in $\mathcal{F}(\mathsf{H}_A)$. The "only if" direction requires we know that the unitaries in $\mathcal{L}(\mathsf{H}_A)$ are connected. Since $\mathcal{L}(\mathsf{H}_A) \cong M\bigl(\mathcal{K}(\mathsf{H}_A)\bigr) \cong M\bigl(\mathcal{K}(A \otimes \ell^2)\bigr) \cong M(A \otimes \mathcal{K})$, this follows from Mingo's ([Min87, Theorem 2.5] or [WO93, Theorem 16.8]) (provided $1 \in A$).[35] Then one shows that that the set of path components $[\mathcal{F}(\mathsf{H}_A)]$ in $\mathcal{F}(\mathsf{H}_A)$ is an abelian group with $[S][T] := [ST]$, $[T]^{-1}$ the class of any parametrix for T, and identity the class of any invertible element. Thus ind : $[\mathcal{F}(\mathsf{H}_A)] \to K_0(A)$ is an isomorphism ([Min87, Proposition 1.13] or [WO93, Theorem 17.3.11]). More generally, the set of homotopy classes $[X, \mathcal{F}(\mathsf{H}_A)]$ from a compact Hausdorff space X into $\mathcal{F}(\mathsf{H}_A)$ is a group isomorphic to $K_0\bigl(C(X) \otimes A\bigr)$ [Min87, Theorem 1.13].

Its now time to give Higson's definition of homotopy between Fredholm operators on (possibly) different Hilbert modules. For this, we need the notion of the internal tensor product of a Hilbert B-module X and a Hilbert A-module Y for which there

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[34] On the other hand, we give an alternate proof in Corollary 9.45.

[35] I was surprised to discover that a corollary of the unitaries in $M(A \otimes \mathcal{K})$ being connected if A is unital or σ-unital, is that the unitary group of $M(A \otimes \mathcal{K})$ is contractible in the *norm topology*. This is a generalization of Kupier result that the unitary group of $B(\mathcal{H})$ is contractible [Kui65].

is a homomorphism $\varphi : B \to \mathcal{L}(\mathsf{Y})$. Unfortunately, this is slightly more general than that treated in [RW98, §3.2]. Instead, a good reference is [Lan94, Proposition 4.5], and I've given a short treatment in Appendix C. There it is shown that

$$\langle\!\langle x_1 \otimes y_1 \,,\, x_2 \otimes y_2 \rangle\!\rangle_A := \bigl\langle y_1 \,,\, \varphi\bigl(\langle x_1 \,,\, x_2\rangle_B\bigr) y_2 \bigr\rangle_A \tag{9.10}$$

is a well defined pre-inner product on the algebraic tensor product $\mathsf{X} \odot \mathsf{Y}$ as in [RW98, Lemma 2.16]. Thus [RW98, Lemma 2.16] implies that the completion $\mathsf{X} \otimes_\varphi \mathsf{Y}$ is a Hilbert A-module. For details, see Proposition C.1 or Appendix C. One of the more important features of this tensor product is that it is B-balanced in that $x \cdot b \otimes y$ and $x \otimes \varphi(b)y$ have the same image in $\mathsf{X} \otimes_\varphi \mathsf{Y}$. If $T \in \mathcal{L}(X, \mathsf{Z})$ for some Hilbert A-module Z, then there is an operator $T \otimes_\varphi 1 \in \mathcal{L}(\mathsf{X} \otimes_\varphi \mathsf{Y}, \mathsf{Z} \otimes_\varphi \mathsf{Y})$ characterized by $T \otimes_\varphi 1(x \otimes_\varphi y) = Tx \otimes_\varphi y$ (Proposition C.2). We primarily need this construct when $\mathsf{Y} = A_A$, $B = C\bigl([0,1], A\bigr)$ and φ is evaluation at t. In these cases, we'll write $\mathsf{X} \otimes_{\epsilon_t} A$ for the resulting Hilbert A-module.

Remark 9.26. As is customary in the subject we will invoke the isomorphism of the C^*-algebras $C_0(X) \otimes A$ and $C_0(X, A)$ without comment [RW98, Proposition B.16]. There is a similar construction with Hilbert modules. Suppose that Y is a Hilbert A-module and X a locally compact Hausdorff space. Then $\mathsf{Z} := C_0(X, \mathsf{Y})$ carries an obvious $C_0(X, A)$-valued inner product and the induced norm on Z is

$$\|f\|_{\mathsf{Z}} = \sup_{x \in X} \|f(x)\|_{\mathsf{Y}}.$$

In particular, Z is a Hilbert $C_0(X, A)$-module. Alternatively, we can form the external tensor product $C_0(X) \otimes \mathsf{Y}$ [RW98, Proposition 3.36], which is also a Hilbert $C_0(X, A)$ module (after identifying $C_0(X, A)$ with $C_0(X) \otimes A$ as above). The usual map of $C_0(X) \odot \mathsf{Y}$ into $C_0(X, \mathsf{Y})$ extends to a Hilbert module isomorphism of $C_0(X) \otimes \mathsf{Y}$ with $C_0(X, \mathsf{Y})$. Notice that

$$\mathcal{K}\bigl(C_0(X) \otimes \mathsf{Y}\bigr) \cong C_0(X) \otimes \mathcal{K}(\mathsf{Y}) \cong C_0(X, \mathcal{K}(\mathsf{Y}))$$

by [RW98, Corollary 3.38].

Example 9.27. Let Y be a Hilbert A-module and $C_0(X, \mathsf{Y})$ the corresponding Hilbert $C_0(X, A)$-module. Then $f \otimes a \mapsto f(t) \cdot a$ extends to a unitary isomorphism

$$U^{(t)} : C_0(X, \mathsf{Y}) \otimes_{\epsilon_t} A \to \mathsf{Y}.$$

Proof. Just compute

$$\begin{aligned} \langle\!\langle f \otimes a \,,\, g \otimes a \rangle\!\rangle_A &= a^* \langle f \,,\, g\rangle_{C(_0(X,A)} (t) b \\ &= a^* \bigl\langle f(t) \,,\, g(t) \bigr\rangle_A b = \bigl\langle f(t) \cdot a \,,\, g(t) \cdot b \bigr\rangle_A. \quad \square \end{aligned}$$

Definition 9.28 ([Hig90, Definition 3.25])**.** Two Fredholm operators $F_0 : \mathsf{E}_0^{(0)} \to \mathsf{E}_1^{(0)}$ and $F_1 : \mathsf{E}_0^{(1)} \to \mathsf{E}_1^{(1)}$ are unitarily equivalent if there are unitaries $U_i \in \mathcal{L}(\mathsf{E}_i^{(0)}, \mathsf{E}_i^{(1)})$ such that $F_0 = U_1^* F_1 U_0$. That is, the diagram

$$\begin{tikzcd} \mathsf{E}_0^{(0)} \arrow[r, "F_0"] \arrow[d, "U_0"'] & \mathsf{E}_1^{(0)} \arrow[d, "U_1"] \\ \mathsf{E}_0^{(1)} \arrow[r, "F_1"] & \mathsf{E}_1^{(1)} \end{tikzcd}$$

commutes.

Definition 9.29. We say that F_0 and F_1 are homotopic if there are Hilbert $C(I,A)$-modules $\widetilde{\mathsf{E}}_0$ and $\widetilde{\mathsf{E}}_1$ and a Fredholm operator $\widetilde{F} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$ such that $\widetilde{F} \otimes_{\epsilon_i} 1 : \widetilde{\mathsf{E}}_0 \otimes_{\epsilon_i} A \to \widetilde{\mathsf{E}}_1 \otimes_{\epsilon_i} A$ and $F_i : \mathsf{E}_0^{(i)} \to \mathsf{E}_1^{(i)}$ are unitarily equivalent for $i = 0, 1$. In this case we write $F_0 \rightsquigarrow_h F_1$.

Thus if $F_0 \rightsquigarrow_h F_1$ via $\widetilde{F}$ as above, then there are Hilbert A-module isomorphisms (a.k.a. unitaries)

$$U_k^{(i)} : \widetilde{\mathsf{E}}_k \otimes_{\epsilon_i} A \to \mathsf{E}_k^{(i)}$$

such that

$$\begin{tikzcd} \widetilde{\mathsf{E}}_0 \otimes_{\epsilon_i} A \arrow[r, "U_0^{(i)}"] \arrow[d, "\widetilde{F}\otimes_{\epsilon_i} 1"'] & \mathsf{E}_0^{(i)} \arrow[d, "F_i"] \\ \widetilde{\mathsf{E}}_1 \otimes_{\epsilon_i} A \arrow[r, "U_1^{(i)}"] & \mathsf{E}_1^{(i)} \end{tikzcd}$$

commutes for $i = 0, 1$.

Definition 9.29 is a bit hard to swallow all at once — at least it was for me. It is not even obvious that it is reflexive or symmetric, and it (apparently) fails to be reflexive, and so it is not an equivalence relation. (Not that this is clear from the literature – quite the opposite is true.) However, we shall see that it generates a rather useful equivalence relation. The following example is the usual way in which one might employ homotopy — a so called *operator homotopy* — and it serves as excellent motivation. In the example, the modules $\widetilde{\mathsf{E}}_i$ are simply $C\big([0,1], \mathsf{E}_i\big)$ as one would expect. The added flexibility of Higson's definition which employs arbitrary $C\big([0,1], A\big)$-modules will be apparent in due course and is crucial to his treatment.

To ease the notational burden in the sequel, it will be convenient to use I in place of $[0,1]$ when dealing with homotopies.

Example 9.30 (Operator homotopies). Suppose that for each $t \in [0,1]$, F_t is a Fredholm operator from E_0 to E_1, and that $t \mapsto F_t$ is norm continuous. Then, as you would hope, F_0 and F_1 are homotopic. To see this, let $\widetilde{\mathsf{E}}_i := C\big(I, \mathsf{E}_i\big)$ with the obvious Hilbert $C(I,A)$-module structure. Then define $\widetilde{F}$ by

$$(\widetilde{F}x)(t) := F_t\big(x(t)\big).$$

Fix $t_0 \in [0,1]$, and let S be a parametrix for F_{t_0}. Let $\pi_i : \mathcal{L}(\mathsf{E}_i) \to \mathcal{Q}(\mathsf{E}_i)$ be the natural map, and notice that $\pi_0(SF_{t_0}) = 1$. Thus, $\pi_0(SF_t)$ is invertible near t_0. A compactness argument allows us to choose operators $S^k \in \mathcal{L}(E_1, E_0)$ and a partition of unity $f_k \in C\big([0,1]\big)$ such that $S_t' := \sum_{k=1}^n f_k(t) S^k$ satisfies $\pi_0(S_t' F_t)$ invertible for all $t \in [0,1]$. Since $C\big(I, \mathcal{Q}(\mathsf{E}_0)\big) \cong C\big(I, \mathcal{L}(\mathsf{E}_0)\big)/C\big(I, \mathcal{K}(\mathsf{E}_0)\big)$, we can find a lift $t \mapsto W_t$ of $t \mapsto \pi_0\big(S_t' F_t\big)^{-1}$ and let $S_t := W_t S_t'$. Then $S_t F_t - 1_{\mathsf{E}_0}$ is compact for all t. By a similar argument, we can find operators $R_t \in \mathcal{L}(E_1, \mathsf{E}_0)$ such that $t \mapsto R_t$ is continuous and $F_t R_t - 1_{\mathsf{E}_1}$ is compact for all t. Then modulo compacts

$$R_t \equiv (S_t F_t) R_t \equiv S_t (F_t R_t) \equiv S_t.$$

In particular, $F_t S_t - 1_{\mathsf{E}_1}$ is compact for all t, and $t \mapsto S_t$ is a parametrix for $\widetilde{F}$ and $\widetilde{F}$ is Fredholm.

Furthermore, as in Example 9.27, $f \otimes a \mapsto f(i) \cdot a$ defines a unitary operator

$$U_k^{(i)} : C(I, \mathsf{E}_k) \otimes_{\epsilon_i} A \to \mathsf{E}_k \tag{9.11}$$

Then it is a simple matter to check that

$$F_0 = U_1^{(0)}(\widetilde{F} \otimes_{\epsilon_0} 1)U_0^{(0)^*} \quad \text{and} \tag{9.12}$$

$$F_1 = U_1^{(1)}(\widetilde{F} \otimes_{\epsilon_1} 1)U_0^{(1)^*}. \tag{9.13}$$

For example, to check (9.12), first note that $U_0^{(0)^*}$ maps $x \in \mathsf{E}_0$ to the class of $\tilde{x} \otimes 1$ in $C(I, \mathsf{E}_0) \otimes_{\epsilon_0} A$, where $\tilde{x}$ is the constant function $\tilde{x}(t) = x$. Thus, $U_1^{(0)}(\widetilde{F} \otimes_{\epsilon_0} 1)U_0^{(0)^*}(x) = (\widetilde{F}\tilde{x})(0) = F_0(x)$.

Remark 9.31. Notice that if $\widetilde{F}$ is a Fredholm operator from $C(I, \mathsf{E}_0)$ to $C(I, \mathsf{E}_1)$, then we can define unitaries $U_k^{(t)} : C(I, \mathsf{E}_k) \otimes_{\epsilon_t} A \to \mathsf{E}_k$ as in (9.11). Then one can show that $F_t := U_1^{(t)}(\widetilde{F} \otimes_{\epsilon_t} 1)U_0^{(t)^*}$ is Fredholm from E_0 to E_1 and that $t \mapsto F_t$ is an operator homotopy.

In general, if $\widetilde{F}$ is a Fredholm from $\widetilde{\mathsf{E}}_0$ to $\widetilde{\mathsf{E}}_1$, then $t \mapsto \widetilde{F} \otimes_{\epsilon_t} 1$ is meant to be a generalized operator homotopy. The dramatic flexibility of Definition 9.29 is illustrated by the next result.

Proposition 9.32 ([Hig90, Proposition 3.27])**.** *Let A be a C^*-algebra with identity and let $F : \mathsf{E}_0 \to \mathsf{E}_1$ be a Fredholm operator between Hilbert A-modules. If F has closed range, then F is homotopic to the zero operator $0 : \ker F \to \ker F^*$.*

For the proof we need a lemma.

Lemma 9.33. *Suppose that E is a finitely generated Hilbert A-module. Then for any compact space X, $C(X, \mathsf{E})$ is a finitely generated Hilbert $C(X, A)$-module.*

Proof. The only real issue is to show that $C(X, \mathsf{E})$ is finitely generated. We can assume that E is a complemented submodule of A^n with basis $\{ \mathbf{e}_i \}$ and that p is the projection of A^n onto E. If $f \in C(X, \mathsf{E}) \subset C(X, A^n)$, then $f = \sum_i \mathbf{e}_i \cdot f_i$ where $f_i \in C(X, A)$. But then for each $t \in X$, $f(t) = p\big(f(t)\big) = \sum_i p(\mathbf{e}_i) \cdot f_i(t)$. Therefore $C(X, \mathsf{E})$ is generated by the constant functions F_i where $F_i(t) = p(\mathbf{e}_i)$. □

Proof of Proposition 9.32. Lemma 9.15 implies that $\ker F$ and $\ker F^*$ are finitely generated projective. Then Corollary 9.11 implies that every operator in $\mathcal{L}(\ker F)$ and $\mathcal{L}(\ker F^*)$ is compact. Therefore, $0 : \ker F \to \ker F^*$ is Fredholm. Furthermore, F has a polar decomposition $F = V|F|$ by Theorem 9.6. Since $t|F| + (1-t)I$ is invertible if $t \neq 1$, if follows that $F_t := V\big(t|F| + (1-t)I\big)$ is Fredholm[36] for all $t \in [0, 1]$. Thus F is operator homotopic to V, and will suffice to prove the result with V in place of F.

Note that $1 - V^*V$ and $1 - VV^*$ are the projections onto $\ker V$ and $\ker V^*$, respectively. Since

$$\widetilde{\mathsf{E}}_0 := \{ f \in C(I, \mathsf{E}_0) : f(1) \in \ker V \}$$

and

$$\widetilde{\mathsf{E}}_1 := \{ f \in C(I, \mathsf{E}_1) : f(1) \in \ker V^* \}$$

[36]This is not completely trivial. It is easy to see that $|F|$ must be Fredholm. To see that V is Fredholm, note that if S is a parametrix for F and W a parametrix for $|F|$, then $U := |F|S$ is one for V. For example, $UV \equiv |F|SV|F|W \equiv 1$.

are closed submodules, they are Hilbert $C(I, A)$-modules, and we can define $\widetilde{V} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$ by

$$(\widetilde{V} f)(t) := V\big(f(t)\big).$$

The adjoint of $\widetilde{V}$ is give by the corresponding formula for V^*. Then it is easy to see that $1 - \widetilde{V}^*\widetilde{V}$ is the projection onto the submodule

$$\widetilde{\mathsf{Y}}_0 := \{\, f \in \widetilde{\mathsf{E}}_0 : f(t) \in \ker V \text{ for all } t \in [0,1] \,\} \cong C(I, \ker V),$$

and $1 - \widetilde{V}\widetilde{V}^*$ is the projection onto

$$\widetilde{\mathsf{Y}}_1 := \{\, f \in \widetilde{\mathsf{E}}_0 : f(t) \in \ker V^* \text{ for all } t \in [0,1] \,\} \cong C(I, \ker V^*),$$

Since $\ker V$ is finitely generated, so is Y_0. Thus Y_0 is projective and $1 \in \mathcal{K}(\mathsf{Y}_0)$ by Corollary 9.11. From this it is not hard to see that $1 - \widetilde{V}^*\widetilde{V}$ is compact. Since a similar argument shows that $1 - \widetilde{V}\widetilde{V}^*$ is compact, $\widetilde{V}$ is Fredholm. Then we can define unitary operators

$$U_k^{(0)} : \widetilde{\mathsf{E}}_k \otimes_{\epsilon_0} A \to \mathsf{E}_k \quad (k = 0, 1)$$

by sending the class of $f \otimes a$ to $f(0) \cdot a$, as well as

$$U_0^{(1)} : \widetilde{\mathsf{E}}_0 \otimes_{\epsilon_1} A \to \ker V,$$

and

$$U_1^{(1)} : \widetilde{\mathsf{E}}_1 \otimes_{\epsilon_1} A \to \ker V^*$$

by sending $f \otimes a$ to $f(1) \cdot a$. Then

$$V = U_1^{(0)}(\widetilde{V} \otimes_{\epsilon_0} 1){U_0^{(0)}}^* \quad \text{and} \quad 0 = U_1^{(1)}(\widetilde{V} \otimes_{\epsilon_1} 1){U_0^{(1)}}^*.$$

Thus $\widetilde{V}$ is the required homotopy between V and the zero operator. □

An even more extreme example is the following.

Proposition 9.34 ([Hig90, Proposition 3.28]). *Let E_0 and E_1 be Hilbert modules over a not necessarily unital C^*-algebra A. If F is an invertible operator in $\mathcal{L}(\mathsf{E}_0, \mathsf{E}_1)$, then F is homotopic to the zero operator $0 : 0 \to 0$ on the trivial Hilbert A-module.*

Proof. Define $\widetilde{\mathsf{E}}_i$ as in the proof of Proposition 9.32 and $\widetilde{F}$ analogously to $\widetilde{V}$. Then one sees easily that $\widetilde{F}$ is invertible and hence Fredholm. Then $\widetilde{F}$ gives the desired homotopy just as above. □

The following lemma is helpful for understanding homotopy, and it will be useful down the road.

Lemma 9.35. *Suppose that $F : \mathsf{E}_0 \to \mathsf{E}_1$ is Fredholm. If $K \in \mathcal{K}(\mathsf{E}_0, \mathsf{E}_1)$ is compact, then $F + K$ is homotopic to F. If X is any Hilbert A-module, then $F \oplus 1 : \mathsf{E}_0 \oplus \mathsf{X} \to \mathsf{E}_1 \oplus \mathsf{X}$ is Fredholm and homotopic to F. In particular, F is homotopic to a Fredholm operator $F' : \mathsf{H}_A \to \mathsf{H}_A$.*

Proof. The first statement is easy: $F_t := F + tK$ is Fredholm and $t \mapsto F_t$ is an operator homotopy from F to $F + K$. For the second, let $S : \mathsf{E}_1 \to \mathsf{E}_0$ be a parametrix for F. Define

$$\widetilde{\mathsf{E}}_k := \{\, (f_1, f_2) \in C(I, \mathsf{E}_k \oplus \mathsf{X}) : f_2(0) = 0 \,\},$$

as well as $\widetilde{F} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$ and $\widetilde{S} : \widetilde{\mathsf{E}}_1 \to \widetilde{\mathsf{E}}_0$ by

$$\widetilde{F}\big((f_1, f_2)\big)(t) := \big(F\big(f_1(t)\big), f_2(t)\big) \quad \text{and} \quad \widetilde{S}\big((g_1, g_2)\big)(t) := \big(S\big(g_1(t)\big), g_2(t)\big).$$

Then $\widetilde{S}$ is a parametrix for $\widetilde{F}$ (see Remark 9.26) and $\widetilde{F}$ is Fredholm. The usual maps induce unitaries

$$U_k^{(0)} : \widetilde{\mathsf{E}}_k \otimes_{\epsilon_0} A \to \mathsf{E}_k, \quad \text{and} \quad U_k^{(1)} : \widetilde{\mathsf{E}}_k \otimes_{\epsilon_1} A \to \mathsf{E}_k \oplus \mathsf{X}$$

such that

$$F = U_1^{(0)}(\widetilde{F} \otimes_{\epsilon_0} 1)\big(U_0^{(0)}\big)^*, \quad \text{and} \quad F \oplus 1 = U_1^{(1)}(\widetilde{F} \otimes_{\epsilon_1} 1){U_0^{(1)}}^*.$$

The second assertion follows from this.

To prove the last assertion, let $\mathsf{X} = \mathsf{H}_A$ in the second assertion. Thus the Kasparov stabilization theorem implies $F \oplus 1$ is unitarily equivalent to an operator on H_A. $\square$

Lemma 9.36. *Suppose that $F_0 : \mathsf{E}_0^{(0)} \to \mathsf{E}_1^{(0)}$ and $F_1 : \mathsf{E}_0^{(1)} \to \mathsf{E}_1^{(1)}$ are Fredholm operators. Then $F_0 \rightsquigarrow_h F_0$. If $F_0 \rightsquigarrow_h F_1$, then we also have $F_1 \rightsquigarrow_h F_0$.*

Proof. Since $t \mapsto F$ is an operator homotopy from F to itself, the first assertion is easy. Suppose that $\widetilde{F} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$ is a homotopy from F_0 to F_1 such that

$$F_k = U_1^{(k)}(\widetilde{F} \otimes_{\epsilon_k} 1){U_0^{(k)}}^*.$$

Let θ be the automorphism of $C(I, A)$ defined by $\theta(f)(t) = f(1 - t)$. Let

$$\widetilde{S} := \widetilde{F} \otimes_\theta 1 : \widetilde{\mathsf{E}}_0 \otimes_\theta C(I, A) \to \widetilde{\mathsf{E}}_1 \otimes_\theta C(I, A),$$

and $\widetilde{U}_i^{(k)} := U_i^{(k)} \otimes 1$. Then $\widetilde{S}$ gives a homotopy[37] from $F_1 \to F_0$. To see this, not that

$$\big(\widetilde{\mathsf{E}}_k \otimes_\theta C(I, A)\big) \otimes_{\epsilon_t} A \cong \widetilde{\mathsf{E}}_k \otimes_{\epsilon_{1-t}} A$$

by the map sending $(x \otimes_\theta f) \otimes_{\epsilon_t} a \mapsto x \otimes_{\epsilon_{1-t}} f(t) \cdot a$. Just compute:

$$\begin{aligned} \langle\!\langle (x \otimes f) \otimes a \,,\, (y \otimes g) \otimes b \rangle\!\rangle_A &= a^* \langle x \otimes_\theta f \,,\, y \otimes_\theta g \rangle_{C(I,A)}(t) b \\ &= \langle\!\langle x \otimes f(t) \cdot a \,,\, y \otimes g(t) \cdot b \rangle\!\rangle_A. \quad \square \end{aligned}$$

Definition 9.37 ([Hig90, Definition 3.29])**.** Let A be a C^*-algebra with identity. Let $\sim_h$ be the equivalence relation generated by $\rightsquigarrow_h$. Then let $K'(A)$ be the set $\mathcal{F}/{\sim_h}$-homotopy classes of (generalized) Fredholm operators $\mathcal{F}$ on countably generated Hilbert A-modules.[38]

Remark 9.38. In view of Lemma 9.36, $F_0 \sim_h F_1$ if and only if there are operators $F_i \in \mathcal{F}$ such that

$$F_0 \rightsquigarrow_h F_2 \rightsquigarrow_h F_3 \rightsquigarrow_h \cdots \rightsquigarrow_h F_n \rightsquigarrow_h F_1. \tag{9.14}$$

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[37]Needs checking.

[38]Since any $F \in \mathcal{F}$ is homotopic to an operator $F' \in \mathcal{F}(\mathsf{H}_A)$, $K'(A)$ is certainly a set.

Even Higson admits that in view of Proposition 9.32, it is reasonable to suspect that $K'(A)$ is small — even $\{0\}$. However, our goal is to prove that $K'(A)$ is naturally isomorphic to $K_0(A)$. The first step in this direction is to show that $K'(A)$ is an abelian group. It is tedious but straightforward to check that $F_0 \rightsquigarrow_h F_1$ and $S_0 \rightsquigarrow_h S_1$ implies that $F_0 \oplus S_0 \rightsquigarrow_h F_1 \oplus S_1$. Now if $F_0 \sim_h F_1$ and $S_0 \sim_h S_1$, then we conclude that there are operators F_i and S_j such that (9.14) holds as well as a similar one for the S_j. Since $\rightsquigarrow_h$ is reflexive, we can assume the two chains have the same length. Thus it follows that $F_0 \oplus S_0 \sim_h F_1 \oplus S_1$, and we can equip $K'(A)$ with a binary operation

$$[F_0] + [F_1] := [F_0 + F_1] \tag{9.15}$$

which makes $K'(A)$ a semigroup with identity equal to the class of the zero operator $0 : 0 \to 0$ on the trivial Hilbert A-module. (This class includes all invertible operators.)

Proposition 9.39. *With the binary operation* (9.15)*, $K'(A)$ is an abelian group. The inverse of $[F]$ is given by the class of any parametrix S for F.*

Proof. Since $F_0 \oplus F_1$ is unitarily equivalent to $F_1 \oplus F_0$, $K'(A)$ is abelian. If $F : \mathsf{E}_0 \to \mathsf{E}_1$ is Fredholm with parametrix $S : \mathsf{E}_1 \to \mathsf{E}_0$. It suffices to see that $F \oplus S$ is homotopic to an invertible operator. For this, it suffices to see that $F \oplus S$ differs from an invertible operator by a compact operator. But, viewed as an operator on $\mathsf{E}_0 \oplus \mathsf{E}_1$,

$$F \oplus S = \begin{pmatrix} 0 & S \\ F & 0 \end{pmatrix};$$

however, modulo compacts

$$\begin{aligned}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}\begin{pmatrix} -1 & S \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1-SF & (1-SF)F + S \\ F & 1-SF \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & S \\ F & 0 \end{pmatrix},\end{aligned}$$

and this suffices as the left-hand side is invertible:

$$\left(\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix}\begin{pmatrix} -1 & S \\ 0 & 1 \end{pmatrix}\right)^{-1} = \begin{pmatrix} -1 & S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix}\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}. \qquad \square$$

The following is helpful for understanding $K'(A)$, although we will not make use of it here.

Proposition 9.40. *If $F \in \mathcal{F}(\mathsf{E}_0, \mathsf{E}_1)$ and $S \in \mathcal{F}(\mathsf{E}_1, \mathsf{E}_2)$, then*

$$[F] + [S] = [S \circ F].$$

Proof. We can choose F' and S' in $\mathcal{F}(\mathsf{H}_A)$ so that $[F] = [F']$ and $[S] = [S']$. Furthermore, we have $F' = U_1(F \oplus 1)U_0^*$ where U_i is an isomorphism of $\mathsf{E}_i \oplus \mathsf{H}_A$ onto H_A. Then $S' \circ F' = U_2(S \circ F \oplus 1)U_0^*$ and $[S \circ F] = [S' \circ F']$. Thus, it suffices to prove the result when $\mathsf{E}_0 = \mathsf{E}_1 = \mathsf{E}_2$.

But now let

$$U_t := \begin{pmatrix} \cos\frac{\pi}{2}t & -\sin\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t & \cos\frac{\pi}{2}t \end{pmatrix}$$

and,

$$W_t := \begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix} U_t \begin{pmatrix} F' & 0 \\ 0 & 1 \end{pmatrix} U_t^*.$$

Then check that

$$W_1 = \begin{pmatrix} S' & 0 \\ 0 & F' \end{pmatrix} \quad \text{and} \quad W_0 = \begin{pmatrix} S' \circ F' & 0 \\ 0 & 1 \end{pmatrix}. \quad \square$$

The next order of business is to see that K' is a functor. To do this, we have produce a homomorphism $K'(f) : K'(A) \to K'(B)$ for any homomorphism $f : A \to B$. But if $F \in \mathcal{F}(\mathsf{E}_0, \mathsf{E}_1)$, then we can construct $F \otimes_f 1 : \mathsf{E}_0 \otimes_f B \to \mathsf{E}_1 \otimes_f B$ as in Proposition C.2. Since $\mathcal{K}(B_B) = B$, it also follows from Proposition C.2 that $K \otimes_f 1 \in \mathcal{K}(\mathsf{E}_i \otimes_f B)$ if $K \in \mathcal{K}(\mathsf{E}_i)$ ($i = 0$ or 1). Thus $F \otimes_f 1$ is Fredholm and if S is a parametrix for F, then $S \otimes_f 1$ is a parametrix for $F \otimes_f 1$.

Now suppose that $\widetilde{F} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$ is a homotopy from $F_0 : \mathsf{E}_0^{(0)} \to \mathsf{E}_1^{(0)}$ to $F_1 : \mathsf{E}_0^{(1)} \to \mathsf{E}_1^{(1)}$. Thus there are unitaries

$$U_i^{(k)} : \mathsf{E}_i^{(k)} \otimes_{\epsilon_k} A \to \mathsf{E}_i^{(k)}$$

such that

$$F_0 = U_1^{(0)} (\widetilde{F} \otimes_{\epsilon_k} 1) {U_0^{(0)}}^*, \quad \text{and} \quad F_1 = U_1^{(1)} (\widetilde{F} \otimes_{\epsilon_k} 1) {U_0^{(1)}}^*.$$

Then one can show that $\widetilde{F} \otimes_f 1 : \widetilde{\mathsf{E}}_0 \otimes_f B \to \widetilde{\mathsf{E}}_1 \otimes_f B$ is a homotopy from $F_0 \otimes_f 1$ to $F_1 \otimes_f 1$. This allows us to define $K'(f)[F] := [F \otimes_f 1]$. Since $F \otimes_f 1 \oplus S \otimes_f 1$ is unitarily equivalent to $(F \oplus S) \otimes_f 1$, $K'(f)$ is a homomorphism. We still need to see that K' preserves compositions. That is, if $f : A \to B$ and $G : D \to A$ are homomorphisms, then we need to show that

$$K'(f \circ g) = K'(f)K'(g).$$

This means that

$$[F \otimes_{f \circ g} 1] = [(F \otimes_g 1) \otimes_f 1],$$

and it will suffice to see that $F \otimes_{f \circ g} 1$ and $(F \otimes_g 1) \otimes_f 1$ are unitarily equivalent. But $x \otimes a \otimes b \mapsto z \otimes f(a)b$ extends to a unitary map[39]

$$U_k : \mathsf{E}_k \otimes_g A \otimes_f B \to \mathsf{E}_k \otimes_{f \circ g} B,$$

and

$$(F \otimes_{f \circ g} 1)U_0 = U_1(F \otimes_g 1 \otimes_f 1).$$

Thus K' is a covariant functor.

Since we are assuming A is unital, each element in $K_0(A)$ is represented by a formal difference $[\mathsf{E}] - [\mathsf{F}]$ of finitely generated projective Hilbert A-modules. Thus $0 : \mathsf{E} \to \mathsf{F}$ is a Fredholm operator, and we get a map

$$J : K_0(A) \to K'(A)$$

[39] It is not so obvious that U_K is onto. But each element $x \in \mathsf{E}_k$ is of the form $z \cdot d$ for some $d \in D$. Thus each $x \otimes_{f \circ g} b$ equals $z \otimes_{f \circ g} f(g(d))b$, which is in the range of U_k.

defined by $J\big([\mathsf{E}]-[\mathsf{F}]\big)=[0:\mathsf{E}\to\mathsf{F}]$. Furthermore, this map is natural: given a *unital* homomorphism $f:A\to B$, the diagram

$$\begin{tikzcd} K_0(A) \arrow[r, "K_0(f)"] \arrow[d, "J^A"'] & K_0(B) \arrow[d, "J^B"] \\ K'(A) \arrow[r, "K'(f)"'] & K'(B) \end{tikzcd}$$

commutes. To see this, note that $K_0(f)\big([pA^n]\big)=[f(p)B^n]$. So we have to see that $pA^n\otimes_f B$ is isomorphic to $f(p)B^n$. But I claim

$$(a_i)\otimes_f b\mapsto \big(f(a_i)b\big) \tag{9.16}$$

defines an isometric map $U:pA^n\otimes_f B\to f(p)B^n$. To see that U is surjective, note that U is the restriction of the corresponding map from $A^n\otimes_f B$ to B^n which is surjective because f is unital: $\sum_i \mathbf{e}_i\otimes_f b_i\mapsto (b_i)$.

Remark 9.41. We will make use of the following easily proved assertions regarding operators $T\in\mathcal{L}(\mathsf{X}_1\oplus\mathsf{X}_2,\mathsf{Y}_1\oplus\mathsf{Y}_2)$. Such operators are in one-to-one correspondence with matrices

$$\begin{pmatrix} T_{\mathsf{X}_1\mathsf{Y}_1} & T_{\mathsf{X}_2\mathsf{Y}_1}\\ T_{\mathsf{X}_1\mathsf{Y}_2} & T_{\mathsf{X}_2\mathsf{Y}_2}\end{pmatrix}$$

where $T_{\mathsf{X}_i\mathsf{Y}_j}\in\mathcal{L}(\mathsf{X}_i,\mathsf{Y}_j)$. Furthermore, T is compact if and only if each $T_{\mathsf{X}_i\mathsf{Y}_j}$ is. Since $\mathcal{K}(\mathsf{X},\mathsf{Y})=\mathcal{L}(\mathsf{X},\mathsf{Y})$ if either X or Y is finitely generated (hence projective), it follows that if E is finitely generated, then $T\in\mathcal{L}(\mathsf{X}\oplus\mathsf{E},\mathsf{Y}\oplus\mathsf{E})$ is compact if and only if T_{XY} is compact. In particular, if T and S belong to $\mathcal{L}(\mathsf{X}\oplus\mathsf{E},\mathsf{Y}\oplus\mathsf{E})$ and $T_{\mathsf{XY}}-S_{\mathsf{XY}}\in\mathcal{K}(\mathsf{X},\mathsf{Y})$, then $T-S$ is compact. Thus $T\in\mathcal{L}(\mathsf{X}\oplus\mathsf{E},\mathsf{Y}\oplus\mathsf{E})$ is Fredholm if and only if T_{XY} is.

As in Definition 9.18, if M is a finitely generated Hilbert A-module, then I'll write $[\mathsf{M}]$ for the associated class in $K_0(A)$ via the isomorphism of Theorem 9.12. If M and N are both finitely generated, we need conditions which force $[\mathsf{M}]$ and $[\mathsf{N}]$ to be equal; equivalently, we want to know when $[\mathsf{M}]-[\mathsf{N}]=0$ in $K_0(A)$. We can assume that $\mathsf{M}=pA^n$ and $\mathsf{N}=qA^r$. There is no harm is taking $n=r$. Thus Proposition 1.19 implies $[\mathsf{M}]-[\mathsf{N}]=0$ if and only if $p\oplus 1^m\approx q\oplus 1^m$ in $M_{n+m}(A)$ (again, increasing n if necessary). In particular, $[\mathsf{M}]=[\mathsf{N}]$ implies that for some m, there is a isomorphism for $\mathsf{M}\oplus A^m$ onto $\mathsf{N}\oplus A^m$. In [Exe93], Exel proves a useful and powerful converse to this assertion. He first makes the following definition.

Definition 9.42 (cf., [Exe93, Defintion 2.5 and Lemma 2.6])**.** Hilbert A-modules X and Y are quasi-stably-isomorphic if there is a countably generated Hilbert A-module Z and an invertible operator $T\in\mathcal{L}(X\oplus\mathsf{Z},\mathsf{Y}\oplus\mathsf{Z})$ such that $1_{\mathsf{Z}}-T_{\mathsf{ZZ}}$ is compact.

Theorem 9.43. *If* M *and* N *are finitely generated Hilbert A-modules which are quasi-stably-isomorphic, then* $[\mathsf{M}]-[\mathsf{N}]=0$ *in* $K_0(A)$*.*

Proof. Let X be a countably generated Hilbert A-module and $T\in\mathcal{L}(\mathsf{M}\oplus\mathsf{X},\mathsf{N}\oplus\mathsf{X})$ an invertible operator with $T_{\mathsf{XX}}-I_{\mathsf{X}}\in\mathcal{K}(\mathsf{X})$. Replacing T by $T\oplus 1_{\mathsf{H}_A}$ and $\mathsf{X}\oplus\mathsf{H}_A$ by H_A (via the Kasparov Stabilization Theorem), we can assume $\mathsf{X}\cong\mathsf{H}_A$. Furthermore, we fix an isomorphism $\varphi : \mathsf{X} \oplus \mathsf{M} \to \mathsf{H}_A$. By Theorem 9.6 we can write $T = V|T|$ with V unitary. Define $F \in \mathcal{L}(\mathsf{H}_A)$ by

$$\mathsf{H}_A \xrightarrow{\varphi} \mathsf{M} \oplus \mathsf{X} \xrightarrow{V} \mathsf{N} \oplus \mathsf{X} \xrightarrow{P} \mathsf{X} \xrightarrow{\iota} \mathsf{M} \oplus \mathsf{X} \xrightarrow{\varphi^*} \mathsf{H}_A,$$

where P and ι are the obvious projection and inclusion. Thus $F = \varphi F' \varphi^*$, where $F' \in \mathcal{L}(\mathsf{M} \oplus \mathsf{X})$ has matrix

$$\begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix}.$$

Then $F^* = \varphi G' \varphi^*$, where $G' \in \mathcal{L}(\mathsf{M} \oplus \mathsf{X})$ has matrix

$$\begin{pmatrix} 0 & V_{\mathsf{MX}}^* \\ 0 & V_{\mathsf{XX}}^* \end{pmatrix}$$

But

$$\begin{aligned} F'G'F' &\sim \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \begin{pmatrix} 0 & V_{\mathsf{MX}}^* \\ 0 & V_{\mathsf{XX}}^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \left\{ \begin{pmatrix} V_{\mathsf{MN}}^* & V_{\mathsf{MX}}^* \\ V_{\mathsf{XN}}^* & V_{\mathsf{XX}}^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \begin{pmatrix} V_{\mathsf{MN}}^* & V_{\mathsf{MX}}^* \\ V_{\mathsf{XN}}^* & V_{\mathsf{XX}}^* \end{pmatrix} \right\} \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \end{aligned}$$

which, since $VV^* = 1$,

$$\begin{aligned} &= \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathsf{XX}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V_{\mathsf{MX}} & V_{\mathsf{XX}} \end{pmatrix} \\ &\sim F' \end{aligned}$$

Therefore $F = FF^*F$, and F is a partial isometry. Moreover, $1_{\mathsf{H}_A} - F^*F$ is the projection onto $\ker F$, which is isomorphic to N. Since N is projective, $1 - F^*F$ is compact, and the class $[1 - F^*F]$, viewed as an element of $K_0(A)$, coincides with that of $[\mathsf{N}]$. Similarly, $1 - FF^*$ is compact and $[1 - FF^*] = [\mathsf{M}]$.

Now let $\pi(F)$ be the image of F in $\mathcal{Q}(\mathsf{H}_A) = \mathcal{L}(\mathsf{H}_A)/\mathcal{K}(\mathsf{H}_A)$. By the above, $\pi(F)$ is unitary. Let

$$\partial : K_1\big(\mathcal{L}(\mathsf{H}_A)/\mathcal{K}(\mathsf{H}_A)\big) \to K_0\big(\mathcal{K}(\mathsf{H}_A)\big)$$

be the index map in K-theory as defined in Theorem 8.1. Now $\mathsf{H}_A \cong A \otimes \ell^2$ [RW98, Lemma 3.43], and so $\mathcal{K}(\mathsf{H}_A) \cong A \otimes \mathcal{K}(\ell^2)$ [RW98, Corollary 3.38]. Thus $K_0\big(\mathcal{K}(\mathsf{H}_A)\big)$ can be identified with $K_0(A)$ via the natural map $K_0(\alpha) : K_0(A) \to K_0\big(\mathcal{K}(\mathsf{H}_A)\big)$ of Theorem 4.1. Thus, we will view ∂ as a map into $K_0(A)$.[40] I intend to show that $\partial\big([\pi(F)]\big) = [\mathsf{N}] - [\mathsf{M}]$ on the one hand, and that $\partial\big([\pi(F)]\big) = 0$ on the other hand. This will suffice.

But, since $\pi(F)$ has a lift to a partial isometry — namely F — we can compute $\partial\big([\pi(F)]\big)$ just as we did in the case $A = \mathbb{C}$ and $\mathsf{H}_A = \mathcal{H}$ in the remarks preceding Theorem 8.1. Let

$$W := \begin{pmatrix} F & 1 - FF^* \\ 1 - F^*F & F^* \end{pmatrix}.$$

[40] Since every projection in $\mathcal{K}(\mathsf{H}_A)$ is equivalent to one in $M_n(A)$ (viewed as subalgebra of $\mathcal{K}(\mathsf{H}_A) \cong A \otimes \mathcal{K}(\ell^2)$), it is not hard to see that our identification of the equivalence class of a projection in $\mathcal{K}(\mathsf{H}_A)$ with a class in $K_0(A)$ in Theorem 9.12 is the inverse of $K_0(\alpha)$. Therefore if p is a projection in $\mathcal{K}(\mathsf{H}_A)$, we can identify the class of p in $K_0\big(\mathcal{K}(\mathsf{H}_A)\big)$ and the class it represents in $K_0(A)$.

Then W is unitary lift of $\pi(F) \oplus \pi(F)^*$, and proceeding as in (8.1), we have

$$\partial\big([\pi(F)]\big) = [1 - F^*F] - [1 - FF^*] = [\mathsf{N}] - [\mathsf{M}].$$

Since $T = V|T|$ is invertible, it is homotopic to V. Thus F' is homotopic to T' given by

$$\mathsf{M} \oplus \mathsf{X} \xrightarrow{T} \mathsf{N} \oplus \mathsf{X} \xrightarrow{P} \mathsf{X} \xrightarrow{\iota} \mathsf{M} \oplus \mathsf{X}$$

which is represented by the matrix

$$\begin{pmatrix} 0 & 0 \\ T_{\mathsf{XM}} & T_{\mathsf{XX}} \end{pmatrix}.$$

Since $T_{\mathsf{XX}} - 1_{\mathsf{X}}$ is compact and M is projective, Remark 9.41 implies that T' is a compact perturbation of the identity. Thus $\pi(F)$ is homotopic to 1, and $\partial\big([\pi(F)]\big) = \partial(0)$ must be zero as required. $\square$

Corollary 9.44 ([Exe93, Theorem 3.6])**.** *If $T \in \mathcal{L}(\mathsf{X})$ is a compact perturbation of 1_{X} and T has closed range, then T is Fredholm with $\operatorname{ind} T = 0$.*

Proof. Any compact perturbation of the identity is trivially Fredholm. Let S be a pseudo-inverse for T such that $1_{\mathsf{X}} - ST$ is the projection onto $\ker T$ and $1_{\mathsf{X}} - TS$ is the projection onto $\ker T^*$ (Lemma 9.15). In particular, $\ker S = \ker T^*$, and ST is an idempotent with range $\mathsf{Y} := \operatorname{Range} S$. Now define

$$U : \ker T \oplus Y \to \ker S \oplus \mathsf{Y}$$

using the matrix

$$\begin{pmatrix} 1_{\mathsf{X}} - TS & 1_{\mathsf{X}} - TS \\ S & S \end{pmatrix}.$$

Then I claim the inverse of U is given by the operator $V : \ker S \oplus \mathsf{Y} \to \ker T \oplus \mathsf{Y}$ with matrix

$$\begin{pmatrix} 1_{\mathsf{X}} - ST & (1_{\mathsf{X}} - ST)T \\ ST & ST^2 \end{pmatrix};$$

for example, UV has matrix

$$\begin{pmatrix} 1_{\mathsf{X}} - TS & (1_{\mathsf{X}} - TS)T \\ S & ST \end{pmatrix},$$

which is the identity on $\ker S \oplus \mathsf{Y} = \ker S \oplus \operatorname{Range} S$. Furthermore,

$$U_{\mathsf{YY}} = S|_{\mathsf{Y}},$$

and

$$1_{\mathsf{Y}} - S|_{\mathsf{Y}} = (ST - S)|_{\mathsf{Y}} = S(T-1)|_{\mathsf{Y}},$$

which is compact (because $1_X - T$ is). Therefore $\ker T$ and $\ker S = \ker T^*$ are quasi-stably-isomorphic. Therefore $\operatorname{ind} T = 0$ by Theorem 9.43. $\square$

Now we (finally) come to the crux of the matter.

Corollary 9.45 ([Exe93, Corollary 3.7])**.** *Suppose that T_1 and T_2 are Fredholm operators from X to Y each of which has closed range. If $T_1 - T_2$ is compact, then*

$$\operatorname{ind} T_1 = \operatorname{ind} T_2.$$

Proof. Let S_i be a pseudo-inverse for T_i. Let U and R be the operators in $\mathcal{L}(\mathsf{X}\oplus\mathsf{Y})$ with matrices

$$U \sim \begin{pmatrix} 1_{\mathsf{X}} - S_1T_1 & S_1 \\ T_1 & 1_Y - T_1S_1 \end{pmatrix} \qquad \text{and} \qquad R \sim \begin{pmatrix} 0 & S_1 \\ T_2 & 0 \end{pmatrix}.$$

Notice that R has closed range, and that $\operatorname{ind} R = \operatorname{ind} T_2 - \operatorname{ind} T_1$. Since $U^2 = 1$, U is invertible and it follows that

$$\operatorname{ind} UR = \operatorname{ind} R = \operatorname{ind} T_2 - \operatorname{ind} T_1.$$

But modulo compacts,

$$UR \sim \begin{pmatrix} S_1T_2 & 0 \\ T_2 - T_1S_1T_2 & T_1S_1 \end{pmatrix}$$

which, since $T_1 - T_2$ is compact,

$$\equiv \begin{pmatrix} S_1T_1 & 0 \\ (1_{\mathsf{X}} - T_1S_1)T_2 & T_1S_1 \end{pmatrix}$$

and, since $1_{\mathsf{X}} - S_1T_1$ is also compact,

$$\equiv \begin{pmatrix} 1_{\mathsf{X}} & 0 \\ 0 & 1_{\mathsf{Y}} \end{pmatrix}.$$

Therefore, $\operatorname{ind} UR = 0$ by Corollary 9.44. $\square$

Lemma 9.46 ([Exe93, Lemma 3.8])**.** *Suppose $1 \in A$, X and Y are Hilbert A-modules, and that $T \in \mathcal{F}(\mathsf{X},\mathsf{Y})$. Then for some $n \in \mathbb{Z}^+$, there is a $F \in \mathcal{F}(X \oplus A^n, \mathsf{Y} \oplus A^n)$ with closed range and such that $F_{\mathsf{XY}} = T$.*

Proof. Choose a parametrix $S \in \mathcal{L}(\mathsf{Y},\mathsf{X})$ for T such that $1_{\mathsf{X}} - ST$ and $1_{\mathsf{Y}} - TS$ are finite rank (Lemma 9.14). Therefore there are $\mathbf{x} := (x_i) \in \mathsf{X}^n$ and $\mathbf{y} := (y_i) \in \mathsf{Y}^n$ such that

$$1_{\mathsf{X}} - ST = \sum_{i=1}^n \theta_{x_i,y_i}.$$

Let $\Omega_{\mathbf{x}} \in \mathcal{L}(A^n, \mathsf{X})$ be defined by $\Omega_{\mathbf{x}}\big((a_i)\big) := \sum_{i=1}^n x_i \cdot a_i$, and notice that $\Omega_{\mathbf{x}}^*(x) = \big(\langle x_i \,,\, x\rangle_{A}\big)$. Then $\Omega_{\mathbf{x}}\Omega_{\mathbf{y}}^* = \sum_{i=1}^n \theta_{x_i,y_i}$. Now define F and G via the matrices

$$F \sim \begin{pmatrix} T & 0 \\ \Omega_{\mathbf{y}}^* & 0 \end{pmatrix} \qquad \text{and} \qquad G \sim \begin{pmatrix} S & \Omega_{\mathbf{x}} \\ 0 & 0 \end{pmatrix}.$$

Now compute

$$\begin{aligned} FGF &\sim \begin{pmatrix} TST + T\Omega_{\mathbf{x}}\Omega_{\mathbf{y}}^* & 0 \\ \Omega_{\mathbf{y}}^*ST + \Omega_{\mathbf{y}}^*\Omega_{\mathbf{x}}\Omega_{\mathbf{y}}^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} T + T(1_{\mathsf{X}} - ST) & 0 \\ \Omega_{\mathbf{y}}^*(ST + (1_{\mathsf{X}} - ST)) & 0 \end{pmatrix} \\ &\sim F. \end{aligned}$$

Similarly, $GFG = G$, and F is Fredholm with closed range such that $F_{\mathsf{XY}} = T$. $\square$

Remark 9.47. We call any Fredholm operator $F \in \mathcal{F}(X \oplus A^n, \mathsf{Y} \oplus A^n)$ having closed range and $F_{\mathsf{XY}} = T$ a *n-regularization* of T. By Remark 9.41, any two n-regularizations of T differ by a compact and therefore have the same index by Corollary 9.45. Furthermore, if F is a n-regularization of T, then $F \oplus 1_{A^k}$ is a $(n+k)$-regularization of T. Since $\ker F \cong \ker(F \oplus 1)$ (and $\ker F^* \cong \ker(F \oplus 1)^*$), we have $\operatorname{ind} F = \operatorname{ind}(F \oplus 1_{A^k})$. Thus if F is a n-regularization of T and G is a m-regularization of T, then $\operatorname{ind} F = \operatorname{ind} G$.

In view of Remark 9.47, we can make the following definition for any Fredholm operator.

Definition 9.48. Suppose that $1 \in A$ and that X and Y are Hilbert A-modules. If $T \in \mathcal{F}(\mathsf{X}, \mathsf{Y})$ the we define $\operatorname{ind} T$ to equal $\operatorname{ind} F$ for any n-regularization of T.

Remark 9.49. Notice that if $T \in \mathcal{F}(\mathsf{X}, \mathsf{Y})$ has closed range, then $F = T \oplus 1_n$ is a regularization, so Definition 9.48 and Definition 9.18 associate the same element of $K_0(A)$ to such operators.

Now having defined an index for all Fredholm operators, we still want to see that we can pass to $K'(A)$. This will involve a bit of overhead even using the "easy" hints in [Hig90].

Theorem 9.50. *Suppose that for $i = 0, 1$, F_i is a Fredholm operator from $\mathsf{E}_0^{(i)}$ to $\mathsf{E}_1^{(i)}$. If $F_0 \sim_h F_1$, then $\operatorname{ind} F_0 = \operatorname{ind} F_1$. In particular, we get a well-defined homomorphism $\operatorname{ind} : K'(A) \to K_0(A)$ defined by $\operatorname{ind}[F] := \operatorname{ind} F$.*

Proof. In view of Remark 9.38, we can assume $F_0 \rightsquigarrow_h F_1$ via a Fredholm operator $\widetilde{F} : \widetilde{\mathsf{E}}_0 \to \widetilde{\mathsf{E}}_1$. Thus we have unitaries

$$U_k^{(i)} : \widetilde{\mathsf{E}}_k \otimes_{\epsilon_i} A \to \mathsf{E}_k^{(i)}$$

such that

$$F_0 = U_1^{(0)}(\widetilde{F} \otimes_{\epsilon_0} 1)U_0^{(0)^*} \quad \text{and} \quad F_1 = U_1^{(1)}(\widetilde{F} \otimes_{\epsilon_1} 1)U_0^{(1)^*}. \tag{9.17}$$

By Lemma 9.46 we have a n-regularization for $\widetilde{F}$:

$$\widehat{F} : \widetilde{\mathsf{E}}_0 \oplus C(I, A)^n \to \widetilde{\mathsf{E}}_1 \oplus C(I, A)^n.$$

Let

$$\widehat{S} : \widetilde{\mathsf{E}}_1 \oplus C(I, A)^n \to \widetilde{\mathsf{E}}_0 \oplus C(I, A)^n$$

be a pseudo-inverse for $\widehat{F}$. Let $V^{(t)} : C(I, A)^n \otimes_{\epsilon_t} A \to A^n$ be the obvious isomorphism. Since

$$\big(\widetilde{\mathsf{E}}_k \oplus C(I, A)^n\big) \otimes_{\epsilon_i} A \quad \text{and} \quad \big(\widetilde{\mathsf{E}}_k \otimes_{\epsilon_i} A\big) \oplus \big(C(I, A)^n \otimes_{\epsilon_i} A\big),$$

are isomorphic, $U_k^{(i)} \oplus V^{(i)}$ give us unitaries

$$W_k^{(i)} : \big(\widetilde{\mathsf{E}}_k \oplus C(I, A)^n\big) \otimes_{\epsilon_i} A \to \mathsf{E}_k^{(i)} \oplus A^n.$$

Therefore can can define Fredholm operators

$$\check{F}_i := W_1^{(i)}(\widehat{F} \otimes_{\epsilon_i} 1)W_0^{(i)^*} \quad \text{and} \quad \check{S}_i := W_0^{(i)}(\widehat{S} \otimes_{\epsilon_i} 1)W_1^{(i)^*}$$

in $\mathcal{F}(\mathsf{E}_0^{(i)} \oplus A^n, \mathsf{E}_1^{(i)} \oplus A^n)$ and $\mathcal{F}(\mathsf{E}_1^{(i)} \oplus A^n, \mathsf{E}_0^{(i)} \oplus A^n)$, respectively. Since $\widehat{F}\widehat{S}\widehat{F} = \widehat{F}$, we have $\check{F}_i\check{S}_i\check{F}_i = \check{F}_i$. Similarly, $\check{S}_i\check{F}_i\check{S}_i = \check{S}_i$. Thus $\check{F}_i$ has closed range and to show it is a n-regularization of F_i, I just have to show that $(\check{F}_i)_{\mathsf{E}_0^{(i)}\mathsf{E}_1^{(i)}} = F_i$. Let

$$\begin{pmatrix} \widetilde{F} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix}$$

be the matrix for $\widehat{F}$. Then

$$\begin{aligned} W_1^{(i)}(\widehat{F} \otimes_{\epsilon_i} 1)\big((e,f) \otimes_{\epsilon_i} a\big) &= \big(U_1^{(i)}\big((\widetilde{F}e + \widehat{B}f) \otimes_{\epsilon_i} a\big), \\ &\qquad\qquad V^{(i)}\big((\widehat{C}e + \widehat{D}f) \otimes_{\epsilon_i} a\big)\big) \\ &= \begin{pmatrix} U_1^{(i)}(\widetilde{F} \otimes_{\epsilon_i} 1)U_0^{(i)^*} & U_1^{(i)}(\widehat{B} \otimes_{\epsilon_i} 1)U_0^{(i)^*} \\ V^{(i)}(\widehat{C} \otimes_{\epsilon_i} 1)V^{(i)^*} & V^{(i)}(\widehat{D} \otimes_{\epsilon_i} 1)V^{(i)^*} \end{pmatrix} W_0^{(i)}\big((e,f) \otimes_{\epsilon_i} a\big). \end{aligned}$$

In view of (9.17), $(\check{F}_i)_{\mathsf{E}_0^{(i)}\mathsf{E}_1^{(i)}} = F_i$ as claimed.

Now by definition

$$\operatorname{ind} F_0 = \operatorname{ind} \check{F}_0 \quad \text{and} \quad \operatorname{ind} F_1 = \operatorname{ind} \check{F}_1.$$

Now consider $K_0(\epsilon_t) : K_0\big(C(I,A)\big) \to K_0(A)$. We want to show that

$$\operatorname{ind} \check{F}_i = K_0(\epsilon_i)(\operatorname{ind} \widehat{F}). \tag{9.18}$$

As in (9.16), we have

$$K_0(\epsilon_t)\big([\mathsf{E}]\big) = [\mathsf{E} \otimes_{\epsilon_t} A].$$

Since $\ker \check{F}_i$ is clearly isomorphic to $\ker(\widehat{F} \otimes_{\epsilon_i} 1)$, (9.18) will follow if we show

$$\ker(\widehat{F} \otimes_{\epsilon_i} 1) = \ker \widehat{F} \otimes_{\epsilon_i} A.$$

Since $\widehat{F}$ has closed range, this follows from Corollary C.4. But $\epsilon_0 \sim_h \epsilon_1$ as homomorphisms from $C(I,A)$ to A (see Definition 5.12 with $\gamma = \mathrm{id}$), so

$$K_0(\epsilon_0) = K_0(\epsilon_1)$$

by Theorem 5.15. Therefore $\operatorname{ind} \check{F}_0 = \operatorname{ind} \check{F}_1$, and we're done by (9.18). $\square$

And now, finally, ...

Theorem 9.51. *The natural map $J : K_0(A) \to K'(A)$ is an isomorphism.*

Proof. The composition

$$K_0(A) \xrightarrow{J} K'(A) \xrightarrow{\operatorname{ind}} K_0(A)$$

is clearly the identity, and

$$K'(A) \xrightarrow{\operatorname{ind}} K_0(A) \xrightarrow{J} K'(A)$$

is the identity by virtue of Proposition 9.32. $\square$

Appendix A. Direct Limits of C^*-Algebras

If A is a $*$-algebra, then a seminorm ρ on A is called a C^*-seminorm if

$$\begin{aligned} &\text{(a)} \quad \rho(ab) \le \rho(a)\rho(b), \\ &\text{(b)} \quad \rho(a^*) = \rho(a), \quad \text{and} \\ &\text{(c)} \quad \rho(a^*a) = \rho(a)^2. \end{aligned}$$

Thus (A, ρ) is C^*-algebra only if ρ is a norm and (A, ρ) is complete. Examples of such things are provided by $*$-homomorphisms $\varphi : A \to B$ where B is a C^*-algebra: $\rho(a) := \|\varphi(a)\|$. A nontrivial example is provided by the left-regular representation $\lambda : L^1(G) \to B\big(L^2(G)\big)$.

In general, $N = \rho^{-1}\big(\{\,0\,\}\big)$ is a self-adjoint two sided ideal in A, and $\|a + N\| := \rho(a)$ is a C^*-norm on A/N. The completion B is a C^*-algebra called the *enveloping C^*-algebra of* (A, ρ).

As the title of this section suggests, the direct limit C^*-algebra is an example of such an enveloping algebra. However, it will be useful to define direct limits via a universal property.

Definition A.1. A direct sequence of C^*-algebras is a collection $\{\,(A_n, \varphi_n)\,\}$ of C^*-algebras A_n and homomorphisms $\varphi_n : A_n \to A_{n+1}$. A family $\{\,\psi_n : A_n \to B\,\}$ of homomorphisms into a C^*-algebra B is said to be compatible if

(A.1)
$$\begin{tikzcd} A_n \arrow[rr, "\varphi_n"] \arrow[dr, "\psi_n"'] & & A_{n+1} \arrow[dl, "\psi_{n+1}"] \\ & B & \end{tikzcd}$$

commutes for all n. A direct limit of $\{\,(A_n, \varphi_n)\,\}$ is a C^*-algebra A together with compatible homomorphisms $\varphi^n : A_n \to A$ such that given compatible homomorphisms ψ_n as in (A.1), then there is a unique homomorphism $\psi : A \to B$ such that

(A.2)
$$\begin{tikzcd} & A \arrow[dd, "\psi"] \\ A_n \arrow[ur, "\varphi^n"] \arrow[dr, "\psi_n"'] & \\ & B \end{tikzcd}$$

commutes.

Remark A.2. General nonsense implies that the direct limit, if it exists, is unique up to isomorphism. Therefore we will refer to the direct limit and use the notation $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$. If $n \ge m$, then we will use the notation $\varphi_{mn} : A_m \to A_n$ for the repeated composition $\varphi_{mn} := \varphi_{n-1} \circ \cdots \circ \varphi_{m+1} \circ \varphi_m$, with the understanding that $\varphi_{nn} := \mathrm{id}_{A_n}$.

Theorem A.3. *Suppose that $\{\,A_n, \varphi_n\,\}$ is a direct sequence of C^*-algebras. Then the direct limit $(A, \varphi^n) = \varinjlim(A_n, \varphi_n)$ always exists. Furthermore, $\varphi^n(A_n) \subset$*

$\varphi^{n+1}(A_{n+1})$, $\bigcup_n \varphi^n(A_n)$ is dense in A, $\|\varphi^n(a)\| = \lim_k \|\varphi_{n,n+k}(a)\|$, and $\varphi^n(a) = \varphi^m(b)$ if and only if for all $\epsilon > 0$ there is a $k \geq \max\{ m, n \}$ such that

$$\|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon. \tag{A.3}$$

Proof. Let $B = \prod_n A_n$ be the C^*-direct product, and let

$$A_0 = \{ \mathbf{a} = (a_n) \in B : a_{k+1} = \varphi_k(a_k) \quad \text{for all } k \text{ sufficiently large} \}.$$

Then A_0 is a $*$-subalgebra of B, and if $\mathbf{a} \in A_0$, then we eventually have $\|a_{k+1}\| \leq \|a_k\|$. Thus we can define

$$\rho(\mathbf{a}) = \lim_n \|a_n\| \quad \text{for all } \mathbf{a} \in A_0.$$

It is not hard to check that ρ is a C^*-seminorm on A_0. We let A be the enveloping C^*-algebra. Furthermore, we can define homomorphisms $\hat{\varphi}^n : A_n \to A_0$ by

$$\big(\hat{\varphi}^n(a)\big)_k = \begin{cases} \varphi_{nk}(a) & \text{if } k \geq n \\ 0 & \text{otherwise.} \end{cases}$$

We then get compatible homomorphisms φ^n by composing with the natural map of A_0 into A (which need not be injective).

Since the remaining assertions are clear, it will suffice to show that (A, φ^n) have the right universal property. So, suppose that $\psi_n : A_n \to B$ are compatible homomorphisms. Since (A.2) forces us to have $\psi\big(\varphi^n(a)\big) = \psi_n(a)$, the map ψ can exist only if $\varphi^n(a) = \varphi^m(b)$ implies that $\psi_n(a) = \psi_m(b)$. Let $\epsilon > 0$ and assume that $\varphi^n(a) = \varphi^m(b)$. There is a k such that $\|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon$. And then

$$\|\psi_n(a) - \psi_m(b)\| \leq \|\psi_k\big(\varphi_{nk}(a) - \varphi_{mk}(b)\big)\| \leq \|\varphi_{nk}(a) - \varphi_{mk}(b)\| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get a well-defined map on $C := \bigcup_n \varphi^n(A_n)$, which is clearly dense in A. Since

$$\begin{aligned} \|\psi\big(\varphi^n(a)\big) &= \|\psi_n(a)\| \\ &= \|\psi_{n+k}\big(\varphi_{n,n+k}(a)\big)\| \quad \text{for any } k \geq 1, \\ &\leq \inf_k \|\varphi_{n,n+k}(a)\| = \lim_k \|\varphi_{n,n+k}(a)\| \\ &= \|\varphi^n(a)\|. \end{aligned}$$

Therefore ψ is norm decreasing and extends to all of A. We have $\psi \circ \varphi^n = \psi_n$ by construction, and ψ is uniquely determined on the dense subalgebra C, so ψ is unique. □

Example A.4. Suppose that A_n is a C^*-subalgebra of A for $n \in \mathbb{Z}^+$, that $A_n \subset A_{n+1}$, and that $\bigcup_n A_n$ is dense in A. Let φ_n be the inclusion of A_n in A_{n+1} and φ^n the inclusion of A_n in A. Then (A, φ^n) is the direct limit of (A_n, φ_n).

Remark A.5. We can also form the algebraic direct limit of $\{ A_n, \varphi_n \}$. It has similar properties to the C^*-direct limit, and can be exhibited as the $*$-algebra quotient A_0/N where N is the ideal of elements of zero length in A_0 in the proof of Theorem A.3. Of course, in Example A.4, the algebraic direct limit is just $\bigcup_n A_n$.

Appendix B. Grothendieck Group

A more general discussion can be found in the appendices to [WO93]. Here we consider only an abelian semigroup V. The Grothendieck group $\mathcal{G}(V)$ of V is the collection of equivalence classes $[v,w]$ in $V\times V$ where

$$(v,w)\sim(r,s)\quad\text{if and only if there is an } x\in V \text{ such that}\qquad v+s+x=r+w+x. \tag{B.1}$$

Then $\mathcal{G}(V)$ is an abelian group under the operation

$$[v,w]+[r,s]:=[v+r,w+s].$$

The neutral element is the class of $[v,v]$ (for any $v\in V$), and $[v,w]^{-1}=[w,v]$.

Remark B.1. The extra term "x" in (B.1) is required to make $\sim$ an equivalence relation in the event V is not cancellative. Of course, if V is cancellative, it plays no rôle and can be omitted.

There is a natural map $\varphi_V:V\to\mathcal{G}(V)$ defined by $\varphi(v):=[v+v,v]$. If V has an identity 0, then $\varphi_V(v)=[v,0]$. If $\varphi:V\to H$ is a semigroup homomorphism into a group H, then there is a unique group homomorphism $\mathcal{G}(\varphi):\mathcal{G}(V)\to H$ such that

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\varphi_V"'] & & H \\ & \mathcal{G}(V) \arrow[ur, "\mathcal{G}(\varphi)"'] & \end{tikzcd}$$

commutes. In fact, $\mathcal{G}(\varphi)\big([v,w]\big)=\psi(v)\psi(w)^{-1}$. More generally, if $\varphi:V\to S$ is a semigroup homomorphism into a semigroup S, then there is a unique group homomorphism $\mathcal{G}(\varphi)$ such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\varphi_V"'] & S \arrow[d, "\varphi_S"] \\ \mathcal{G}(V) \arrow[r, "\mathcal{G}(\varphi)"'] & \mathcal{G}(S) \end{tikzcd}$$

commutes. Here, $\mathcal{G}(\varphi)\big([v,w]\big)=[\varphi(v),\varphi(w)]$. If S is a group, then $\mathcal{G}(S)=S$ and $\varphi_S=\mathrm{id}_S$, and this diagram reduces to the previous one.

We will almost always write φ in place of $\mathcal{G}(\varphi)$.

Example B.2. We have $\mathcal{G}(\mathbb{N},+)\cong(\mathbb{Z},+)$. On the other hand, $\mathcal{G}(\mathbb{N}\cup\{\infty\},+)\cong\{0\}\cong\mathcal{G}(\mathbb{N},\cdot)$. While $\mathcal{G}(\mathbb{Z}^+,\cdot)\cong(\mathbb{Q}^+,\cdot)$.

Appendix C. The Internal Tensor Product

In [RW98], we only considered a very special case of the internal tensor product which sufficed for the study of imprimitivity bimodules. Here we want to consider C^*-algebras A and B together a Hilbert A-module X, a Hilbert B-module Y, and a homomorphism $\varphi:A\to\mathcal{L}(\mathsf{Y})$. We can view Y as a left A-module — $a\cdot y:=\varphi(a)y$ — and form the A-balanced module tensor product $\mathsf{X}\odot_A\mathsf{Y}$; recall that this is simply the quotient of the vector space tensor product $\mathsf{X}\odot\mathsf{Y}$ by the subspace N generated by

$$\{\,x\cdot a\otimes y-x\otimes\varphi(a)y : x\in\mathsf{X},\ y\in\mathsf{Y},\text{ and } a\in A\,\}.$$

The B-module structure is given by $(x \otimes_A y) \cdot b := x \otimes_A y \cdot b$. Our object here is to equip $\mathsf{X} \odot_A \mathsf{Y}$ with a B-valued inner product and use [RW98, Lemma 2.16] to pass to the completion $\mathsf{X} \otimes_\varphi \mathsf{Y}$ which is a Hilbert B-module called the internal tensor product. I'll give a minimal treatment here just sufficient for our purposes in Section 9.3. A more complete treatment can be found in Lance [Lan94]. Here we take a slight short-cut, and merely equip $\mathsf{X} \odot \mathsf{Y}$ with a B-valued *pre*-inner product. As it turns out, elements in N all have 0-length, and we can, and do, view $\mathsf{X} \otimes_\varphi \mathsf{Y}$ as a completion of $\mathsf{X} \odot_A \mathsf{Y}$.[41]

Proposition C.1 ([Lan94, Proposition 4.5])**.** *Let X be a Hilbert A-module and Y and Hilbert B-module with a homomorphism $\varphi : A \to \mathcal{L}(\mathsf{Y})$. Then there is a unique B-valued pre-inner product on $\mathsf{X} \odot \mathsf{Y}$ such that*

$$\langle\!\langle x \otimes y \,,\, z \otimes w \rangle\!\rangle_B := \big\langle y \,,\, \varphi\big(\langle x \,,\, z\rangle_A\big) w\big\rangle_B. \tag{C.1}$$

The completion $\mathsf{X} \otimes_\varphi \mathsf{Y}$ is a Hilbert B-module which satisfies

$$x \cdot a \otimes_\varphi y = x \otimes_\varphi \varphi(a) y$$

for all $x \in \mathsf{X}$, $y \in \mathsf{Y}$, and $a \in A$.

Proof. As in Propositions 2.64, 3.16, and 3.36 of [RW98], the universal property of the algebraic tensor product $\odot$ implies that (C.1) determines a unique sequilinear form on $\mathsf{X} \odot \mathsf{Y}$. The only real issue is to prove that this form is positive.

So let $t := \sum x_i \otimes y_i$. Then

$$\langle\!\langle t \,,\, t \rangle\!\rangle_B = \sum_{i,j} \big\langle y_i \,,\, \varphi\big(\langle x_i \,,\, x_j\rangle_A\big) y_j\big\rangle_B. \tag{C.2}$$

But [RW98, Lemma 2.65] implies that $M := \big(\langle x_i \,,\, x_j\rangle_A\big)$ is a positive matrix in $M_n(A)$. Thus there is a matrix D such that $M = D^*D$, and there are $d_{kl} \in A$ such that

$$\langle x_i \,,\, x_j\rangle_A = \sum_k d_{ki}^* d_{kj}.$$

Thus (C.2) equals

$$\begin{aligned}\sum_{i,j,k} \big\langle y_i \,,\, \varphi(d_{ki}^* d_{kj}) y_j\big\rangle_B &= \sum_{i,j,k} \big\langle \varphi(d_{ki}) y_i \,,\, \varphi(d_{kj}) y_j\big\rangle_B \\ &= \sum_k \Big\langle \Big(\sum_i \varphi(d_{ki}) y_i\Big) \,,\, \Big(\sum_i \varphi(d_{ki}) y_i\Big)\Big\rangle_B \\ &\geq 0 \quad \square\end{aligned}$$

Now we want to see that each $T \in \mathcal{L}(\mathsf{X})$ determines an operator $T \otimes_\varphi 1$ on the internal tensor product $\mathsf{X} \otimes_\varphi \mathsf{Y}$. In general, $T \otimes_\varphi 1$ can fail to be compact when T is unless we also assume that $\varphi(A) \subset \mathcal{K}(\mathsf{Y})$ (see [Lan94, p. 43]).

Proposition C.2 ([Lan94, Proposition 4.7])**.** *Let A, B, X, Y, and φ be as above, and let Z be a Hilbert A-module. If $T \in \mathcal{L}(\mathsf{X}, \mathsf{Z})$, then there is unique operator $T \otimes_\varphi 1 \in \mathcal{L}(\mathsf{X} \otimes_\varphi \mathsf{Y}, \mathsf{Z} \otimes_\varphi \mathsf{Y})$ such that*

$$T \otimes_\varphi 1(x \otimes_\varphi y) = Tx \otimes_\varphi y.$$

If $\varphi(A) \subset \mathcal{K}(\mathsf{Y})$ and $T \in \mathcal{K}(\mathsf{X})$, then $T \otimes_\varphi 1 \in \mathcal{K}(\mathsf{X} \otimes_\varphi \mathsf{Y})$.

[41] The difference is that Lance goes the extra mile and proves that the pre-inner product is actually an inner product (that is, definite) on $\mathsf{X} \odot_A \mathsf{Y}$.

Proof. We clearly have a well-defined operator $T \otimes 1$ on $\mathsf{X} \odot \mathsf{Y}$, and if $t \in \mathsf{X} \odot \mathsf{Y}$ and $s \in \mathsf{Z} \odot \mathsf{Y}$, then straightforward calculations reveal that

$$\langle\!\langle (T \otimes 1)t \, , \, s \rangle\!\rangle_B = \langle\!\langle t \, , \, (T^* \otimes 1)s \rangle\!\rangle_B.$$

Similarly, if $S \in \mathcal{L}(\mathsf{Z}, \mathsf{X})$, then on the algebraic tensor products, $(T \otimes 1) \circ (S \otimes 1) = (TS \otimes 1)$. Since there is a $R \in \mathcal{L}(X)$ such that $\|T\|^2 1_{\mathsf{X}} - T^*T = R^*R$, for all $t \in \mathsf{X} \odot \mathsf{Y}$,

$$\begin{aligned} \|T\|^2 \|t\|^2_{\mathsf{X} \otimes_\varphi \mathsf{Y}} - \|(T \otimes 1)t\|^2_{\mathsf{X} \otimes_\varphi \mathsf{Y}} &= \langle\!\langle \|T\|^2 t \, , \, t \rangle\!\rangle_B - \langle\!\langle (T^*T \otimes 1)t \, , \, t \rangle\!\rangle_B \\ &= \langle\!\langle ((\|T\|^2 1_{\mathsf{X}} - T^*T) \otimes 1)t \, , \, t \rangle\!\rangle_B \\ &= \|(R \otimes 1)t\|^2_{\mathsf{X} \otimes_\varphi \mathsf{Y}} \geq 0. \end{aligned}$$

It follows that $T \otimes 1$ is bounded, and extends to an operator $T \otimes_\varphi 1 \in \mathcal{L}(\mathsf{X} \otimes_\varphi \mathsf{Y}, \mathsf{Z} \otimes_\varphi \mathsf{Y})$ such that $(T \otimes_\varphi 1)^* = T^* \otimes_\varphi 1$ and $\|T \otimes_\varphi 1\| \leq \|T\|$. Furthermore, if $\mathsf{Z} = \mathsf{X}$ and $\varphi_*(T) := T \otimes_\varphi 1$, then φ_* is a homomorphism from $\mathcal{L}(X)$ into $\mathcal{L}(\mathsf{X} \otimes_\varphi \mathsf{Y})$.

For the second assertion, note that if $x \in \mathsf{X}$, then

$$y \mapsto x \otimes_\varphi y \tag{C.3}$$

is a linear map of Y into $\mathsf{X} \otimes_\varphi \mathsf{Y}$. Since

$$\|\langle\!\langle x \otimes_\varphi y \, , \, x \otimes_\varphi y \rangle\!\rangle_B\| = \big\langle y \, , \, \varphi\big(\langle x \, , \, x \rangle_A\big) y \big\rangle_B \leq \|x\|^2_A \|y\|^2_B,$$

(C.3) determines an operator $\upsilon_x : \mathsf{Y} \to \mathsf{X} \otimes_\varphi \mathsf{Y}$ with $\|\upsilon_x\| \leq \|x\|_A$. Since

$$\langle\!\langle \upsilon_x(y) \, , \, x \otimes w \rangle\!\rangle_B = \big\langle y \, , \, \varphi\big(\langle x \, , \, z \rangle_A\big) w \big\rangle_B,$$

$\upsilon_x \in \mathcal{L}(\mathsf{Y}, \mathsf{X} \otimes_\varphi \mathsf{Y})$ with $\upsilon_x^*(z \otimes_\varphi w) = \varphi(\langle x \, , \, z \rangle_A) w$. Using this, we compute that

$$\begin{aligned} (\theta_{x \cdot a, y} \otimes 1)(z \otimes_\varphi w) &= \theta_{x \cdot a, y}(z) \otimes_\varphi w \\ &= x \cdot a \langle y \, , \, z \rangle_A \otimes_\varphi w \\ &= x \otimes_\varphi \varphi\big(a \langle y \, , \, z \rangle_A\big) w \\ &= x \otimes_\varphi \varphi(a) \varphi\big(\langle y \, , \, z \rangle_A\big) w \\ &= \upsilon_x\big(\varphi(a) \varphi\big(\langle y \, , \, z \rangle_A\big) w\big) \\ &= \upsilon_x \circ \varphi(a) \circ \upsilon_y^*(z \otimes_\varphi w). \end{aligned}$$

It follows that if $\varphi(a) \in \mathcal{K}(\mathsf{Y})$, then $\theta_{x \cdot a, y} \otimes 1 = \upsilon_x \varphi(a) \upsilon_y^*$ is compact. Since the Cohen Factorization Theorem ([RW98, Proposition 2.31] suffices) implies every $x \in \mathsf{X}$ is of the form $z \cdot a$ for some $z \in \mathsf{X}$ and $a \in A$, it follows that $\varphi_*(\theta_{x,y}) \in \mathcal{K}(\mathsf{X} \otimes_\varphi \mathsf{Y})$ for all $x, y \in \mathsf{X}$. Since φ_* is a homomorphism and $\mathcal{K}(\mathsf{X})$ is generated by the $\theta_{x,y}$, we have $\varphi_*\big(\mathcal{K}(\mathsf{X})\big) \subset \mathcal{K}(\mathsf{X} \otimes_\varphi \mathsf{Y})$ as desired □

Remark C.3. In [Lan94, Proposition 4.7], Lance also shows that the homomorphism φ_* is injective if φ is, and surjective if φ is.

Corollary C.4. *Suppose that A, B, X, Y, Z, and φ are as in Proposition C.2. If $T \in \mathcal{L}(\mathsf{X}, \mathsf{Z})$ has closed range, then*

$$\ker(T \otimes_\varphi 1) = \ker T \otimes_\varphi \mathsf{Y}.$$

Proof. By Theorem 9.6, $\mathsf{X} = \ker T \oplus (\ker T)^{\perp}$. Therefore we can identify $\mathsf{X} \otimes_{\varphi} \mathsf{Y}$ with $\ker T \otimes_{\varphi} \mathsf{Y} \oplus (\ker T)^{\perp} \otimes_{\varphi} \mathsf{Y}$, and it will suffice to see that $T \otimes_{\varphi} 1$ restricted to $(\ker T)^{\perp} \otimes_{\varphi} \mathsf{Y}$ is injective. To see this, it suffices to see that $T|_{(\ker T)^{\perp}} \otimes_{\varphi} 1$ is injective as an operator from $(\ker T)^{\perp} \otimes_{\varphi} \mathsf{Y}$ to $T(\mathsf{X}) \otimes_{\varphi} \mathsf{Y}$. But the open mapping theorem that $T|_{(\ker T)^{\perp}} : (\ker T)^{\perp} \to T(\mathsf{X})$ has a bounded inverse S, and as in the proof of Lemma 9.15, $T|_{(\ker T)^{\perp}}$ is adjointable. Then $S \otimes_{\varphi} 1$ is an inverse for $T|_{(\ker T)\perp} \otimes_{\varphi} 1$. □

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