The Peter-Weyl Theorem for Compact Groups

The following notes are from a series of lectures I gave at Dartmouth College in the summer of 1989. The general outline is provided by an introductory section in [1] but with considerable detail added by myself. The mistakes of course are mine.

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§1 Preliminaries.

We begin with some warm-up exercises on locally compact groups; a.k.a., a long series of definitions! At this point, G is meant to be an arbitrary locally compact group.

Definition 1: *A (unitary) representation of G is a continuous homomorphism π from G to the unitary group $\mathcal{U}(\mathcal{H}_\pi)$ on a (complex) Hilbert space $\mathcal{H}_\pi$ equipted with the strong operator topology.*

Remark 2: The condition that π be continuous merely means that $g \mapsto \pi(g)\xi$ is continuous from G to $\mathcal{H}_\pi$ for each $\xi \in \mathcal{H}_\pi$. There are many equivalent conditions: the weakest I'm aware of is to insist that $g \mapsto \langle \pi(g)\xi, \eta \rangle$ be Borel for each $\xi, \eta \in \mathcal{H}_\pi$.

Example 3: Let $\mathcal{H} = L^2(G)$. The *left regular representation* $\lambda : G \to \mathcal{B}(\mathcal{H})$ is given by $\lambda(g)f(t) = f(g^{-1}t)$.

Definition 4: *Two representations π_1 and π_2 are said to be (unitarily) equivalent if there is a unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that $\pi_1(g) = U\pi_2(g)U^*$ for all $g \in G$. In this event, we write $\pi_1 \cong \pi_2$ and let $[\pi]$ denote the (unitary) equivalence class of π.*

If $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is a representation of G, then we'll write d_π for the dimension (in $\{0, 1, 2, \ldots, \infty\}$) of $\mathcal{H}_\pi$. Fortunately, whenever $\pi_1 \cong \pi_2$, then it is clear that $d_{\pi_1} = d_{\pi_2}$. Thus we will often write $d_{[\pi]}$ to denote the dimension of each representation in the same equivalence class as π.

Definition 5: *A non-zero representation π is called irreducible if $\mathcal{H}_\pi$ has no no-trivial closed invariant subspaces*

Remark 6: If $d_\pi < \infty$, then the word "closed" in redundent in the above definition.

Definition 7: *The symbol $\widehat{G}$ is used to denote the collection of equivalence classes of irreducible representations of G.*

Example 8: If G is abelian, then every irreducible representation is one-dimensional. In particular, $\widehat{G}$ coincides with the character group of G.

Definition 9: *If π and η are representations of G, then $\pi \oplus \eta$ denotes the representation on $\mathcal{H}_\pi \oplus \mathcal{H}_\eta$ defined by*

$$\pi \oplus \eta(g)(\xi, \zeta) = (\pi(g)\xi, \eta(g)\zeta).$$

If $n \in \mathbb{Z}^+ \cup \{\infty\}$, then $n \cdot \pi = \oplus_{i=1}^n \pi$.

Definition 10: *If π is a representation of G, then*

$$\pi(G)' = \{ A \in \mathcal{B}(\mathcal{H}_\pi) : A\pi(g) = \pi(g)A, \text{ for all } g \in G \}.$$

Remark 11: Since π is unitary, $\pi(G)'$ is a self-adjoint. In fact it is not hard to check that $\pi(G)'$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ which is closed in the weak operator topology.

Theorem 12: *Suppose that π is a representation of G. Then the following are equivalent.*

(1) *π is irreducible.*
(2) *$\pi(G)' = \mathbb{C}I$.*
(3) *Every non-zero $\xi \in \mathcal{H}_\pi$ is cyclic for π (i.e., $\overline{\pi(G)\xi} = \overline{\operatorname{span}}\{ \pi(g)\xi : g \in G \} = \mathcal{H}_\pi$).*

Proof: Suppose that π is irreducible. Let $A \in \pi(G)'$ and suppose for the moment that A is normal: $A^*A = AA^*$. Then the norm closed unital *-subalgebra generated by A—that is the C*-algebra $C^*(I, A)$—is contained in $\pi(G)'$. Since A is normal, $C^*(I, A)$ is commutative and is isomorphic to $C\big(\sigma(A)\big)$ by the spectral theorem. If $\sigma(A) \neq \{\text{pt}\}$, then we can find nonzero *self-adjoint* operators B_1 and B_2 in $C^*(I, A)$ so that $B_1B_2 = B_2B_1 = 0$. Thus $\langle B_1\xi, B_2\eta\rangle = 0$ for all $\xi, \eta \in \mathcal{H}_\pi$. In particular, the *closures* V_1 and V_2 of the ranges of B_1 and B_2, respectively, are closed, non-zero, orthogonal, invariant subspaces for π. This contradics the irreducibility of π; therefore $\sigma(A)$ must be a single point and therefore $A = \alpha I$ for some $\alpha \in \mathbb{C}$. For a general $A \in \pi(G)'$, we apply the above reasoning to AA^* and A^*A. Thus, we have $AA^* = \alpha I$ and $A^*A = \beta I$ with $\alpha, \beta > 0$ (since $A \neq 0$). Since $\alpha A = A(A^*A) = \beta A$, we have $\alpha = \beta$ and A is normal. This shows that (1) implies (2).

Since $V = \overline{\pi(G)\xi}$ is a closed, non-zero, invarinat subspace, in order to show that (2) implies (3) it will suffice to show that the orthogonal projection P onto any closed, non-zero, invarinat subspace V is in $\pi(G)'$. But since π is unitary, $V^\perp$ is also invariant. Thus if $\xi, \eta \in \mathcal{H}_\pi$, then

$$\begin{aligned}\langle P\pi(g)\xi, \eta\rangle &= \langle \pi(g)\xi, P\eta\rangle \\ &= \langle \pi(g)P\xi + \pi(g)(I-P)\xi, P\eta\rangle,\end{aligned}$$

which, since $\pi(g)(I-P)\xi \in V^\perp$ and $P\eta \in V$,

$$
\begin{aligned}
&= \langle \pi(g)P\xi, P\eta \rangle \\
&= \langle P\pi(g)P\xi, \eta \rangle \\
&= \langle \pi(g)P\xi, \eta \rangle,
\end{aligned}
$$

because $\pi(g)P\xi \in V$.

That (3) implies (1) is clear. $\square$

§2 The Peter-Weyl Theorem.

Now we'll specialize to compact groups G. Compact groups are characterized by the fact that any Haar measure μ on G satisfies $\mu(G) < \infty$. It is customary to normalize Haar measure on a compact group by choosing the unique measure such that $\mu(G) = 1$. Since there is now no possibility of confusion, I'll simply write

$$\int_G f(g)\, dg$$

for the integral of $f \in L^1(G)$.

Now suppose that π is a finite dimensional representation of G. If $b = \{ e_1, \dots, e_{d_\pi} \}$ is an orthonormal basis for $\mathcal{H}_\pi$, then for each $g \in G$ the matrix of $\pi(g)$ with respect to b has ij^{th} coordinate $\langle \pi(g)e_j, e_i\rangle$. The function $\phi_{ij}(g) = \langle e_i, \pi(g)e_j\rangle$ is called a coordinate function for π. (It will be convenient to use this convention—even though ϕ_{ij} is the complex conjugate of what you might expect. This usage and terminolgy will be justified, somewhat, by Remark 14 below.) Notice that $\phi_{ij} \in C(G)$. We'll write $\mathcal{E}_G$, or just $\mathcal{E}$ when no confusion is likely to arise, for the linear span of the all the functions $\phi(g) = \langle \xi, \pi(g)\eta\rangle$, where π ranges over all *irreducible* representations of G and ξ and η range over $\mathcal{H}_\pi$. Since *every* finite dimensional representation is the direct sum of irreducibles, notice that $\psi(g) = \langle \xi, \pi(g)\eta\rangle$ defines an element of $\mathcal{E}$ for every finite dimensional representation π—irreducible or not.

Remark 13: When G is abelian, and sometimes in general, the functions in $\mathcal{E}$ are called the trigonometric polynomials. The motivation for this probably comes from the case where $G = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then each $f \in \mathcal{E}_\mathbb{T}$ has the form

$$f(\theta) = \sum_{n=-k}^{n=k} c_n \exp(in\theta) = \sum_{n=0}^{k} d_n \cos(n\theta) + b_n \sin(n\theta).$$

□

Remark 14: It is clear that $\mathcal{E}$ is self-adjoint; that is, if $f \in \mathcal{E}$, then so is $f^* \in \mathcal{E}$, where $f^*(g) = \overline{f(g^{-1})}$. This is because matrix coefficients are themselves self-adjoint: $\phi^\pi_{ij} = (\phi^\pi_{ij})^*$. It is also true that if $f \in \mathcal{E}$, then so is $\check{f}$, where $\check{f}(g) = f(g^{-1})$. To see this we need to introduce the conjugate Hilbert space $\widetilde{\mathcal{H}}$ to a given Hilbert space $\mathcal{H}$. The space $\widetilde{\mathcal{H}}$ coincides with $\mathcal{H}$ as an additive group. If $j : \mathcal{H} \to \widetilde{\mathcal{H}}$ denotes the identity map, then the Hilbert space structure on $\widetilde{\mathcal{H}}$ is given by the formulas $\alpha j(\xi) = j(\overline{\alpha}\xi)$, and

$\langle j(\xi), j(\eta)\rangle_{\widetilde{\mathcal{H}}} = \langle \eta, \xi\rangle_{\mathcal{H}}$. If π is a given representation of G on $\mathcal{H}$, then we can define a representation $\widetilde{\pi}$ on $\widetilde{\mathcal{H}}$ in the obvious way: $\widetilde{\pi}(g)j(\xi) = j\big(\pi(g)\xi\big)$. The assertion follows from the fact that $\phi^{\widetilde{\pi}}_{ij}(g) = \phi^{\pi}_{ij}(g^{-1})$. Since $\phi^{\pi}_{ij}(g^{-1}) = \overline{\phi^{\pi}_{ji}(g)}$, it is reasonable to call $g \mapsto \langle \xi, \pi(g)\eta\rangle$ a coordinate function. □

Definition 15: *Let M_n be the $n \times n$ complex matrices. If $A = (a_{ij}) \in M_n$, then the Hilbert-Schmidt norm of A is*

$$\|(a_{ij})\|_{\text{H.S.}} = \sum_{ij} |a_{ij}|^2.$$

Remark 16: If $A \in M_n$, then $\|A\|_{\text{H.S.}} = \operatorname{tr}(A^*A)$. In particular, if $B = U^*AU$ for some unitary matrix U, then $\|A\|_{\text{H.S.}} = \|B\|_{\text{H.S.}}$. It follows that if π is finite dimensional, then $\|\pi(g)\|_{\text{H.S.}}$ is well defined and depends only on $[\pi]$. □

Our object here is to prove the following theorem known as the *Peter-Weyl Theorem.*

Theorem 17: *Let G be a compact group.*

(1) *Every irreducible representation of G is finite dimensional.*

(2) *If λ is the left-regular representation of G, then*

$$\lambda \cong \bigoplus_{[\pi]\in\widehat{G}} d_\pi \cdot \pi$$

(3) *Given $g \in G$, there is a $[\pi] \in \widehat{G}$ such that $\pi(g) \neq I$.*

(4) *$\mathcal{E}$ is dense in $C(G)$ (and hence in $L^p(G)$ for $1 \leq p < \infty$).*

(5) *If $f \in L^2(G)$, then*

$$\|f\|_2^2 = \sum_{[\pi]\in\widehat{G}} d_\pi \cdot \operatorname{tr}\big(\pi(f)\pi(f)^*\big) = \sum_{[\pi]\in\widehat{G}} d_\pi \cdot \|\pi(f)\|_{\text{H.S.}}.$$

We'll need the following preliminary results, some of which may be of interest by themselves.

Lemma 18: *Let π_1 and π_2 be representations of a locally compact group G. If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator satisfying $A\pi_1(g) = \pi_2(g)A$ for all $g \in G$, then $A^*A\pi_1(g) = \pi_1(g)A^*A$ for all $g \in G$.*

Proof: One computes as follows:

$$\begin{aligned}
\langle A^*A\pi_1(g)\xi, \eta\rangle &= \langle A\pi_1(g)\xi, A\eta\rangle \\
&= \langle \pi_2(g)A\xi, A\eta\rangle \\
&= \langle A\xi, \pi_2(g^{-1})A\eta\rangle \\
&= \langle A\xi, A\pi_1(g^{-1})\eta\rangle \\
&= \langle \pi_1(g)A^*A\xi, \eta\rangle
\end{aligned}$$

□

Lemma 19: *Suppose that π_1 and π_2 are representations of a locally compact group G, that π_1 is irreducible, and that $A : \mathcal{H}_1 \to \mathcal{H}_2$ is any* non-zero *bounded linear operator such that $A\pi_1(g) = \pi_2(g)A$ for all $g \in G$. Then $A\mathcal{H}_1$ is a* closed *invariant subspace for π_2, and $\pi_1 \cong \pi_2|_{A\mathcal{H}_1}$, where $\pi_2|_{A\mathcal{H}_1}$ is the subrepresentation of π_2 corresponding to $A\mathcal{H}_1$.*

Proof: By Lemma 18, $A^*A \in \pi_1(G)'$. By Theorem 12, $A^*A = \lambda I$. Thus $B = \lambda^{-\frac{1}{2}}A$ is an isometery, and hence a unitary from $\mathcal{H}_1$ onto $A\mathcal{H}_1$. (Note that $A\mathcal{H}_1$ is actually closed in $\mathcal{H}_2$ since A is a multiple of an isometery.) Again Lemma 18 shows that $BB^* \in \pi_2(G)'$. Since BB^* is the orthogonal projection onto $B\mathcal{H}_1 = A\mathcal{H}_1$, it follows that $A\mathcal{H}_1$ is invariant for π_2 and the assertion follows. □

The next result is crucial, and depends heavily on the fact that G is compact.

Proposition 20: *Suppose that π_1 and π_2 are irreducible representations of a compact group G. Fix orthonormal bases $\{ e^i_k \}_{k=1}^{d_{\pi_i}}$ for $\mathcal{H}_i$, and put $\phi^i_{kl}(g) = \langle e^i_k, \pi_i(g)e^i_l\rangle$.*

(1) *If $\pi_1 \ncong \pi_2$, then*

$$\int_G \phi^1_{ij}(g)\overline{\phi^2_{jl}(g)}\,dg = 0,$$

for all $1 \le i, j \le d_{\pi_1}$, and $1 \le k, l \le d_{\pi_2}$.

(2) *If π_1 is finite dimensional, then*

$$\int_G \phi^1_{ij}(g)\overline{\phi^1_{kl}(g)}\,dg = \delta_{ik}\delta_{jl}\frac{1}{d_{\pi_1}},$$

for all $1 \leq i, j, k, l \leq d_{\pi_1}$.

Proof: Let B be any bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then we can define

$$A = \int_G \pi_2(g) B \pi_1(g^{-1})\, ds.$$

Then

$$\begin{aligned} A\pi_1(r) &= \int_G \pi_2(g) B \pi_1(g^{-1}r)\, dg \\ &= \int_G \pi_2(rg) B \pi_1(g^{-1})\, dg = \pi_2(r) A. \end{aligned}$$

If $\pi_1 \not\cong \pi_2$, then $A = 0$ by Lemma 19. Now suppose that $B = B_{ij}$ is the rank-one operator defined by $B_{ij}(\xi) = \langle \xi, e_j^1 \rangle e_l^2$. Then,

$$\begin{aligned} 0 = \langle Ae_i^1, e_k^2 \rangle &= \int_G \langle B_{ij} \pi_1(g^{-1}) e_i^1, \pi_2(g^{-1}) e_k^2 \rangle\, dg \\ &= \int_G \langle \pi_1(g^{-1}) e_i^1, e_j^1 \rangle \langle e_l^2, \pi_2(g^{-1}) e_k^2 \rangle\, dg \\ &= \int_G \phi_{ij}^1(g) \overline{\phi_{kl}^2(g)}\, ds. \end{aligned}$$

This proves (1).

Now assume that $d_{\pi_1} < \infty$. By the above and Lemma 19,

$$A = \int_G \pi_1(g) B \pi_1(g^{-1})\, dg = \lambda I,$$

for any B. Taking traces,

$$\begin{aligned} \mathrm{tr}(A) &= \sum_{k=1}^{d_{\pi_1}} \langle Ae_k^1, e_k^1 \rangle \\ &= \int_G \sum_{k=1}^{d_{\pi_1}} \langle B\pi_1(g^{-1}) e_k^1, \pi_1(g^{-1}) e_k^1 \rangle\, dg \\ &= \int_G \mathrm{tr}(B)\, dg = \mathrm{tr}(B). \end{aligned}$$

Since $\mathrm{tr}(A) = \mathrm{tr}(\lambda I) = \lambda d_{\pi_1}$ and $\mathrm{tr}(B_{jl}) = \delta_{jl}$, we have $\lambda = \delta_{jl} \frac{1}{d_{\pi_1}}$ when $B = B_{jl}$. On the other hand,

$$\int_G \phi_{ij}^1(g) \overline{\phi_{kl}^1(g)}\, dg = \langle Ae_i^1, e_k^1 \rangle = \lambda \langle e_i^1, e_k^1 \rangle = \delta_{ik} \delta_{jl} \frac{1}{d_{\pi_1}}.$$

□

Proof of Theorem 17: Let $\mathcal{E}_f$ be the subset of $\mathcal{E}$ consisting of the collection of matrix coefficients of the form $\phi(g) = \langle \xi, \pi(g)\eta \rangle$ for π irreducible and $d_\pi < \infty$. The first part of the proof will consist of showing that it suffices to show that $\mathcal{E}_f$ is dense in $C(G)$. Since $C(G)$ is dense in $L^2(G)$, it follows that $L^2(G) = \mathcal{H}_\lambda$ has an orthonormal basis consisting of normalized matrix coefficients $d_\pi^{\frac{1}{2}} \phi_{ij}^\pi(g) = d_\pi^{\frac{1}{2}} \langle e_i^\pi, \pi(g) e_j^\pi \rangle$ (where, of course, $\{ e_1^\pi, \ldots, e_{d_\pi}^\pi \}$ denotes an orthonormal basis for $\mathcal{H}_\pi$).

In fact, if π is any irreducible representation of G and if $\{ e_\alpha \}_{\alpha \in A}$ is an orthonormal basis for $\mathcal{H}_\pi$, then it follows from Proposition 20 that $\phi_{\alpha\beta}^\pi \perp \phi_{ij}^\rho$, where $\phi_{\alpha\beta}^\pi(g) = \langle e_\alpha^\pi, \pi(g) e_\beta^\pi \rangle$ and $d_\rho < \infty$. Thus we must have $\phi_{\alpha\beta}^\pi = 0$, and (1) follows.

If $d_\pi < \infty$, then Proposition 20 shows that the d_π^2 functions $\{ \sqrt{d_\pi} \phi_{ij}^\pi \}$ are an orthonormal basis for a subspace $\mathcal{H}_{[\pi]}$ of $L^2(G)$. Now observe that

$$\begin{aligned} \phi_{ij}^\pi(g^{-1}t) &= \langle \pi(g) e_i^\pi, \pi(t) e_j^\pi \rangle \\ &= \sum_{k=1}^{d_\pi} \langle \pi(g) e_i^\pi, e_k^\pi \rangle \langle e_k^\pi, \pi(t) e_j^\pi \rangle \\ &= \sum_{k=1}^{d_\pi} \phi_{ik}^\pi(g^{-1}) \phi_{kj}^\pi(t). \end{aligned}$$

Therefore if, for each $1 \le j \le d_\pi$, we define $A_j : \mathcal{H}_\pi \to \mathcal{H}_{[\pi]}$ by $A_j e_i^\pi = \phi_{ij}^\pi$, then

$$\begin{aligned} \lambda(g)(A e_i^\pi)(t) &= \phi_{ij}^\pi(g^{-1}t) \\ &= \sum_{k=1}^{d_\pi} \phi_{ik}^\pi(g^{-1}) \phi_{kj}^\pi(t) \\ &= \sum_{k=1}^{d_\pi} \langle e_i^\pi, \pi(g^{-1}) e_k^\pi \rangle A_j e_k^\pi(t) \\ &= A\left(\sum_{k=1}^{d_\pi} \langle \pi(g) e_i^\pi, e_k^\pi \rangle e_k^\pi \right)(t) \\ &= A\big(\pi(g) e_i^\pi \big)(t). \end{aligned}$$

That is, A intertwines the irreducible representation π and the subrepresentation $\lambda|_{\mathcal{H}_{[\pi],j}}$, where $\mathcal{H}_{[\pi],j} = \operatorname{span}\{ \phi_{1j}^\pi, \phi_{2j}^\pi, \ldots, \phi_{d_\pi j}^\pi \}$. Therefore Lemma 19 implies that $\pi \cong \lambda|_{\mathcal{H}_{[\pi],j}}$, and thus, $\lambda|_{\mathcal{H}_{[\pi]}} \cong d_\pi \cdot \pi$. Since we're assuming that $\mathcal{E}_f$ is dense, we have

$$\mathcal{H} \cong \bigoplus_{[\pi] \in \widehat{G}} \mathcal{H}_{[\pi]},$$

and we have proved (2).

We have now shown that $\{\, d_{[\pi]}^{\frac{1}{2}}\phi_{ij}^{\pi}\,\}_{[\pi]\in\widehat{G}}$ forms an orthonormal basis for $L^2(G)$. (Under the assumption that $\mathcal{E}_f$ is dense in $C(G)$.) Thus, if $f \in L^2(G)$, we can write

$$f = \sum c\big([\pi], i, j\big)\, d_{[\pi]}^{\frac{1}{2}}\phi_{ij}^{\pi}.$$

Furthermore,

$$\|f\|_2^2 = \sum_{[\pi]\in\widehat{G}} \sum_{i,j=1}^{d_{[\pi]}} |c\big([\pi], i, j\big)|^2.$$

Now (5) follows from the fact that

$$\begin{aligned} c\big([\pi], i, j\big) &= \int_G f(g)\overline{d_{[\pi]}^{\frac{1}{2}}\phi_{ij}^{\pi}(g)}\, dg \\ &= d_{[\pi]}^{\frac{1}{2}} \int_G f(g)\langle \pi(g)e_j^{\pi}, e_i^{\pi}\rangle\, dg \\ &= d_{[\pi]}^{\frac{1}{2}} \langle \pi(f)e_j^{\pi}, e_i^{\pi}\rangle\, dg \\ &= d_{[\pi]}^{\frac{1}{2}} \big[\pi(f)\big]_{ij}. \end{aligned}$$

Of course, (3) follows from (4) (otherwise, $\mathcal{E}$ wouldn't separate e and g), so it only remains to prove that $\mathcal{E}_f$ is dense in $C(G)$. Towards this end, we need to recall some basic facts about so-called Hilbert-Schmidt operators. If $(X, \mathcal{M}, \mu)$ is a measure space and if $K \in L^2(X \times X, \mu \times \mu)$, then we can define a bounded linear operator $T : L^2(X) \to L^2(X)$ by

$$Tf(x) = \int_X K(x, y) f(y)\, dy.$$

It is not hard to see that T is self-adjoint if $K(x, y) = \overline{K(y, x)}$. An operator of this form is called a Hilbert-Schmidt operator and all such operators are self-adjoint compact operators[(1)]. In particular, each eigenspace

$$\mathcal{H}_\alpha = \{\, f \in L^2(X) : Tf = \alpha f\,\}$$

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(1) In our case, we'll only be interested in the case where $X = G$, μ is normalized Haar measure, and K is continuous. Then the Stone-Weierstrass Theorem implies that there are functions $\psi_i \in C(G)$ such that

$$K(x, y) = \sum_{i\in I} \alpha_i \psi_i(x)\overline{\psi_i(y)} \qquad (*)$$

is finite dimensional and there is an orthonormal sequence $\{ \phi_i \}$ of eigenvectors with eigenvalues α_i so that every $f \in L^2(X)$ can be written uniquely as

$$f = \sum c_i \phi_i + \phi_0,$$

where $T\phi_0 = 0$ and $c_i = \langle f, \phi_i \rangle$.

Our interest in such operators is as follows. Let k be any element of $C(G)$ which satisfies $k(g) = \overline{k(g^{-1})}$. Therefore

$$f * k(g) = \int_G f(r) k(r^{-1}g)\, dr = \int_G K(g,r) f(r)\, dr,$$

where $K(g,r) = k(r^{-1}g)$, is a self-adjoint Hilbert-Schmidt operator. Let $C = \|K\|_\infty = \max_{x,y \in G} |k(y^{-1}x)|$. Notice that $\|Tf\|_\infty \leq C\|f\|_1 \leq C\|f\|_2$. A moments reflections allows one to see that this implies that $Tf \in C(G)$. (Of course, Tf is only defined almost everywhere, but I mean it agrees almost everywhere with a continuous function on G. Since this function is uniquely determined, it makes sense to treat Tf as a continuous function. This is standard practice.) It follows that each eigenfunction of T is continuous.

Lemma 21: *Let k be as above and let T be the Hilbert-Schmidt operator on $L^2(G)$ defined by $Tf = f * k$. Then for each $\mu \in \mathbb{C} \setminus \{ 0 \}$*

$$\mathcal{H}_\mu = \{ f \in L^2(G) : Tf = \mu f \} \subseteq \mathcal{E}_f.$$

Proof: By the above remarks, $\mathcal{H}_\mu$ is finite dimensional and consists of continuous functions. Suppose that $f \in \mathcal{H}_\mu$. Then $T\big(\lambda(g)f\big) = \big(\lambda(g)f\big) * k = \lambda(g)\big(f * k\big) = \lambda(g)(Tf) = \mu\big(\lambda(g)f\big)$. That is, $\mathcal{H}_\mu$ is invariant for λ. Let $\{ f_1, \ldots, f_r \}$ be an orthonormal basis for $\mathcal{H}_\mu$. Define continuous functions ψ_{ki} by

$$\psi_{ki}(g) = \langle \lambda(g) f_i, f_k \rangle. \tag{1}$$

Since $\lambda(g) f_i \in \mathcal{H}_\mu$,

$$f_i(g^{-1}t) = \sum_{k=1}^{r} \psi_{ki}(g) f_k(t). \tag{2}$$

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uniformly. Notice that for each finite subset $F \subset I$ the operator T_F corresponding to

$$K_F(x,y) = \sum_{i \in F} \alpha_i \psi_i(x) \overline{\psi_i(y)}$$

is a finite rank operator. Since the convergence in $(*)$ is uniform, it follows that $T_F \to T$ in the operator norm; hence T is compact. (The remaining assertions in the paragraph follow from the Spectral Theorem.)

(A priori, Equation (2) is an equality in $L^2(G)$, and so would yeild pointwise equality only almost everywhere. But since both sides are continuous, the equality must hold everywhere.) Now define $\pi(g)$ to be the operator on $\mathcal{H}_\mu$ whose $r \times r$ matrix with respect to the basis $\{ f_1, \dots, f_r \}$ is $\big(\psi_{ij} \big)$. Since we have $\psi_{ij}(g) = \overline{\psi_{ji}(g^{-1})}$, it follows from Equation (1) that $\pi(g)^* = \pi(g^{-1})$. Similarly,

$$\begin{aligned} \psi_{ij}(gt) &= \langle \lambda(t) f_j, \lambda(g^{-1}) f_i \rangle \\ &= \sum_{k=1}^{r} \langle f_k, \lambda(g^{-1}) f_i \rangle \langle \lambda(t) f_j, f_k \rangle \\ &= \sum_{k=1}^{r} \psi_{ik}(g) \psi_{kj}(t). \end{aligned}$$

Therefore $\pi(gt) = \pi(g)\pi(t)$. It follows that π is a finite dimensional (unitary) represention of G. Using Equation (2), we see that

$$f_i(g) = \sum_{k=1}^{r} \psi_{ki}(g^{-1}) f_k(e) = \sum_{k=1}^{r} f_k(e) \phi(g),$$

where $\phi_{ki}(g) = \psi_{ki}(g^{-1}) = \langle f_i, \lambda(g) f_k \rangle$. We have shown that each f_i, and hence $\mathcal{H}_\mu$, is in the span of the matrix coefficients of finite dimensional representations of G. Since every finite dimensional representation is the direct sum of irreducible (finite dimensional) representaions, we have $\mathcal{H}_\mu \subseteq \mathcal{E}_f$ as desired. □

Lemma 22: *Let k and T be as above. If $f \in L^2(G)$, then $Tf \in \overline{\mathcal{E}_f}$.*

Proof: Let $\{ \phi_i \}$ be a complete orthonormal set of eigenvectors for the eigenspaces with non-zero eigenvalues of T. By the spectral theorem, we can write

$$f = \sum_k c_k \phi_k + \phi_0,$$

where $T\phi_0 = 0$ and $\|f\|_2^2 \le \sum_k |c_k|^2$. Let $T\phi_k = \alpha_k \phi_k$. Given $\epsilon > 0$, there is an N so that $\| \sum_{k>N} c_k \phi_k \|_2 < \epsilon$. In particular, $\|T\Big(\sum_{k>N} c_k \phi_k \Big)\|_2 \le C\epsilon$. But

$$T\Big(\sum_{k=1}^{N} c_k \phi_k \Big) = \sum_{k=1}^{N} \alpha_k c_k \phi_k \in \mathcal{E}_f$$

by Lemma 21. This suffices as

$$\|Tf - \sum_{k=1}^{N} \alpha_k c_k \phi_k\|_\infty = \|T\Big(\sum_{k>N} c_k \phi_k\Big)\|_\infty \leq C\epsilon.$$

□

The Peter-Weyl theorem now follows as $C(G)$ always contains a self-adjoint approximate identity. Specifically, we have the following.

Proposition 23: *If G is a compact group, then there is a net $\{ k_\alpha \}$ in $C(G)$ which satisfies $k_\alpha^* = k_\alpha$, and such that both $\{ k_\alpha * f \}$ and $\{ f * k_\alpha \}$ converge uniformly to f for each $f \in C(G)$.*

Proof: For each neighborhood U of $e \in G$, let k_U be a continuous non-negative function which satisfies $k_U(e) = 1$, $\int_G k_U(g)\,dg = 1$, $k_U^* = k_U$, and $\operatorname{supp} k \subseteq U$. Then $\{ k_U \}_U$ is a net in $C(G)$ directed by reverse inclusion: i.e., $U \geq V$ if and only if $U \subseteq V$.

Fix $f \in C(G)$. Since G is compact, given $\epsilon > 0$, there is a neighborhood W of $e \in G$ such that $|f(t^{-1}g) - f(g)| < \epsilon$ for all $g \in G$ and $t \in W$. Therefore if $U \geq W$, then

$$\begin{aligned} |k_U * f(g) - f(g)| &= \left| \int_G k_U(t) f(t^{-1}g)\,dt - f(g) \right| \\ &\leq \int_G k_U(t) \big| f(t^{-1}g) - f(g) \big|\,dt \\ &\leq \epsilon \int_G k_U(t)\,dt = \epsilon. \end{aligned}$$

It follows that $k_U * f \to f$ uniformly, as claimed. On the other hand, $(k_U * f)^* = f^* * k_U$ implies that $f * k_U \to f$ uniformly as well. This proves the lemma. □

Let T_U be the Hilbert-Schmidt operator corresponding to k_U. Thus, $T_U f$ converges to f uniformly for each $f \in C(G)$. Thus, $f \in \mathcal{E}_f$ by Lemma 22, and Theorem 17 is proved.

□

References

[1] Elias Stein, *Topics in harmonic analysis, related to the Littlewood-Paley theory*, Annals of mathematics studies, no. 63, Princeton University Press, New Jersey, 1970