

Introduction

This book is meant to provide the tools necessary to begin doing research involving crossed product C^* -algebras. Crossed products of operator algebras can trace their origins back to statistical mechanics, where crossed products were called covariance algebras, and to the group measure space constructions of Murray and von Neumann. Now the subject is fully developed with a vast literature. Crossed products provide both interesting examples of C^* -algebras and are fascinating in their own right. Simply put, a crossed product C^* -algebra is a C^* -algebra $A \rtimes_{\alpha} G$ built out of a C^* -algebra A and a locally compact group G of automorphisms of A . When the C^* -algebra A is trivial, that is, when A is the complex numbers, then the crossed product construction reduces to the group C^* -algebra of harmonic analysis. In fact, the subject of crossed product C^* -algebras was introduced to me as the harmonic analysis of functions on a locally compact group taking values in a C^* -algebra. This is a valuable analogy and still serves as motivation for me and for the approach in this book.

The subject of crossed products is now too massive to be covered in a single volume. This is especially true for this treatment as I have tried to write at a level suitable for graduate students who are just beginning to search out research areas, and as I also want to make the treatment reasonably self-contained modulo a modest set of prerequisites (to be described below). As a result, it has been necessary to leave out many important topics. A list of what is covered is given below in the “Reader’s Guide”. (A brief discussion of what is not covered is given under “Further Reading” below.) In choosing what to include I have been guided by my own interests and bounded by my ignorance. Thus the central theme of this book is to uncover the ideal structure of crossed products via the Mackey machine as extended to C^* -algebras by Rieffel and to crossed products by Green. Crossed products, just as group C^* -algebras, arise via a subtle completion process, and a detailed structure theorem determining the isomorphism classes of such algebras is well out of reach. Instead, we try to understand these algebras via their representations. Thus a key objective is to determine the primitive ideals of a given crossed product and the topology on its primitive ideal space. (We settle for primitive ideals in the general case, since it is not practicable to try to collect the same sort of information about irreducible representations of non-type I algebras.) If we have an abelian C^* -algebra $B = C_0(X)$, then the spectrum and primitive ideal space coincide with X as *topological* spaces. Thus, a description of X and its topology contains everything there is to know about $C_0(X)$. If B is arbitrary, then the primitive ideal space

$\text{Prim } B$ is a modest invariant, but it still contains a great deal of information about B . For example, we can recover the lattice of ideals of B . (Hence the term “ideal structure”.)

The basic paradigm for studying $\text{Prim}(A \rtimes_{\alpha} G)$ is to recover the primitive ideals via induction from the algebras $A \rtimes_{\alpha|_H} H$ where H is a closed subgroup of G . Loosely speaking, this process is the “Mackey machine” for crossed products. The centerpiece for this is the GRS-Theorem (“GRS” stands for Gootman, Rosenberg and Sauvageot), and we devote a fair chunk of this book to its proof (Chapter 9 and several appendices).

Errata

There are mistakes and typographical errors in this book. I only wish that the previous sentence were one of them. As I become aware of typographical errors and other mistakes, I will post them at <http://math.dartmouth.edu/cpsa>. If you find a mistake that is not listed there, I would be grateful if you would use the e-mail link provided there to send me a report so that I can add your contribution to the list.

Prerequisites and Assumptions

I have tried to keep the required background to a minimum in order to meet the goal of providing a text with which a graduate student in operator algebras can initiate an investigation of crossed products without having to consult outside sources. I do assume that such a student has had the equivalent of a basic course in C^* -algebras including some discussion of the spectrum and primitive ideal space. For example, the first few chapters of Murphy’s book [110] together with Appendix A of [139] should be enough. Since it is a bit harder to pick up the necessary background on locally compact groups, I have included a very brief introduction in Chapter 1. Of course, this material can also be found in many places, and Folland’s book [56] is a good source. Rieffel’s theory of Morita equivalence is assumed, and I will immodestly suggest [139] as a reference. Anyone interested in the details of the proof of the GRS-Theorem will have to sort out Borel structures on analytic Borel spaces. While some of that material can be found in the appendices, there is no better resource than Chapter 3 of Arveson’s beautiful little book [2]. Of course, I also assume a good deal of basic topology and functional analysis, but I have tried to give references when possible.

I have adopted the usual conventions when working in the subject. In particular, all homomorphisms between C^* -algebras are assumed to be $*$ -preserving, and ideals in C^* -algebras are always closed and two-sided. Unless otherwise stated, a representation of a C^* -algebra on a Hilbert space is presumed to be nondegenerate, and our Hilbert spaces are all complex.

I have tried to use fairly standard notation, and I have provided a “Notation and Symbol Index” as well as the usual index. The canonical extension of a representation π of an ideal I in an algebra A to A is usually denoted by $\bar{\pi}$. When using the notation sH for the cosets becomes too cumbersome, I will use \dot{s} instead.

If X is a locally compact Hausdorff space, then $C(X)$, $C^b(X)$, $C_0(X)$, $C_c(X)$ and $C_c^+(X)$ denote, respectively, the algebra of all continuous complex-valued continuous functions on X , the subalgebra of bounded functions in $C(X)$, the subalgebra of functions in $C^b(X)$ which vanish at infinity, the subalgebra of all functions in $C_0(X)$ which have compact support and the cone in $C_c(X)$ of nonnegative functions.

In general, I have not made separability assumptions unless, such as in the case in the proof of the GRS-Theorem, they cannot be avoided. With the possible exception of Section 4.5, this causes little extra effort. Some arguments do use nets in place of sequences, but this should cause no undue difficulty. (I suggest Pedersen's [127, §1.3] as a good reference for nets and subnets.)

Reader's Guide

Chapter 1 provides a very quick overview of the theory of locally compact groups and Haar measure. As we will repeatedly want to integrate continuous compactly supported functions taking values in a Banach space, we review the basics of the sort of vector-valued integration needed. More information on groups can be found in Appendix D, and for those who want a treatment of vector-valued integration complete with measurable functions and the like, I have included Appendix B. Anyone who is comfortable with locally compact groups, and/or the integrals in the text, may want to skip or postpone this chapter.

In Chapter 2, we define dynamical systems and their associated crossed products. We show that the representations of the crossed product are in one-to-one correspondence with covariant representations of the associated dynamical system. We also prove that a crossed product $A \rtimes_{\alpha} G$ can be defined as the C^* -algebra generated by a universal covariant homomorphism. We talk briefly about examples, although it is difficult to do much at this point. Working out the details of some of the deeper examples will have to wait until we have a bit more technology.

In Chapter 3, partially to give something close to an example and partially to provide motivation, we take some time to work out the structure of the group C^* -algebra of both abelian and compact groups. We also look at some basic tools needed to work with crossed products. In particular, we show that if G is a semidirect product $N \rtimes_{\varphi} H$, then the crossed product $A \rtimes_{\alpha} G$ decomposes as an iterated crossed product $(A \rtimes_{\alpha|_N} N) \rtimes_{\beta} H$. We also prove that G -invariant ideals I in A correspond naturally to ideals $I \rtimes_{\alpha} G$ in $A \rtimes_{\alpha} G$.

In Chapter 4, we turn to the guts of the Mackey machine. One of the main results is Raeburn's Symmetric Imprimitivity Theorem, which provides a common generalization of many fundamental Morita equivalences in the subject and implies the imprimitivity theorems we need to define and work with induced representations. Since this finally provides the necessary technology, we give some basic examples of crossed products. In particular, we show that for any group G , $C_0(G) \rtimes_{\text{lt}} G$ is isomorphic to the compact operators on $L^2(G)$. More generally, we also show that $C_0(G/H) \rtimes_{\text{lt}} G$ is isomorphic to $C^*(H) \otimes \mathcal{K}(L^2(G/H, \beta))$, where β is any quasi-invariant measure on the homogeneous space G/H . This result is a bit unusual, as it requires we use a suitably measurable cross section for the natural map of G onto G/H . This briefly pulls us away from continuous compactly supported functions

and requires some fussing with measure theory.

In Chapter 5, we define induced representations of crossed products and develop the properties we'll need in the sequel. We also introduce the important concept of inducing ideals, and we show that the induction processes are compatible with the decomposition of crossed products with respect to invariant ideals.

In Chapter 6, we expand on our preliminary discussion of orbit spaces in Section 3.5 and add quasi-orbits and the quasi-orbit space to the mix. A particularly important result is the Mackey-Glimm dichotomy for the orbit space for a second countable locally compact group G action on a second countable, not necessarily Hausdorff, locally compact space X . Simply put, the orbit space is either reasonably well-behaved or it is awful. When the orbit space is awful, it is often necessary to pass to quasi-orbits, and we discuss this and its connection with the restriction of representations.

In Chapter 7, we get to the heart of crossed products and prove a number of fundamental results. In Section 7.1, we prove the Takai Duality Theorem, which is the analogue for crossed products by abelian groups of the Pontryagin Duality Theorem. In Section 7.2, we look at the reduced crossed product and show that in analogy with the group C^* -algebra case, the reduced and universal crossed products coincide when G is amenable. (The necessary background on amenable groups is given in Appendix A.) In Section 7.3, we look at the special case where the algebra A is the algebra of compact operators. Since this leads naturally to a discussion of projective and cocycle representations (discussed in Appendix D.3), I couldn't resist including a short aside on twisted crossed products. This also allows us to show that, in analogy with the decomposition result for semidirect products in Section 3.3, we can decompose $A \rtimes_{\alpha} G$ into an iterated *twisted* crossed product whenever G contains a normal subgroup N . In Section 7.5, we give a very preliminary discussion of when a crossed product is GCR or CCR. This is a ridiculously difficult problem in general, so our results in this direction are very modest. We have more to say in Section 8.3 about the case of a transformation group C^* -algebra $C_0(X) \rtimes_{\text{lt}} G$ with G abelian.

In Chapter 8, we take on the ideal structure of crossed products. In Section 8.1, we see that one can obtain fairly fine information if the action of G on $\text{Prim } A$ is "nice". (In the text, the formal term for nice is "regular". In the literature, the term "smooth" is also used.) In Section 8.2, we face up to the general case. When (A, G, α) is separable and G is amenable, we can use the GRS-Theorem to say quite a bit. However, the proof of this result is very difficult and occupies all of Chapter 9 (and a few appendices). In Chapter 8, we merely concentrate on some of its wide ranging implications. In particular, we devote Section 8.3 to a detailed analysis of the ideal structure of $C_0(X) \rtimes_{\text{lt}} G$ when G is abelian. Although extending the results in Section 8.3 to cases where either G or A is nonabelian appears to be very formidable, this section provides a blueprint from which to start.

The remainder of this book consists of appendices that are meant to be read "as needed" to provide supplements where material is needed which is not part of the prerequisites mentioned above. As a result, I have definitely *not* tried to completely avoid overlap. Appendix A gives a brief overview of the properties of amenable groups which are needed in our treatment of reduced crossed products, and in the proof of the GRS-Theorem. Appendix B is a self-contained treatment of

vector-valued integration on locally compact spaces. Although we make passing use of this material in Chapter 9, I have included this appendix primarily to satisfy those who would prefer to think of $A \rtimes_{\alpha} G$ as a completion of $L^1(G, A)$. This material may be useful to anyone who wishes to extend the approach here to Busby-Smith twisted crossed products.

Appendix C is a self-contained treatment of $C_0(X)$ -algebras. These algebras are well known to be a convenient way to view an algebra as being fibred over X . In addition to developing the key properties of these kinds of algebras, we prove an old result of Hofmann's which implies, at least after translating his results onto contemporary terminology, that any $C_0(X)$ -algebra is the section algebra of a bona fide bundle which we call an upper semicontinuous-bundle of C^* -algebras over X .

Appendix D contains additional information about groups — particularly Borel structure issues. Appendices E and F give considerable background on representations of C^* -algebras and on direct integrals in particular. The direct integral theory is needed for the proof of the GRS-Theorem. In particular, we need direct integrals to discuss Effros's decomposition result for representations of C^* -algebras, called an ideal center decomposition, which is proved in Appendix G and which is essential to the proof of the GRS-Theorem.

Appendix H is devoted to a discussion of the Fell topology on the closed subsets of a locally compact space and its restriction to the closed subgroups of a locally compact group. This material is essential to our discussion of the topology on the primitive ideal space of $C_0(X) \rtimes_{\text{lt}} G$ with G abelian and to the proof of the GRS-Theorem.

Further Reading

As I mentioned above, no book at this level, or perhaps any level, could adequately cover all there is to talk about when it comes to crossed products. Even a modest expository discussion of the material that I wished I could have included would involve several chapters, or even a short book by itself. Instead, I will just give a short list of topics and references to some of the major omissions. There is a good deal in Pedersen's classic book [126] that won't be found here, and Green's original paper [66] and Echterhoff's *Memoir* [38] are full of ideas and results about the Mackey Machine for crossed products. In fact, both [66] and [38] work with twisted crossed products $A \rtimes_{\alpha}^{\tau} G$, which are quotients of $A \rtimes_{\alpha} G$, and which get only a brief aside in this text. There is even another flavor of twisted crossed products, called Busby-Smith twisted crossed products, which is not dealt with at all [15, 100, 119–121]. One serious research question about crossed products is under what conditions is a crossed product simple. This is particularly important today with the intense study of separable, nuclear, purely infinite nuclear C^* -algebras [89, 90, 128, 152]. One of the tools for this sort of question is the Connes spectrum. Its relationship to crossed products was studied extensively in [63, 114–116]. Many of the key points can be found in [126]. The fundamental obstacle to extending many of the results for transformation group C^* -algebras to the general case is the appearance of Mackey obstructions as touched on in Section 7.3. Understanding the ideal structure in this case requires a subtle analysis involving the symmetrizer subgroup of the stability

group. For example, see [42, 83] and [38] in particular. We have nothing here to say about the K -theory of crossed products. This is a difficult subject, and the key results here are the Pimsner-Viculescu six-term exact sequence for \mathbf{Z} -actions, the Thom isomorphism for \mathbf{R} -actions, and a variety of results about embedding certain transformation group C^* -algebras into AF C^* -algebras. Summaries and references for these results can be found in Blackadar's treatise [8]. As we shall see in Chapter 4, proper actions of groups on spaces are especially important and there is a bit of an industry on extending this notion to actions of groups on noncommutative C^* -algebras [79, 80, 107, 150, 151]. Also, there was no time to discuss Morita equivalence of dynamical systems and its implications [16, 19, 20, 41, 79, 95, 122, 172], nor was there space to talk about imprimitivity theorems for the reduced crossed product [78, 131]. We have not touched on coactions, their crossed products, and the powerful theory of noncommutative duality. An overview of non-abelian duality for reduced crossed products is provided in [40, Appendix A]. The theory for full crossed products is less well-developed, but this is a topic of current interest (see, for example, [85]). There is also a short introduction to the main ideas of the subject in [135].

Acknowledgments

This is survey of a huge subject, so almost all of the results in the book are due to someone besides the author. I have tried to give references for further reading, but I have not always been able to cite original sources. No doubt there are some glaring omissions, and I apologize for all omissions — glaring or not. I am very indebted to the people who taught me the subject. In particular, I am grateful to my advisor, Marc Rieffel, who pointed me to the subject in the first place, and to Phil Green whose work was both groundbreaking and inspirational. I owe a special debt to my friend Gert Pedersen whose inspiration extended far beyond mathematics. Naturally, I could have done nothing without the love and support of my wife and family.

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