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Equivariant Brauer and Picard groups and a Chase–Harrison–Rosenberg exact sequence ☆

David Crocker [a], Iain Raeburn [b], Dana P. Williams [c,*]

[a] *School of Mathematics, The University of New South Wales, NSW 2052, Australia*
[b] *School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia*
[c] *Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA*

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Abstract

The classical Chase–Harrison–Rosenberg exact sequence relates the Picard and Brauer groups of a Galois extension S of a commutative ring R to the group cohomology of the Galois group. We associate to each action of a locally compact group G on a locally compact space X two groups which we call the equivariant Picard group and the equivariant Brauer group. We then prove an analogue of the Chase–Harrison–Rosenberg exact sequence in the which the roles of the Picard and Brauer groups are played by their equivariant analogues.

Keywords: Equivariant Brauer group; Equivariant Picard group; Principal **T**-bundle

0. Introduction

Associated to each commutative ring R are two abelian groups: the Picard group $P(R)$ and the Brauer group $B(R)$. Loosely, $P(R)$ consists of the rank-one projective R-modules, and $B(R)$ consists of the Azumaya R-algebras A with center $Z(A) = R1_A$; in both cases, the group operation is based on $\otimes_R$.

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* Corresponding author.
E-mail addresses: d.crocker@unsw.edu.au (D. Crocker), iain.raeburn@newcastle.edu.au (I. Raeburn), dana.williams@dartmouth.edu (D.P. Williams).

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We now consider a subring R of a commutative ring S, and we suppose that S is a *Galois extension* of R: there is an action of a finite group G by automorphisms of S such that, among other things, R is the set S^G of fixed points for the action. In the case of interest to us, S is the ring of continuous functions on a compact space, the action of G is given by a continuous action of G on X, and S is a Galois extension of R when $R = C(G\backslash X)$ and G acts freely on X. The change-of-base map $A \mapsto S \otimes_R A$ induces a homomorphism of $B(R)$ into $B(S)$, whose kernel is called the *relative Brauer group* $B(S/R)$. The Chase–Harrison–Rosenberg exact sequence relates $P(R)$ and $B(S/R)$ to the group cohomology of G with coefficients in the group $U(S)$ of units in S: there are homomorphisms α_i such that

$$\begin{array}{l} 1 \longrightarrow H^1\big(G, U(S)\big) \xrightarrow{\alpha_1} P(R) \xrightarrow{\alpha_2} P(S)^G \xrightarrow{\alpha_3} \\ H^2\big(G, U(S)\big) \xrightarrow{\alpha_4} B(S/R) \xrightarrow{\alpha_5} H^1\big(G, P(S)\big) \xrightarrow{\alpha_6} H^3\big(G, U(S)\big) \end{array} \tag{1}$$

is an exact sequence. This is proved in [2, Corollary 5.5] or [4, Theorem IV.1.1], and a more general version is proved in [1, Corollary 6.2.3 and Eq. (6.17)].

When $S = C(X)$, the Picard group $P(C(X))$ is isomorphic to the Čech cohomology group $\check{H}^2(X; \mathbf{Z})$ of the underlying space, and the Brauer group $B(C(X))$ is isomorphic to the torsion subgroup of $\check{H}^3(X; \mathbf{Z})$ [9]. The elements of $B(C(X))$ can be viewed as the homogeneous C^*-algebras with spectrum X, which are all unital because X is compact. When one adds more general nonunital continuous-trace C^*-algebras to the mix, one obtains a Brauer group $\mathrm{Br}(X)$ which is isomorphic to all of $\check{H}^3(X; \mathbf{Z})$, and elements of the Picard group $P(C(X))$ can be concretely realized as $C(X)$-automorphisms of elements of $\mathrm{Br}(X)$. (These are classical theorems of Dixmier and Douady [5] and Phillips and Raeburn [14], and are given more modern proofs in [17, Theorems 6.3 and 5.42].)

In [3], we showed that the Brauer group has a natural equivariant version: for each transformation group (G, X), there is a *equivariant Brauer group* $\mathrm{Br}_G(X)$ whose elements are represented by actions of G on continuous-trace algebras with spectrum X which induce the given action of G on X. The main theorem of [3] is a structure theorem for $\mathrm{Br}_G(X)$ which identifies a filtration of $\mathrm{Br}_G(X)$ in terms of group cohomology [3, Theorem 5.1]. Here we define an equivariant Picard group $\mathrm{Pic}_G(X)$, and prove that there is an analogue of the Chase–Harrison–Rosenberg exact sequence in which the role of $P(S)$ is played by $\mathrm{Pic}_G(X)$ and the role of the relative Brauer group is played by the kernel of the homomorphism $F : \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ which forgets the G-action.

Our exact sequence is valid for more or less arbitrary continuous actions of a locally compact group on a locally compact space, though when G is not discrete, we have to use in place of ordinary group cohomology the theory developed by Moore [10,11] for actions of locally compact groups on Polish modules (see Theorem 4). When the action of G is free and proper, we can think of the algebra $C_0(X)$ (which is a C^*-algebra with spectrum X) as a Galois extension of $C_0(G\backslash X)$, and we recover a sequence which looks exactly like the original sequence (1) (see Corollary 7). Our constructions and arguments, however, rely heavily on our knowledge of actions of locally compact groups on continuous-trace C^*-algebras. It is intriguing to wonder whether there are purely algebraic versions of our equivariant groups for which more traditional algebraic methods will suffice.

We begin in Section 1 by introducing our equivariant Picard group, which is based on the realization of $P(C(X))$ as isomorphism classes of principal $\mathbf{T}$-bundles over X. We state and prove our main theorem in Section 2. Many of the homomorphisms needed for our exact sequence were constructed in [3], and we are able to deduce much of what we need from the main theorem of [3]. In the last section, we discuss what happens for free and proper actions and for trivial actions, and we also obtain a curious purely topological result about actions of **R**.

0.1. Conventions

Throughout, (G, X) will be a second countable locally compact transformation group in which G acts on the left. We shall refer to a locally trivial principal **T**-bundle simply as a **T**-bundle; this should not be confusing, since a theorem of Gleason [7] implies that every free action of **T** gives rise to a locally trivial bundle, and hence the various definitions of principal bundle coincide for the group **T**. We write ze for the action of $z \in \mathbf{T}$ on $e \in E$, and reserve the notation $s \cdot x$ for the action of $s \in G$ on $x \in X$.

1. The equivariant Picard group

We denote by $\operatorname{Pic}(X)$ the set of isomorphism classes $[p, E]$ of principal **T**-bundles $p : E \to X$. If $p : E \to X$ and $q : F \to X$ are principal **T**-bundles, then the quotient of the fibred product

$$E \underset{\mathbf{T}}{\times} F = \{(e, f) \in E \times F \colon\ p(e) = q(f)\}$$

by the **T**-action $z(e, f) := (ze, \bar{z}f)$ is also a principal **T**-bundle $(p * q, E * F)$ with **T**-action $z[e, f] := [ze, f]$. With $[p, E][q, F] := [p * q, E * F]$, $\operatorname{Pic}(X)$ becomes an abelian group, which we call the *Picard group of* X. A continuous map $f : X \to Y$ induces via the pull-back construction a homomorphism $f^* : \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$, and of course this assignment is functorial.

Remark 1. When X is a compact space, $\operatorname{Pic}(X)$ is isomorphic to the usual Picard group $P(C(X))$ of the ring $C(X)$ of complex-valued continuous functions on X. To see this, first use Swan's theorem to identify projective $C(X)$-modules with complex vector bundles over X, and observe that the rank-one modules correspond to the complex line bundles. These have structure group $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$, and are in one-to-one correspondence with the principal $\mathbf{C}^*$-bundles over X. Now the inclusion of **T** in $\mathbf{C}^*$ induces an isomorphism of the group of isomorphism classes of principal **T**-bundles over X onto the group of isomorphism classes of principal $\mathbf{C}^*$-bundles over X. (One way to see this last point is to verify by direct calculations that the corresponding inclusion $\mathcal{T} \to \mathcal{C}^*$ of sheaves induces an isomorphism of $H^1(X, \mathcal{T})$ onto $H^1(X, \mathcal{C}^*)$.) Since $H^1(X, \mathcal{T})$ is naturally isomorphic to the Čech cohomology group $\check{H}^2(X, \mathbf{Z})$ (see [3, Theorem 4.42], for example), we have $\operatorname{Pic}(X) \cong \check{H}^2(X, \mathbf{Z})$.

We say that a **T**-bundle $p : E \to X$ is *equivariant* if there is a left G-action, usually denoted by lt, of G on the total space E such that

$$\mathrm{lt}_s(ze) = z\,\mathrm{lt}_s(e) \quad \text{and} \quad p\big(\mathrm{lt}_s(e)\big) = s \cdot p(e)$$

for $z \in \mathbf{T}$, $e \in E$ and $s \in G$. Two equivariant **T**-bundles $p : E \to X$ and $q : F \to X$ are isomorphic if there is a G-equivariant bundle isomorphism $\varphi : E \to F$.

Proposition 2. *Suppose (p, E, lt) and (q, F, lt') are equivariant* $\mathbf{T}$*-bundles. Then the formula* $(\mathrm{lt} * \mathrm{lt}')_s([e, f]) = [\mathrm{lt}_s(e), \mathrm{lt}'_s(f)]$ *induces a left action of G on the* $\mathbf{T}$*-bundle $p * q$, and $(p * q, E * F, \mathrm{lt}'')$ is then equivariant. The set $\mathrm{Pic}_G(X)$ of isomorphism classes of equivariant* $\mathbf{T}$*-bundles is an abelian group with $[p, E, \mathrm{lt}][q, F, \mathrm{lt}'] = [p * q, E * F, \mathrm{lt} * \mathrm{lt}']$, which we call the* equivariant Picard group of (G, X).

The following observation makes it easy to tell when two equivariant bundles are isomorphic; it is a minor variation on [12, Lemma 1.12], for example. We use it without comment in the proof of Proposition 2.

Lemma 3. *Suppose that (p, E, lt) and (q, F, lt') are equivariant* $\mathbf{T}$*-bundles over X, and that $\varphi : E \to F$ is a continuous equivariant map such that $q \circ \varphi = p$. Then φ is a bundle isomorphism and $[p, E, \mathrm{lt}] = [q, F, \mathrm{lt}']$ in $\mathrm{Pic}_G(X)$.*

Proof of Proposition 2. Given (p, E, lt) and (q, F, lt') as above, there is a natural G action $\mathrm{lt} * \mathrm{lt}'$ on the product $(p * q, E * F)$ given by

$$(\mathrm{lt} * \mathrm{lt}')_s[e, f] := \big[\mathrm{lt}_s(e), \mathrm{lt}'_s(f)\big]. \tag{2}$$

Thus $(p * q, E * F, \mathrm{lt} * \mathrm{lt}')$ defines a class in $\mathrm{Pic}_G(X)$. Using Lemma 3, it is not hard to check that this class depends only on the classes of (p, E, lt) and (q, F, lt'), and that (2) defines a commutative and an associative product on $\mathrm{Pic}_G(X)$.

If $E = X \times \mathbf{T}$ is the trivial bundle, then we can let G act on the left of E by translation in the first factor. It takes a few straightforward computations to see that the class of this bundle acts as a multiplicative identity in $\mathrm{Pic}_G(X)$.

If (p, E, lt) is an equivariant $\mathbf{T}$-bundle, let $(p^{\mathrm{op}}, E^{\mathrm{op}}, \mathrm{lt}^{\mathrm{op}})$ be the equivariant $\mathbf{T}$-bundle built as follows. As a topological space, E^{op} is equal to E and $p^{\mathrm{op}}(e) = p(e)$. If e^{op} denotes $e \in E$ viewed as an element in E^{op}, then the $\mathbf{T}$- and G-actions are given by

$$\mathrm{lt}^{\mathrm{op}}_s\big(e^{\mathrm{op}}\big) := \big(\mathrm{lt}_s(e)\big)^{\mathrm{op}} \quad \text{and} \quad z e^{\mathrm{op}} = (\bar{z} e)^{\mathrm{op}}.$$

Since $e \mapsto [e, e^{\mathrm{op}}]$ is a $\mathbf{T}$-invariant map from E into $E * E^{\mathrm{op}}$, we get a section of $X \cong G\backslash E$ into $E * E^{\mathrm{op}}$. Thus the latter is trivial and $(\mathbf{T}e, z) \mapsto [ze, e]$ is a bundle isomorphism of $G\backslash E \times \mathbf{T}$ onto $E * E^{\mathrm{op}}$. Since $s \cdot (\mathbf{T}e) = \mathbf{T}(\mathrm{lt}_s(e))$, this map intertwines the natural G-action on $(G\backslash E) \times \mathbf{T}$ (letting G act via the given action on the first factor) and $\mathrm{lt} * \mathrm{lt}^{\mathrm{op}}$ on $E * E^{\mathrm{op}}$. Thus the class of $(p^{\mathrm{op}}, E^{\mathrm{op}}, \mathrm{lt}^{\mathrm{op}})$ is an inverse for the class of (p, E, lt) in $\mathrm{Pic}_G(X)$, and the latter is an abelian group. □

2. The exact sequence

Let $\mathrm{Br}_G(X)$ be the equivariant Brauer group defined in [3], whose elements are the Morita equivalence classes of pairs (A, α) consisting of a continuous-trace C^*-algebra A with spectrum X and an action α of G on A which induces the given action of G on the spectrum. The role of the relative Brauer group in our analogue of the Chase–Harrison–Rosenberg sequence is taken by the kernel of the forgetful homomorphism $F : \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ which sends $[A, \alpha]$ to $[A]$.

The cohomology groups appearing in our theorem are the ones defined and studied by Moore in [10,11]; there is a brief discussion of these groups and their properties in [17, §7.4]. The hypothesis of countability on $\mathrm{Pic}(X)$ implies that $\mathrm{Pic}(X)$ is a Polish group in the discrete topology, and is needed to ensure that $H^1(G, \mathrm{Pic}(X))$ makes sense. The hypothesis of second countability likewise ensures that $C(X, \mathbf{T})$ is Polish in the topology of uniform convergence on compacta. By the functoriality of Pic each homeomorphism lt_s of X induces an automorphism lt_s^* of $\mathrm{Pic}(X)$, and $\mathrm{Pic}(X)^G$ is the subgroup of elements of $\mathrm{Pic}(X)$ which are fixed by all these automorphisms.

Theorem 4. *Suppose that (G, X) is a second countable locally compact transformation group with $\mathrm{Pic}(X)$ countable. Then there are homomorphisms η_i such that the following sequence is exact*:

$$1 \longrightarrow H^1\big(G, C(X,\mathbf{T})\big) \xrightarrow{\eta_1} \mathrm{Pic}_G(X) \xrightarrow{\eta_2} \mathrm{Pic}(X)^G \xrightarrow{\eta_3}$$
$$H^2\big(G, C(X,\mathbf{T})\big) \xrightarrow{\eta_4} \ker F \xrightarrow{\eta_5} H^1\big(G, \mathrm{Pic}(X)\big) \xrightarrow{\eta_6} H^3\big(G, C(X,\mathbf{T})\big).$$

The proof of this theorem will occupy the rest of the section.

We begin by observing that, since $\mathrm{Pic}(X) \cong \check{H}^2(X, \mathbf{Z})$ (see Remark 1), countability of $\mathrm{Pic}(X)$ is equivalent to countability of $\check{H}^2(X, \mathbf{Z})$, and Theorem 5.1 of [3] applies. Now for η_3 we take the map d_2'' defined in the proof of [3, Theorem 5.1] (see also Section 2.2 below), and for η_4 the map ξ defined in part 3 of [3, Theorem 5.1], which also says that $\mathrm{im}\,\eta_3 = \ker \eta_4$. For η_5 we take the map η defined in part 2 of [3, Theorem 5.1], and part 3 says that $\mathrm{im}\,\eta_4 = \ker \eta_5$. For η_6 we take the homomorphism d_2' defined in [3, pp. 174–175], which by part 2 of [3, Theorem 5.1] satisfies $\mathrm{im}\,\eta_5 = \ker \eta_6$. So it remains for us to define the maps η_1 and η_2 and prove exactness of the sequence at the first three stops. Everything except exactness at $\mathrm{Pic}(X)^G$ is quite straightforward.

2.1. *The maps η_1 and η_2*

If $[p, E, \mathrm{lt}]$ is in $\mathrm{Pic}_G(X)$, then for every $s \in G$, the map $(e, x) \mapsto (\mathrm{lt}_{s^{-1}}(e), x)$ is an isomorphism of $\mathrm{lt}_s^*(p, E)$ onto $\mathrm{lt}_e^*(p, E) = (p, E)$ for all $s \in G$. Thus $[p]$ is fixed by every lt_s^*, and we can define

$$\eta_2 : \mathrm{Pic}_G(X) \to \mathrm{Pic}(X)^G \quad \text{by } \eta_2\big([p, \mathrm{lt}]\big) = [p].$$

To define η_1, we first define

$$\mu : Z^1\big(G, C(X, \mathbf{T})\big) \to \mathrm{Pic}_G(X);$$

for $c \in Z^1(G, C(X, \mathbf{T}))$ we define a left action lt^c on the trivial bundle $X \times \mathbf{T}$ by

$$\mathrm{lt}_s^c(x, z) := \big(s \cdot x, c(s)(s \cdot x)z\big),$$

and then define $\mu(c) = [X \times \mathbf{T}, \mathrm{lt}^c]$. (A Borel 1-cocyle with values in a Polish module is continuous by [10, Theorem 3]. Hence $c \in Z^1(G, C(X, \mathbf{T}))$ is continuous.) Notice that since the classes in $\ker \eta_2$ are precisely those whose underlying $\mathbf{T}$-bundle is trivial, we have $\mathrm{im}\,\mu \subset \ker \eta_2$. A computation shows that μ is a homomorphism. Since bundle automorphisms of $X \times \mathbf{T}$ are given by multiplication by continuous functions $w : X \to \mathbf{T}$, $\mu(c)$ is the identity in $\mathrm{Pic}_G(X)$ if and only if there is a continuous function $w : X \to \mathbf{T}$ such that $\varphi(x, z) := (x, zw(x))$ converts the

G-actions given by lt^c into lt^1, or in other words, such that w satisfies

$$\big(s\cdot x, zw(x)\big) = \mathrm{lt}^1_s\big(\varphi(x,z)\big) = \varphi\big(\mathrm{lt}_s(x,z)\big) = \big(s\cdot x, c(s)(s\cdot x)zw(s\cdot x)\big).$$

Thus $\mu(c)$ is the identity in $\mathrm{Pic}_G(X)$ if and only if there is a continuous function $w: X \to \mathbf{T}$ such that $w(x) = c(s)(s\cdot x)w(s\cdot x)$. Replacing $s\cdot x$ by y shows that $\mu(c)$ is the identity if and only if

$$c(s)(y) = w\big(s^{-1}\cdot y\big)w(y)^{-1} \quad \text{for } s \in G \text{ and } y \in X,$$

and hence if and only if c is a coboundary. Thus μ induces an injective homomorphism of $H^1(G, C(X,\mathbf{T}))$ into $\ker\eta_2 \subset \mathrm{Pic}_G(X)$, which we take to be η_1.

The injectivity of η_1 gives exactness at $H^1(G, C(X,\mathbf{T}))$. To check exactness at $\mathrm{Pic}_G(X)$, let $[p, \mathrm{lt}] \in \ker\eta_2$. Then we can assume that $p: X\times\mathbf{T} \to X$ is the trivial bundle, and there is therefore a continuous function $b: G\times X \to \mathbf{T}$ such that

$$\mathrm{lt}_s(x,z) = \big(s\cdot x, b(s,x)z\big). \tag{3}$$

Define $c: G \to C(X,\mathbf{T})$ by

$$c(s)(x) := b\big(s, s^{-1}\cdot x\big).$$

A straightforward computation using the relation $\mathrm{lt}_{st} = \mathrm{lt}_s \circ \mathrm{lt}_t$ shows that $c(st) = c(s)s\cdot c(t)$, so $c \in Z^1(G, C(X,\mathbf{T}))$, and (3) implies that $\eta_1([c]) = \mu(c) = [p, \mathrm{lt}]$.

2.2. *The map* η_3

We now recall from [3, Theorem 5.1] the construction of the map $\eta_3 := d_2''$. Let $A = C_0(X, \mathcal{K})$. Then [14, Theorem 2.1] allows us to identify the quotient $\mathrm{Aut}_{C_0(X)} A/\mathrm{Inn}\, A$ with $\check{H}^2(X;\mathbf{Z})$, and in view of Remark 1, with $\mathrm{Pic}(X)$. For $\varphi \in \mathrm{Aut}_{C_0(X)} A$, the associated class $\zeta(\varphi) \in \mathrm{Pic}(X)$ is represented by the spectrum $(A \rtimes_\varphi \mathbf{Z})^\wedge$ of the crossed product by the (necessarily locally unitary) action of $\mathbf{Z}$ generated by φ. Indeed, elements of $(A \rtimes_\varphi \mathbf{Z})^\wedge$ are represented by covariant pairs (π, V) consisting of an irreducible representation $\pi: A \to B(\mathcal{H})$ and a unitary $V \in U(\mathcal{H})$ such that

$$V\pi(a)V^* = \pi\big(\varphi(a)\big) \quad \text{for all } a \in A;$$

then the $\mathbf{T}$-action is given by $z(\pi, V) = (\pi, zV)$, and the bundle map is $\mathrm{Res}: \pi \times U \mapsto \pi \in \hat{A} = X$ (see [15, §2] for this description of $\zeta(\varphi)$). Denote by $\tau: G \to \mathrm{Aut}\, A$ the action of G defined by $\tau_s(f)(x) = f(s^{-1}\cdot x)$. Then it follows from [3, Lemma 4.4] that this identification intertwines the G-action on $\mathrm{Pic}(X)$ with the G-action on $\mathrm{Aut}_{C_0(X)} A/\mathrm{Inn}\, A$ given by conjugation with τ_s:

$$\mathrm{lt}^*_{s^{-1}}\big(\zeta(\varphi)\big) = \zeta\big(\tau_s \circ \varphi \circ \tau_s^{-1}\big). \tag{4}$$

Suppose that $\mathfrak{c} \in \mathrm{Pic}(X)^G$, and choose $\varphi \in \mathrm{Aut}_{C_0(X)} A$ such that $\zeta(\varphi) = \mathfrak{c}$. Since $\mathfrak{c}$ is G-invariant, it follows from (4) that φ and $\tau_s \circ \varphi \circ \tau_s^{-1}$ differ by an inner automorphism. Then

$$s \mapsto \tau_s \circ \varphi \circ \tau_s^{-1} \circ \varphi^{-1}$$

is a continuous function from G into $\mathrm{Aut}_{C_0(X)} A$ taking values in $\mathrm{Inn}\, A$. Since $\mathrm{Ad} : UM(A) \to \mathrm{Inn}\, A$ is a homomorphism of Polish groups, it has a Borel section. Hence there is a Borel map $s \mapsto u_s : G \to UM(A)$ such that

$$\mathrm{Ad}\, u_s \circ \tau_s \circ \varphi = \varphi \circ \tau_s. \tag{5}$$

Then we compute that

$$\begin{aligned} \mathrm{Ad}\, u_{st} \circ \tau_{st} \circ \varphi &= \varphi \circ \tau_s \circ \tau_t \\ &= \mathrm{Ad}\, u_s \circ \tau_s \circ \varphi \circ \tau_t \\ &= \mathrm{Ad}\, u_s \circ \tau_s \circ \mathrm{Ad}\, u_t \circ \tau_t \circ \varphi \\ &= \mathrm{Ad}\big(u_s \tau_s(u_t)\big) \circ \tau_{st} \circ \varphi. \end{aligned}$$

Thus $\mathrm{Ad}\, u_{st} = \mathrm{Ad}(u_s \tau_s(u_t))$, and there exists $\omega \in Z^2(G, C(X, \mathbf{T}))$ such that

$$u_s \tau_s(u_t) = \omega(s,t) u_{st}.$$

It is shown in [3, p. 179] that $\mathfrak{c} \mapsto \omega$ is a well-defined homomorphism η_3 of $\mathrm{Pic}(X)^G$ into $H^2(G, C(X, \mathbf{T}))$.

Remark 5. Notice that ω is only a Borel function on $G \times G$. However, if ω is a coboundary, say $\omega = \partial b$, then $s \mapsto b(s) u_s$ is a Borel 1-cocycle taking values in a Polish space. Hence it is continuous by [10, Theorem 3].

2.3. $\ker \eta_3 \subset \mathrm{im}\, \eta_2$

Suppose, retaining the notation of the previous subsection, that $\zeta(\varphi) \in \ker \eta_3$. Then we can adjust the given choice of u so that there is a 1-cocycle $u : G \to C(X, \mathbf{T})$ satisfying (5); that is, u is continuous and satisfies (5) as well as $u_{st} = u_s \tau_s(u_t)$. To see that $\zeta(\varphi) \in \mathrm{im}\, \eta_2$, it will suffice to produce a G-action on $(A \times_\varphi \mathbf{Z})^\wedge$ making the latter into a G-equivariant $\mathbf{T}$-bundle. We define

$$\mathrm{lt}_s(\pi, V) := \big(s \cdot \pi, s \cdot \pi(u_s) V\big), \tag{6}$$

where $s \cdot \pi := \pi \circ \tau_s^{-1}$. To see that this suffices, we need to verify that the right-hand side of (6) is a covariant representation, that $\mathrm{lt}_{st} = \mathrm{lt}_s \circ \mathrm{lt}_t$ and that (6) defines a continuous map from $G \times (A \rtimes_\varphi \mathbf{Z})^\wedge \to (A \rtimes_\varphi \mathbf{Z})^\wedge$.

To check covariance, first note that (5) implies that

$$\tau_s^{-1} \circ \varphi = \varphi \circ \tau_s^{-1} \circ \mathrm{Ad}\, \varphi^{-1}(u_s). \tag{7}$$

Now we compute using (7):

$$\begin{aligned} s \cdot \pi\big(\varphi(a)\big) &= \pi\big(\tau_s^{-1}\big(\varphi(a)\big)\big) \\ &= \pi\big(\varphi\big(\tau_s^{-1}\big(\varphi^{-1}(u_s) a \varphi^{-1}(u_s)^*\big)\big)\big) \\ &= V \pi\big(\tau_s^{-1}\big(\varphi^{-1}(u_s) a \varphi^{-1}(u_s)^*\big)\big) V^* \\ &= V \pi\big(\tau_s^{-1}\big(\varphi^{-1}(u_s)\big)\big) s \cdot \pi(a) \big(V \pi\big(\tau_s^{-1}\big(\varphi^{-1}(u_s)\big)\big)\big)^*. \end{aligned}$$

Now we can use (5) in the form $\tau_s^{-1} \circ \varphi^{-1} = \varphi^{-1} \circ \tau_s^{-1} \circ \operatorname{Ad} u_s^*$ and the equation $\operatorname{Ad} u_s^*(u_s) = u_s$ to get

$$\begin{aligned} s \cdot \pi\big(\varphi(a)\big) &= V\pi\big(\varphi^{-1}\big(\tau_s^{-1}(u_s)\big)\big)s \cdot \pi(a)\big(V\pi\big(\varphi^{-1}\big(\tau_s^{-1}(u_s)\big)\big)\big)^* \\ &= \pi\big(\tau_s^{-1}(u_s)\big)Vs \cdot \pi(a)\big(\pi\big(\tau_s^{-1}(u_s)\big)V\big)^*. \end{aligned} \tag{8}$$

This shows that the right-hand side of (6) is a covariant representation. To see that (6) defines an action of G, we compute as follows:

$$\begin{aligned} \mathrm{lt}_s\big(\mathrm{lt}_t(\pi, V)\big) &= \mathrm{lt}_s\big(t \cdot \pi, \pi\big(\tau_t^{-1}(u_t)V\big)\big) \\ &= \big(st \cdot \pi, t \cdot \pi\big(\tau_s^{-1}(u_s)\big)\pi\big(\tau_t^{-1}(u_t)V\big)\big) \\ &= \big(st \cdot \pi, \pi \circ \tau_{st}^{-1}\big(u_s\tau_s(u_t)\big)V\big) \\ &= \big(st \cdot \pi, st \cdot \pi(u_{st})V\big) \\ &= \mathrm{lt}_{st}(\pi, V). \end{aligned}$$

To show that (6) defines a continuous action, we want to take advantage of the fact that φ is locally unitary so that $(A \rtimes_\varphi \mathbf{Z})^\wedge$ is locally trivial. Thus for each $\rho \in \hat{A}$, there is a neighborhood N of ρ and a unitary $v \in M(A)$ such that

$$\pi\big(\varphi(a)\big) = \pi(v)\pi(a)\pi(v)^* \quad \text{for all } \pi \in N.$$

In particular,

$$(\pi, z) \mapsto \big(\pi, z\pi(v)\big)$$

is a $\mathbf{T}$-equivariant isomorphism of $N \times \mathbf{T}$ onto $p^{-1}(N)$, where $p : (A \rtimes_\varphi \mathbf{Z})^\wedge \to \hat{A}$ is the restriction map. The covariance calculation (8) shows that

$$\begin{aligned} s \cdot \pi\big(\varphi(a)\big) &= \pi\big(\tau_s^{-1}(u_s)v\big)s \cdot \pi(a)\pi\big(\tau_s^{-1}(u_s)v\big)^* \\ &= s \cdot \pi(u_s v)s \cdot \pi(a)s \cdot \pi(u_s v)^* \end{aligned}$$

for all $\pi \in s^{-1} \cdot N$. This shows that $u_s v$ implements φ over $s^{-1} \cdot N$. Now suppose that $s_n \to s$ and $(\pi_n, V_n) \to (\pi_0, V_0)$. We can assume that each $\pi_n \in N$, and then (π_n, V_n) has the form $(\pi_n, z_n\pi_n(v))$. Thus $(\pi_n, z_n) \to (\pi_0, z_0)$ in $\hat{A} \times \mathbf{T}$. Then

$$\mathrm{lt}_{s_n}(\pi_n, V_n) = \big(s_n \cdot \pi_n, z_n s_n \cdot \pi_n(u_s v)\big),$$

and $\mathrm{lt}_{s_n}(\pi_n, V_n) \to (\pi_0, V_0)$ as required. This completes the proof that (6) makes $(A \rtimes_\varphi \mathbf{Z})^\wedge$ into an equivariant $\mathbf{T}$-bundle. Thus $\ker \eta_3 \subset \operatorname{im} \eta_2$, as claimed.

2.4. *Exactness at* $\mathrm{Pic}(X)^G$

It remains for us to see that $\operatorname{im}\eta_2 \subset \ker\eta_3$. So let $[p, E, \mathrm{lt}] \in \mathrm{Pic}_G(X)$; we want to see that $[p, E] = \eta_2([p, E, \mathrm{lt}])$ belongs to the kernel of η_3. We have to realize $[p, E]$ as the $\zeta(\varphi)$ for some $C_0(X)$-automorphism of $C_0(X, \mathcal{K})$, and we do this using the construction of [15]. Let φ_1 be the generator of the dual action on the crossed product $C_0(E) \rtimes \mathbf{T}$. Since $p : E \to X$ is a locally trivial principal $\mathbf{T}$-bundle, it follows from Theorem 14 and Corollary 15 of [8] that there is a $C_0(X)$-automorphism Θ of $C_0(E) \rtimes \mathbf{T}$ onto $C_0(X, \mathcal{K})$, and it follows from [15, Corollary 3.5] that $\zeta(\varphi_1) = [p, E]$. We take $\varphi := \Theta \circ \varphi_1 \circ \Theta^{-1}$.

Next we consider the action of G by left translation on $C_0(E)$. This commutes with the action of $\mathbf{T}$ used to define the crossed product, and hence induces an action $\mathrm{lt} \rtimes \mathrm{id}$ of G on $C_0(E) \rtimes \mathbf{T}$. It follows from [15, Lemma 3.3] that the induced action of G on $X = (C_0(E) \rtimes \mathbf{T})^\wedge$ is the given one. The action $\mathrm{lt} \rtimes \mathrm{id}$ fixes the copy of $\mathbf{T}$ in the crossed product (strictly, in $M(C_0(E) \rtimes G)$), and hence commutes with the dual automorphism φ_1; moving $\mathrm{lt} \rtimes \mathrm{id}$ over to $C_0(X, \mathcal{K})$ gives an action $\alpha := \Theta \circ (\mathrm{lt} \rtimes \mathrm{id}) \circ \Theta^{-1}$ of G on $C_0(X, \mathcal{K})$ which induces the given action on X and commutes with φ.

Theorem 14 of [8], which we used above to realize $C_0(E) \rtimes \mathbf{T}$ as $C_0(X, \mathcal{K})$, can also be viewed as saying that $C_0(E) \rtimes \mathbf{T}$ is Morita equivalent over X to $C_0(X)$. We want to argue now that the system $(C_0(E) \rtimes \mathbf{T}, G, \mathrm{lt} \rtimes \mathrm{id})$ is Morita equivalent to $(C_0(X), G, \tau)$. To see this, we recall from [18] that an imprimitivity bimodule can be obtained by completing $C_c(E)$, with

$$
\begin{aligned}
(g \cdot y)(e) &= \int_{\mathbf{T}} g(z, e) y(\bar{z}e)\, dz, \\
(y \cdot f)(e) &= y(e) f\big(p(e)\big), \\
\langle y, w \rangle_{C_0(X)}\big(p(e)\big) &= \int_{\mathbf{T}} \overline{y(\bar{z}e)} w(\bar{z}e)\, dz, \\
{}_{C_0(E) \rtimes \mathbf{T}}\langle y, w \rangle(z, e) &= y(e)\overline{w(\bar{z}e)},
\end{aligned}
$$

for $y, w \in C_c(E)$, $g \in C_c(\mathbf{T} \times E) \subset C_0(E) \rtimes \mathbf{T}$, and $f \in C_0(X)$. Since $\mathrm{lt} \rtimes \mathrm{id}$ is given on $C_c(\mathbf{T} \times E)$ by $(\mathrm{lt} \rtimes \mathrm{id})_s(g)(z, e) = g(z, \mathrm{lt}_s^{-1}(e))$, it is a routine matter to check that the maps $U_s : C_c(E) \to C_c(E)$ defined by $U_s(y)(e) = y(\mathrm{lt}_s^{-1}(e))$ extend to $\overline{C_c(E)}$ and then give an action of G on $\overline{C_c(E)}$ which implements the desired equivalence.

We now have two systems $(C_0(X, \mathcal{K}), G, \alpha)$ and $(C_0(X, \mathcal{K}), G, \tau)$ which define the same class in $\mathrm{Br}_G(X)$ as $(C_0(X), G, \tau)$ (which is the identity of $\mathrm{Br}_G(X)$). Thus it follows from [6, Proposition 5.1] that α and τ are exterior equivalent: there is a continuous function $v : G \to UM(C_0(X, \mathcal{K}))$ such that $v_s \alpha(v_t) = v_{st}$ and $\tau_s = \operatorname{Ad} v_s \circ \alpha_s$. We are finally ready to compute $\eta_3(\zeta(\varphi))$; the crucial point is that every α_s commutes with φ, because every $(\mathrm{lt} \rtimes \mathrm{id})_s$ commutes with φ_1. Thus

$$
\tau_s \circ \varphi \circ \tau_s^{-1} = \operatorname{Ad} v_s \circ \alpha_s \circ \varphi \circ \alpha_s^{-1} \circ \operatorname{Ad} v_s^* = \operatorname{Ad}\big(v_s \varphi\big(v_s^*\big)\big) \circ \varphi,
$$

and the unitary u_s in (5) is $u_s := \varphi(v_s) v_s^*$. We now let $s, t \in G$ and compute:

$$\begin{aligned} u_s\tau_s(u_t) &= \varphi(v_s)v_s^*\tau_s\big(\varphi(v_t)v_t^*\big) \\ &= \varphi(v_s)\alpha_s\big(\varphi(v_t)v_t^*\big)v_s^* \quad \text{(using that } \tau_s = \operatorname{Ad} v_s \circ \alpha_s\text{)} \\ &= \varphi\big(v_s\alpha_s(v_t)\big)\big(v_s\alpha_s(v_t)\big)^* \quad \text{(since } \alpha_s \circ \varphi = \varphi \circ \alpha_s\text{)} \\ &= \varphi(v_{st})v_{st}^* \\ &= u_{st}. \end{aligned}$$

Thus the cocycle ω representing $\eta_3(\zeta(\varphi))$ is identically 1, and $\eta_3(\zeta(\varphi)) = 0$.

This completes the proof of exactness at $\operatorname{Pic}(X)^G$, and hence also the proof of Theorem 4.

3. Applications

3.1. Free and proper actions

We suppose in this subsection that the group G acts freely and properly on X. Then the quotient $G\backslash X$ is also a locally compact space; let $q : X \to G\backslash X$ be the quotient map. We showed in [3, §6.2] that the pullback construction of [3, §6.2] gives an isomorphism $Q : [A] \mapsto [q^*A, q^*\,\mathrm{id}]$ of $\operatorname{Br}(G\backslash X)$ onto $\operatorname{Br}_G(X)$. When G is noncompact, the fixed-point algebra $C_0(X)^G$ contains only the zero function, but it is widely understood that the algebra $C_0(G\backslash X)$ is the natural substitute (see, for example, the discussion in [19]). Since the composition $F \circ Q$ is the homomorphism $q^* : \operatorname{Br}(G\backslash X) \to \operatorname{Br}(X)$ induced by q, we deduce that $\ker F$ is the correct analogue of the relative Brauer group $B(C_0(X)/C_0(G\backslash X)) := \ker q^*$. Thus in this situation we expect our exact sequence to take a form more directly resembling the original sequence of Chase, Harrison and Rosenberg.

For such a result to be completely satisfactory, we would want to replace also the equivariant Picard group with the analogue of $P(C_0(X)^G)$. So we look for an isomorphism based on the pull-back map. For $[p, E] \in \operatorname{Pic}(G\backslash X)$, the pull-back bundle

$$q^*E := \big\{(x, e)\colon\ q(x) = p(e)\big\}$$

is an equivariant $\mathbf{T}$-bundle over X with the action of G given by

$$\mathrm{lt}_s(x, e) := (s \cdot x, e). \tag{9}$$

Proposition 6. *Suppose that (G, X) is a second countable locally compact transformation group with G acting freely and properly. Let $q : X \to G\backslash X$ be the orbit map. Then there is a group isomorphism of* $\operatorname{Pic}(G\backslash X)$ *onto* $\operatorname{Pic}_G(X)$ *which sends the class of a* $\mathbf{T}$*-bundle (p, E) over $G\backslash X$ to* $[q^*(p, E), \mathrm{lt}]$ *with* lt *defined as in* (9).

Proof. Let $[p, E, \mathrm{lt}] \in \operatorname{Pic}_G(X)$. Then G must act freely on E, and since $\mathbf{T}$ is compact, it must act properly as well. In particular, $G\backslash E$ is locally compact Hausdorff. The natural maps

$$E \xrightarrow{\ p\ } X \xrightarrow{\ q\ } G\backslash X$$

induce a natural map

$$p_G : G\backslash E \to G\backslash X. \tag{10}$$

There is a well-defined $\mathbf{T}$-action on $G\backslash E$ given by $z(G \cdot e) := G \cdot (ze)$. This action is clearly continuous, and a little reflection shows that it is free as well: if $z(G \cdot e) = G \cdot e$, then there is an $s \in G$ such that $ze = s \cdot e$. This forces $p(e) = s \cdot p(e)$. Since G acts freely on X, $s = e$. But then $ze = e$ and $z = 1$. Since $\mathbf{T}\backslash E$ is homeomorphic to X, it is not hard to check that $\mathbf{T}\backslash(G\backslash E)$ is homeomorphic to $G\backslash X$. Thus p_G is a $\mathbf{T}$-bundle over $G\backslash X$, and $(p, E, \mathrm{lt}) \mapsto (p_G, G\backslash E)$ induces a map from $\mathrm{Pic}_G(X)$ to $\mathrm{Pic}(G\backslash X)$.

If we start with $[p, E, \mathrm{lt}] \in \mathrm{Pic}_G(X)$, then we get a map $\psi : E \to q^*(G\backslash E)$ given by

$$\psi(e) := \big(p(e), G \cdot e\big).$$

On the other hand, if $[L] \in \mathrm{Pic}(G\backslash X)$, then we clearly have $G\backslash q^*L \cong L$. Thus the pull-back map establishes a bijection Θ between the two groups.

If $[L], [L'] \in \mathrm{Pic}(G\backslash X)$, then

$$\begin{aligned} q^*\big([L][L']\big) &= \big[q^*(L * L')\big] \\ &= \big[q^*L * q^*L'\big] \\ &= \big[q^*L\big]\big[q^*L'\big]. \end{aligned}$$

Thus Θ is an isomorphism and the groups are isomorphic. □

Corollary 7. *Suppose that (G, X) is a second countable locally compact transformation group such that G acts freely and properly on X and $\mathrm{Pic}(X)$ is countable. Then there are homomorphisms η_i such that the following sequence is exact*:

$$1 \longrightarrow H^1\big(G, C(X, \mathbf{T})\big) \xrightarrow{\eta_1} \mathrm{Pic}(G\backslash X) \xrightarrow{\eta_2} \mathrm{Pic}(X)^G \xrightarrow{\eta_3}$$
$$H^2\big(G, C(X, \mathbf{T})\big) \xrightarrow{\eta_4} \mathrm{Br}(G\backslash X) \xrightarrow{\eta_5} H^1\big(G, \mathrm{Pic}(X)\big) \xrightarrow{\eta_6} H^3\big(G, C(X, \mathbf{T})\big).$$

3.2. *Trivial actions*

When the group G acts trivially on X, Packer has shown that the equivariant Brauer group has a direct sum decomposition [13] (see also [6]). A similar thing happens to the equivariant Picard group. Indeed, if G acts trivially on X, then it acts trivially on $\mathrm{Pic}(X)$, so $\mathrm{Pic}(X)^G = \mathrm{Pic}(X)$, and the map which takes a $\mathbf{T}$-bundle (p, E) to the equivariant bundle (p, E, id) gives a splitting for η_2. On the other hand, the group $H^1(G, C(X, \mathbf{T}))$ is then the group $\mathrm{Hom}(G, C(X, \mathbf{T}))$ of continuous homomorphisms. Thus $\mathrm{Pic}_G(X)$ is isomorphic to $\mathrm{Hom}(G, C(X, \mathbf{T})) \oplus \mathrm{Pic}(X)$. Indeed, it is easy to write down an isomorphism: given $\varphi : G \to C(X, \mathbf{T})$ and a $\mathbf{T}$-bundle (p, E), the formula

$$\mathrm{lt}_s(e) := \varphi\big(p(e)\big)(s) \cdot e$$

gives an equivariant action of G on E.

3.3. An application to actions of $\mathbf{R}$

Since $H^2(\mathbf{R}, C(X,\mathbf{T})) = \{0\}$ by [16, Theorem 4.1], our exact sequence collapses to give a short exact sequence

$$1 \longrightarrow H^1\big(\mathbf{R}, C(X,\mathbf{T})\big) \xrightarrow{\eta_1} \mathrm{Pic}_{\mathbf{R}}(X) \xrightarrow{\eta_2} \mathrm{Pic}(X)^{\mathbf{R}} \longrightarrow 1. \tag{11}$$

In particular, η_2 is surjective. On the other hand, $\mathbf{R}$ is connected, and by homotopy invariance, acts trivially on $\check{H}^2(X;\mathbf{Z}) \cong \mathrm{Pic}(X)$. Thus $\mathrm{Pic}(X)^{\mathbf{R}} = \mathrm{Pic}(X)$. Therefore exactness at $\mathrm{Pic}(X)^{\mathbf{R}}$ in (11) implies the following proposition.

Proposition 8. *Suppose that X is a second countably locally compact $\mathbf{R}$-space and that $p: Y \to X$ is a principal $\mathbf{T}$-bundle. Then Y admits an $\mathbf{R}$-action covering the given action on X.*

It is not obvious to us how one could prove Proposition 8 directly.

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