

## CONTINUOUS-TRACE GROUPOID $C^*$ -ALGEBRAS. III

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ABSTRACT. Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with a Haar system and with abelian isotropy. We show that the groupoid  $C^*$ -algebra  $C^*(\mathcal{G}, \lambda)$  has continuous trace if and only if there is a Haar system for the isotropy groupoid  $\mathcal{A}$  and the action of the quotient groupoid  $\mathcal{G}/\mathcal{A}$  is proper on the unit space of  $\mathcal{G}$ .

### 1. INTRODUCTION

Throughout  $\mathcal{G}$  will be a second countable locally compact Hausdorff groupoid with unit space  $\mathcal{G}^{(0)}$  and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . We are primarily interested in the case where the isotropy subgroupoid  $\mathcal{A} = \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma)\}$  is abelian (in the obvious sense: each fibre  $A_u = \mathcal{A}^u = \mathcal{A}_u = \mathcal{A}_u^u$  is an abelian group). Notice that  $\mathcal{A}$  acts freely and properly on the left and right of  $\mathcal{G}$ . The quotient  $\mathcal{R} = \mathcal{A} \backslash \mathcal{G} = \mathcal{G}/\mathcal{A}$  is a principal groupoid which may be identified (set theoretically, but usually not topologically) with an equivalence relation in  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ .

The collection  $\Sigma^{(0)}$  of closed subgroups of  $\mathcal{G}$  is a compact Hausdorff space in the Fell topology (see [16, §1]), and a groupoid in its own right. The map  $S : \mathcal{G}^{(0)} \rightarrow \Sigma^{(0)}$  need not be continuous in general, but the second author has shown that the continuity of  $S$  is equivalent to the existence of a Haar system  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  for  $\mathcal{A}$  [16, Lemmas 1.1 and 1.2].

Our object is to prove the following theorem.

**Theorem 1.1.** *Suppose that  $\mathcal{G}$  is a second countable locally compact Hausdorff groupoid with unit space  $\mathcal{G}^{(0)}$ , abelian isotropy, and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Then  $C^*(\mathcal{G}, \lambda)$  has continuous trace if and only if*

- (1) *the stabilizer map  $u \mapsto A_u$  is continuous from  $\mathcal{G}^{(0)}$  to  $\Sigma^{(0)}$ , and*
- (2) *the action of  $\mathcal{R}$  on  $\mathcal{G}^{(0)}$  is proper.*

Theorem 1.1 should be thought of as a natural groupoid generalization of [20, Theorem 5.1] which says that a transformation group  $C^*$ -algebra  $C^*(G, \Omega)$ , with all the stability groups contained in a *fixed abelian subgroup* of  $G$ , has continuous trace if and only if the stability groups vary continuously and  $G$  acts  $\sigma$ -properly. (Recall that  $G$  acts  $\sigma$ -properly if the map  $(s, \omega) \mapsto (s \cdot \omega, \omega)$  induces a proper map from the quotient  $\mathcal{R} = G \times \Omega / \sim$ , where  $(s, \omega) \sim (r, \omega)$  if  $s^{-1}r \in G_\omega$ .) In fact, as an immediate corollary of Theorem 1.1, we recover a generalization of [20, Theorem 5.1] that was recently obtained by Echterhoff [3, Corollary 1] using different methods.

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**Corollary 1.2** (Echterhoff). *Suppose that  $(G, \Omega)$  is a second countable locally compact transformation group with abelian stability groups. Then  $C^*(G, \Omega)$  has continuous trace if and only if the stabilizers vary continuously and  $G$  acts  $\sigma$ -properly.*

Our strategy for proving Theorem 1.1 is to show that  $C^*(\mathcal{G}, \lambda)$  is isomorphic to the restricted groupoid  $C^*$ -algebra  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  of a  $\mathbb{T}$ -groupoid  $\mathcal{D}$  over a principal groupoid  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  as defined in [10, 15]. Here  $\widehat{\mathcal{A}}$  is the spectrum of  $C^*(\mathcal{A}, \beta)$  for a Haar system  $\beta$ . It turns out that  $\widehat{\mathcal{A}}$  is independent of  $\beta$ , and is only defined when condition (1) of Theorem 1.1 is satisfied. Then we apply the main theorem of [10] which characterizes exactly when such  $C^*$ -algebras have continuous trace. The isomorphism between  $C^*(\mathcal{G}, \lambda)$  and  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  can be established in a quite general framework, where the only hypothesis is the existence of Haar system for  $\mathcal{A}$ , and should be viewed as a version of Mackey’s normal subgroup analysis. For our purposes, it is sufficient to prove this isomorphism with the additional hypothesis that  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff, which makes the proof more elementary. Therefore we can proceed as follows. We note that if  $C^*(\mathcal{G}, \lambda)$  is CCR, then orbits must be closed (Proposition 2.8). Since  $\mathcal{R}$  acting properly on  $\mathcal{G}^{(0)}$  always implies that orbits are closed, we can assume throughout that orbits are closed. Next we observe that if the spectrum of  $C^*(\mathcal{G}, \lambda)$  is Hausdorff, then condition (1) of Theorem 1.1 is satisfied (Proposition 3.1). We are then reduced to proving condition (2) of Theorem 1.1 under the assumption that condition (1) holds and that orbits are closed. We can then apply our isomorphism result.

The isomorphism result is of independent interest. As a consequence, every continuous-trace groupoid  $C^*$ -algebra of a groupoid with abelian isotropy has a realization as the restricted groupoid  $C^*$ -algebra of a  $\mathbb{T}$ -groupoid over an equivalence relation (Remark 4.9). This should prove useful for investigating the “fine” structure of such groupoids and  $C^*$ -algebras. As evidence for this, we note that in [11, Theorem 1.1] the first and third authors have given a six term exact sequence, modeled after that of Kumjian’s [6], that describes the Dixmier-Douady class of these  $C^*$ -algebras in terms of the topology of the relation.

Our work is organized as follows. In Section 2, we give a careful development of representations induced from an abelian subgroup of a stabilizer. We use this to obtain our results on closed orbits (Proposition 2.8) mentioned above. Our proof relies on the amenability of the stabilizers—which is trivially guaranteed by our assumption of abelian isotropy. Section 3 is devoted to showing that  $\mathcal{A}$  has a Haar system if  $(C^*(\mathcal{G}, \lambda))^\wedge$  is Hausdorff (Proposition 3.1), and to showing that  $\widehat{\mathcal{A}}$  is a locally compact group bundle<sup>1</sup> (Corollary 3.4). The crucial step here is to characterize the topology on  $\widehat{\mathcal{A}}$  as a suitable generalized compact-open topology (Proposition 3.3). Section 4 is devoted to the construction of the  $\mathbb{T}$ -groupoid  $\mathcal{D}$  (Proposition 4.3), and the proof that there is an isomorphism of  $C^*(\mathcal{G}, \lambda)$  with  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  in our special situation (Proposition 4.5).

We use the notation and terminology of [14] except that we write  $s$  for the source map in place of  $d$ . All our groupoids are locally compact, second countable, and Hausdorff. If  $\mathcal{G}$  is a groupoid with unit space  $\mathcal{G}^{(0)}$  and isotropy groupoid  $\mathcal{A}$ , then  $\mathcal{R} = \mathcal{G}/\mathcal{A} = \mathcal{A} \backslash \mathcal{G}$  is the principal quotient of  $\mathcal{G}$ . We identify  $\mathcal{R}$  with a subset of  $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$  via the map  $j : \mathcal{G} \rightarrow \mathcal{R}$  given by  $j(\gamma) = (r(\gamma), s(\gamma))$ . We will often write  $\dot{\gamma}$  for  $j(\gamma)$ .

<sup>1</sup>Here a “group bundle” always refers to a groupoid whose range and source maps coincide.

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2. INDUCED REPRESENTATIONS

In this section we want to give a straightforward definition of the representation  $\text{Ind}(u, A, \pi)$  of  $C^*(\mathcal{G}, \lambda)$  induced from a representation  $\pi$  of an abelian<sup>2</sup> subgroup  $A$  of  $\mathcal{G}_u^u$ . To do this, we shall show that restriction defines a map

$$(2.1) \quad P : C_c(\mathcal{G}) \rightarrow C_c(A)$$

which is essentially a *generalized conditional expectation* in the sense of Rieffel [17, Definition 4.12]. Unfortunately we cannot use Rieffel’s definition directly as  $C_c(A)$  acts on the right of  $C_c(\mathcal{G}_u)$  rather than  $C_c(\mathcal{G})$ , and the former is not an algebra. However, this turns out not to be a real obstruction, and we obtain an induced representation using Rieffel’s techniques. Notice that  $A$  acts freely and properly on the right of  $\mathcal{G}_u$ .

**Lemma 2.1.** *Suppose that  $u \in \mathcal{G}^{(0)}$ , that  $A$  is an abelian subgroup of  $\mathcal{G}_u^u$ , and that  $\beta$  is a Haar measure on  $A$ .*

(1) *The formula*

$$\mathcal{Q}(f)(\dot{\gamma}) = \int_A f(\gamma a) d\beta(a)$$

*defines a surjection from  $C_c(\mathcal{G})$  onto  $C_c(\mathcal{G}_u/A)$ .*

(2) *There is a non-negative, bounded, continuous function  $b$  on  $\mathcal{G}_u$  such that for any compact set  $K \subseteq \mathcal{G}_u$  the support of  $b$  and  $KA$  have compact intersection, and*

$$(2.2) \quad \int_A b(\gamma a) d\beta(a) = 1$$

*for all  $\gamma \in \mathcal{G}_u$ .*

(3) *There is a Radon measure  $\sigma$  on  $\mathcal{G}_u/A$  such that*

$$(2.3) \quad \int f(\gamma) d\lambda_u(\gamma) = \int_{\mathcal{G}_u/A} \int_A f(\gamma a) d\beta(a) d\sigma(\dot{\gamma}).$$

*Proof.* The properness of the  $A$ -action implies that  $\mathcal{Q}$  takes values in  $C_c(\mathcal{G}_u/A)$ . The existence of a function  $b'$  satisfying the requirements of (2) with the exception of (2.2) follows from Lemme 1 on page 96 of [1]. Now (2) follows by normalizing, and the rest of (1) follows from (2). Part (3) will follow if we can show that the equation

$$\sigma(\mathcal{Q}(f)) = \int f(\gamma) d\lambda_u(\gamma)$$

yields is a well defined, positive linear functional on  $C_c(\mathcal{G}_u/A)$ . But this amounts to showing that if

$$(2.4) \quad \int_A f(\gamma a) d\beta(a) = 0 \quad \text{for all } \gamma \in \mathcal{G}_u,$$

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<sup>2</sup>Here we *do not* assume that  $\mathcal{G}_u^u$  is abelian.

then the left-hand side of (2.3) is 0. However if (2.4) holds, then for any  $h \in C_c(\mathcal{G})$ ,

$$\begin{aligned} \int h * f(a) d\beta(a) &= \int_A \int_{\mathcal{G}} h(a\gamma) f(\gamma^{-1}) d\lambda^u(\gamma) d\beta(a) \\ &= \int_{\mathcal{G}} h(\gamma) \left( \int_A f(\gamma^{-1}a) d\beta(a) \right) d\lambda^u(\gamma) = 0. \end{aligned}$$

On the other hand,

$$0 = \int h * f(a) d\beta(a) = \int_{\mathcal{G}} \left( \int_A \overline{h^*(\gamma^{-1}a)} d\beta(a) \right) f(\gamma^{-1}) d\lambda^u(\gamma).$$

(Where we have replaced  $a$  by  $a^{-1}$  in the final formula, and used the fact that  $A$  is abelian and hence unimodular.) It follows from (2) and the Tietze extension theorem that there is an  $h \in C_c^+(\mathcal{G})$  such that the final  $A$  integral is equal to one for every  $\gamma^{-1}$  in the support of  $f$ . The lemma follows.  $\square$

Recall that  $C_c(\mathcal{G}_u)$  is a  $C_c(\mathcal{G})$ - $C_c(A)$ -bimodule;  $C_c(A)$  acts on the right of  $C_c(\mathcal{G}_u)$  by the formula

$$f \cdot \phi(\gamma) = \int_A f(\eta a) \phi(a^{-1}) d\beta(a),$$

while  $C_c(\mathcal{G})$  acts on the left by convolution:

$$g \cdot f(\gamma) = f * g(\gamma) = \int_{\mathcal{G}} g(\gamma\eta) f(\eta^{-1}) d\lambda^u(\eta).$$

Of course, if  $\mathcal{G}$  were transitive, then  $\mathcal{G}_u$  would be an  $\mathcal{G}$ - $A_u$ -equivalence (see [8, Example 2.2]), and we could have circumvented some of the details in the proof of the following proposition. While it may be the case that [8, Example 2.2] extends to a setting sufficiently general to satisfy our needs, the proof of Proposition 2.2 is sufficiently elementary to be of independent interest.

**Proposition 2.2.** *Suppose that  $u \in \mathcal{G}^{(0)}$ , and that  $A$  is an abelian subgroup of  $\mathcal{G}_u^u$ . Then*

$$\langle f, g \rangle_A = P(f^* * g)$$

defines a  $C^*(A)$ -valued sesquilinear form on  $C_c(\mathcal{G}_u)$  such that for all  $g, h \in C_c(\mathcal{G}_u)$  and  $\phi \in C_c(A)$

$$(2.5) \quad \langle g, h \rangle_A^* = \langle h, g \rangle_A, \quad \langle g, h \cdot \phi \rangle_A = \langle g, h \rangle_A * \phi, \text{ and}$$

$$(2.6) \quad \langle g, g \rangle_A \text{ is a positive element of } C^*(A).$$

Furthermore, there is an approximate identity  $\{e_k\}$  in  $C_c(\mathcal{G})$  such that

$$(2.7) \quad \langle g * e_k - g, g * e_k - g \rangle_A \rightarrow 0 \text{ in } C^*(A), \text{ and}$$

$$(2.8) \quad \langle f * g, f * g \rangle_A \leq \|f\|_{C^*(\mathcal{G}, \lambda)}^2 \langle g, g \rangle_A,$$

for all  $f \in C_c(\mathcal{G})$ , and  $g \in C_c(\mathcal{G}_u)$ .

*Proof.* The formulas in (2.5) follow from routine calculations. For (2.6), it will suffice to check that  $\chi(\langle g, g \rangle_A) \geq 0$  for all  $\chi \in \widehat{A}$ :

$$\chi(\langle g, g \rangle_A) = \int_A \chi(a) g^* * g(a) d\beta(a) = \int_A \int_{\mathcal{G}} \chi(a) \overline{g(\gamma a^{-1})} g(\gamma) d\lambda_u(\gamma) d\beta(a);$$

which, using (2.3), is

$$= \int_A \int_{\mathcal{G}_u/A} \int_A \chi(a) \overline{g(\gamma b a^{-1})} g(\gamma b) d\beta(b) d\sigma(\dot{\gamma}) d\beta(a);$$

which, using the Fubini Theorem and sending  $a \mapsto a^{-1}b$ , is

$$= \int_{\mathcal{G}_u/A} \left( \int_A \overline{\chi(a)g(\gamma a)} d\beta(a) \right) \left( \int_A \chi(b)g(\gamma b) d\beta(b) \right) d\sigma(\dot{\gamma}) \geq 0.$$

This proves (2.6). Assertion (2.7) follows from [8, Proposition 2.10], and the fact that the inductive limit topologies are stronger than the  $C^*$ -norm topologies on  $C^*(\mathcal{G}, \lambda)$  and  $C^*(A)$ .

To prove the final assertion, we claim that it suffices to notice that

$$f \mapsto P(h^* * f * h)$$

is bounded and defines a positive operator  $P_h : C^*(\mathcal{G}, \lambda) \rightarrow C^*(A)$  for each  $h \in C_c(\mathcal{G})$ . To see this note that  $e_k^* * g^* * f^* * f * g * e_k \leq \|f\|_{C^*(\mathcal{G}, \lambda)}^2 e_k^* * g^* * g * e_k$  in  $C^*(\mathcal{G}, \lambda)$ . If  $P_{g^*e_k}$  is positive, then  $\langle f * g^* e_k, f * g^* e_k \rangle_A \leq \|f\|_{C^*(\mathcal{G}, \lambda)}^2 \langle g^* e_k, g^* e_k \rangle_A$ . Then (2.8) now follows from (2.7).

To prove the claim, recall that  $C^*(\mathcal{G}, \lambda)$  and  $C^*(A)$  are, respectively, the enveloping  $C^*$ -algebras of  $L^I(\mathcal{G}, \lambda)$  (as defined in [14]) and  $L^1(A)$ . Moreover,

$$\begin{aligned} \|P_g(f)\|_1 &= \int_A |g^* * f * g(a)| d\beta(a) \\ &\leq \int_A \int_{\mathcal{G}} \int_{\mathcal{G}} |g(\eta^{-1}\gamma^{-1}a)f(\eta^{-1})g(\gamma^{-1})| d\lambda^{s(\gamma)} d\lambda^u(\gamma) d\beta(a) \\ &\leq \|\mathcal{Q}(|g|)\|_{\infty} \|f\|_I \|g\|_I. \end{aligned}$$

It follows that  $P_g$  extends to a continuous map of  $L^I(\mathcal{G}, \lambda)$  into  $L^1(A)$ . But if  $\rho$  is a state of  $L^1(A)$ , then  $\rho \circ P$  is a positive form on  $L^I(\mathcal{G}, \lambda)$  of norm no larger than  $M = \|\mathcal{Q}(|g|)\|_{\infty} \|g\|_I$ . Then by [2, Proposition 2.7.1], if  $f \geq 0$ ,  $M\|f\|_{C^*(\mathcal{G}, \lambda)} \geq \rho \circ P_g(f)$ . Taking the supremum over all states  $\rho$ , we get

$$M\|f\|_{C^*(\mathcal{G}, \lambda)} \geq \|P_g(f)\|_{C^*(A)}$$

for  $f \in C^*(\mathcal{G}, \lambda)^+$ ; this suffices. □

It now follows that, given a representation  $\pi$  of  $A$  on  $\mathcal{H}_{\pi}$ ,

$$(2.9) \quad \langle f \otimes \xi, g \otimes \zeta \rangle = (\pi(\langle g, f \rangle_A) \xi | \zeta)_{\mathcal{H}_{\pi}}$$

defines a pre-inner product on  $C_c(\mathcal{G}_u) \otimes \mathcal{H}_{\pi}$  on which  $C_c(\mathcal{G})$  acts by bounded operators via convolution. Consequently, we can make the following definition.

**Definition 2.3.** Suppose that  $u \in \mathcal{G}^{(0)}$ , that  $A$  is an abelian subgroup of  $\mathcal{G}_u^u$ , and that  $\pi$  is a representation of  $A$  on  $\mathcal{H}_{\pi}$ . Then we write  $\text{Ind}(u, A, \pi)$  for the representation of  $C^*(\mathcal{G}, \lambda)$  on the Hilbert space completion of  $C_c(\mathcal{G}_u) \otimes \mathcal{H}_{\pi}$  with respect to the inner product determined by (2.9) given by the formula

$$\text{Ind}(u, A, \pi)(f)[g \otimes \xi] = [f * g \otimes \xi].$$

*Remark 2.4.* The only hard part of defining the above representations induced from the stability groups is showing the boundedness: Proposition 2.2. We could have appealed to [15, Proposition 4.2] for this, but have chosen instead to give a more elementary proof in our situation.

The remainder of this section closely parallels Lemma 2.4 and Proposition 2.5 of [9]. Recall that  $\mathcal{G}$  acts on the right of  $\mathcal{G}^{(0)}$  via  $r(\gamma) \cdot \gamma = s(\gamma)$ . Furthermore, if  $\chi \in \widehat{A}_{r(\gamma)}$ , then  $\chi \cdot \gamma$  will denote the element of  $\widehat{A}_{s(\gamma)}$  given by  $a \mapsto \chi(\gamma a \gamma^{-1})$ . Note that both  $u \cdot \gamma$  and  $\chi \cdot \gamma$  depend only on  $\dot{\gamma} = j(\gamma)$  in  $\mathcal{R}$ . (Recall that  $j : \mathcal{G} \rightarrow \mathcal{R}$  is defined by  $j(\gamma) = (r(\gamma), s(\gamma))$ .)

**Lemma 2.5.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Then  $\text{Ind}(u, A_u, \chi)$  is an irreducible representation for all  $u \in \mathcal{G}^{(0)}$  and  $\chi \in \widehat{A}_u$ . Furthermore, if  $\gamma \in \mathcal{G}_u$ , then  $\text{Ind}(u, A_u, \chi)$  is unitarily equivalent to  $\text{Ind}(u \cdot \gamma, \widehat{A}_{s(\gamma)}, \chi \cdot \gamma)$ . In particular,  $\text{Ind}(u, A_u, 1)$  is equivalent to  $\text{Ind}(v, A_v, 1)$  if and only if  $[u] = [v]$ .*

*Proof.* Note that  $L^{(\chi, u)} := \text{Ind}(u, A_u, \chi)$  acts on the completion  $V_{(\chi, u)}$  of  $C_c(\mathcal{G}_u)$  with respect to the inner product:

$$\begin{aligned} (f|g)_{(\chi, u)} &= \int_{A_u} \chi(a)g^* * f(a) d\beta^u(a) = \int_{A_u} \int_{\mathcal{G}} \chi(a)\overline{g(\gamma a)}f(\gamma) d\lambda_u(\gamma) d\beta^u(a) \\ &= \int_{A_u} \int_{\mathcal{G}_u/A_u} \int_{A_u} \chi(a)\overline{g(\gamma ba)}f(\gamma b) d\beta^u(b) d\sigma^u(\dot{\gamma}) d\beta^u(a) \\ &= \int_{\mathcal{G}_u/A_u} \left( \int_{A_u} \chi(b)f(\gamma b) d\beta^u(b) \right) \left( \int_{A_u} \overline{\chi(a)g(\gamma a)} d\beta^u(a) \right) d\sigma^u(\dot{\gamma}). \end{aligned}$$

As in [10, §3], we let  $\mathcal{H}_{(\chi, u)}^0$  be the collection of bounded Borel functions  $\xi$  on  $\mathcal{G}_u$  such that  $\xi(\gamma a) = \chi(a)\xi(\gamma)$  for all  $a \in A_u$  and  $\gamma \in \mathcal{G}_u$ , and such that  $\dot{\gamma} \mapsto |\xi(\gamma)|$  has compact support on  $\mathcal{G}_u/A_u$ . We let  $\mathcal{H}_{(\chi, u)}$  be the Hilbert space completion of  $\mathcal{H}_{(\chi, u)}^0$  with respect to the inner product defined by

$$\langle \xi, \zeta \rangle_{(\chi, u)} = \int_{\mathcal{G}_u/A_u} \xi(\gamma)\overline{\zeta(\gamma)} d\sigma^u(\dot{\gamma}).$$

Then using (2.3) it follows that the equation

$$U^{(\chi, u)}(f)(\gamma) = \int_{A_u} \overline{\chi(a)}f(\gamma a) d\beta^u(a)$$

defines a unitary operator  $U^{(\chi, u)} : V_{(\chi, u)} \rightarrow \mathcal{H}_{(\chi, u)}$  which intertwines  $L^{(\chi, u)}$  with the representation given by the formula

$$(2.10) \quad T^{(\chi, u)}(f)\xi(\gamma) = \int_{\mathcal{G}} f(\gamma\eta)\xi(\eta^{-1}) d\lambda^u(\eta), \quad \xi \in \mathcal{H}_{(\chi, u)}.$$

Since  $\mathcal{G}_u$  is second countable and the action of  $A_u$  is proper, there is a Borel cross section  $c$  for the natural map from  $\mathcal{G}_u$  to  $\mathcal{G}_u/A_u$ . This allows us to introduce a Borel function  $\delta : \mathcal{G}_u \rightarrow A_u$  such that  $\gamma = c(\dot{\gamma})\delta(\gamma)$  for all  $\gamma \in \mathcal{G}_u$ . Notice that  $\delta(\gamma a) = \delta(\gamma)a$  for  $a \in A_u$ . The point is that we can define an isometric operator  $W^{(\chi, u)}$  from  $\mathcal{H}_u$  into  $L^2(\mathcal{G}_u/A_u, \sigma^u)$  by  $W^{(\chi, u)}(f)(\dot{\gamma}) = f(c(\dot{\gamma}))$ . Since  $R(\xi)(\gamma) = \chi(\delta(\gamma))\xi(\dot{\gamma})$  defines an inverse,  $W^{(\chi, u)}$  is a unitary operator which intertwines  $L^{(\chi, u)}$  with the

representation  $M^{(\chi,u)}$  on  $L^2(\mathcal{G}_u/A_u, \sigma^u)$  given by the formula

$$M^{(\chi,u)}(f)\xi(\dot{\gamma}) = \int_{\mathcal{G}} \chi(\delta(\eta)\delta(\gamma)^{-1})f(\gamma\eta^{-1})\xi(\dot{\eta}) d\lambda_u(\eta).$$

Again, since  $\mathcal{G}_u$  is second countable, the restriction of  $r$  defines a Borel isomorphism of  $\mathcal{G}_u/A_u$  with  $[u]$ . We let  $\sigma_*^u = r_*(\sigma^u)$  so that

$$\int \phi(v) d\sigma_*^u(v) = \int_{\mathcal{G}_u/A_u} \phi(r(\gamma)) d\sigma^u(\dot{\gamma}).$$

Then  $M^{(\chi,u)}$  is equivalent to the representation  $R^{(\chi,u)}$  on  $L^2([u], \sigma_*^u)$  given by

$$(2.11) \quad R^{(\chi,u)}(f)\xi(\gamma \cdot u) = \int_{\mathcal{G}} \chi(\delta(\eta\gamma)\delta(\gamma)^{-1})f(\eta^{-1})\xi(\eta\gamma \cdot u) d\lambda_{r(\gamma)}(\eta).$$

Fortunately,  $\chi(\delta(\eta\gamma)\delta(\gamma)^{-1})$  depends only on  $v = \gamma \cdot u$  and  $\eta$ , and we can write  $\theta(\eta, v)$  for the corresponding Borel function. Thus we can rewrite (2.11) as

$$R^{(\chi,u)}(f)\xi(v) = \int_{\mathcal{G}} \theta(\eta, v)f(\eta^{-1})\xi(\eta \cdot v) d\lambda_v(\eta).$$

Now let  $N^u$  be the representation of  $C_0(\mathcal{G}^{(0)})$  on  $L^2([u], \sigma_*^u)$  given by

$$N^u(\phi)(\xi)(v) = \phi(v)\xi(v), \quad v \in [u], \phi \in C_0(\mathcal{G}^{(0)}), \text{ and } \xi \in L^2([u]).$$

Notice that  $R^{(\chi,u)}(\phi \cdot f) = N^u(\phi)R^{(\chi,u)}(f)$  for all  $\phi \in C_c(\mathcal{G}^{(0)})$  and  $f \in C_c(\mathcal{G})$ . Just as in the proof of [9, Lemma 2.4], we see that  $R^{(\chi,u)}$ , and hence  $L^{(\chi,u)}$ , is irreducible. More precisely, any projection commuting with  $R^{(\chi,u)}(C^*(\mathcal{G}, \lambda))$  must also commute with  $N^u(C_0(\mathcal{G}^{(0)}))''$ , and since  $C_c(\mathcal{G}^{(0)})|_{[u]}$  separates points of  $[u]$ ,  $N^u(C_0(\mathcal{G}^{(0)}))''$  is a maximal abelian subalgebra of operators on  $L^2([u])$ . Therefore any projection commuting with  $R^{(\chi,u)}(C^*(\mathcal{G}, \lambda))$  must be of the form  $N^u(\phi)$  with  $\phi = 1_E$  and  $E \subseteq [u]$ . Since  $N^u(\phi)$  commutes with every  $R^{(\chi,u)}(f)$ , we have

$$\phi(v) \int_{\mathcal{G}} \theta(\eta, v)f(\eta^{-1})\xi(\eta \cdot v) d\lambda_v(\eta) = \int_{\mathcal{G}} \theta(\eta, v)f(\eta^{-1})\phi(\eta \cdot v)\xi(\eta \cdot v) d\lambda_v(\eta)$$

for  $\sigma_*^u$ -almost every  $v$ , all  $\xi \in L^2$ , and all  $f \in C_c(\mathcal{G})$ . Thus for some  $v \in [u]$ ,

$$\phi(v) = \phi(\eta \cdot v)$$

for  $\lambda_v$ -almost all  $\eta$ . Then, of course,  $\phi$  is constant (a.e.) on  $[u]$ . This proves the irreducibility.

Now let  $(v, A_v, \rho) = (u \cdot \gamma, \widehat{A}_s(\gamma), \chi \cdot \gamma)$ . Then  $a \mapsto \gamma a \gamma^{-1}$  is an isomorphism of  $A_u$  onto  $A_v$ . Since the unitary equivalence class of  $L^{(\chi,u)}$  is certainly independent of our choice of Haar measure  $\beta^u$  on  $A_u$ , we may as well assume that

$$(2.12) \quad \int_{A_u} f(a) d\beta^u(a) = \int_{A_v} f(\gamma a \gamma^{-1}) d\beta^v(a)$$

for all  $f \in C_c(A_u)$ . Then we can define a unitary map  $Q : \mathcal{H}_{(\chi,u)} \rightarrow \mathcal{H}_{(v,\rho)}$  by  $Q(f)(\eta) = f(\eta\gamma^{-1})$ , and  $Q$  clearly intertwines  $\text{Ind}(u, A_u, \chi)$  and  $\text{Ind}(v, A_v, \rho)$ .

Conversely, suppose that  $\text{Ind}(u, A_u, \chi)$  is equivalent to  $\text{Ind}(v, A_v, \rho)$ . Then  $N^u$  and  $N^v$  are certainly equivalent. Just as in the proof of [9, Proposition 2.5], if  $[u] \cap [v] = \emptyset$ , then  $N^u$  cannot be equivalent to  $N^v$  by [19, Lemma 4.15]. The result follows.  $\square$

*Remark 2.6.* Notice that we have proved a bit more. If  $\text{Ind}(u, A_u, \chi)$  is equivalent to  $\text{Ind}(v, A_v, \rho)$ , then  $v = u \cdot \gamma$  and  $\text{Ind}(v, A_v, \rho)$  is equivalent to  $\text{Ind}(u, A_u, \rho \cdot \gamma^{-1})$ . At this point, it does not seem to be easy to see that this forces  $\rho$  to equal  $\chi \cdot \gamma$ . Fortunately, this will follow when orbits are closed (Lemma 2.11), and as this is the only situation in which we will require the result, we will settle for that.

The next result is standard; see for example, Rieffel’s Theorem 5.9 (Induction in Stages) and Proposition 6.26 in [17]. We omit the proof.

**Lemma 2.7.** *Suppose that  $u \in \mathcal{G}^{(0)}$ , that  $A$  and  $B$  are abelian subgroups of  $\mathcal{G}_u^u$  with  $A \subseteq B$ , and that  $\pi$  and  $\rho$  are representations of  $A$ .*

- (1)  $\text{Ind}(u, A, \pi)$  is unitarily equivalent to  $\text{Ind}(u, B, \text{Ind}_A^B(\pi))$ , and
- (2) if  $\pi$  weakly contains  $\rho$ , then  $\text{Ind}(u, A, \pi)$  weakly contains  $\text{Ind}(u, A, \rho)$ .

Now we use [16, Lemma 1.3] to fix once and for all a continuous choice of Haar measures for the subgroups of  $\mathcal{A}$ . That is, for each  $A \in \Sigma_{\mathcal{A}}^{(0)}$ , we choose a measure  $\beta^A$  such that the map

$$A \mapsto \int_A f(a) d\beta^A(a)$$

is continuous on  $\Sigma_{\mathcal{A}}^{(0)}$  for each  $f \in C_c(\mathcal{A})$ . We will continue to write  $\beta^u$  in place of  $\beta^{A_u}$ .

Our next result closely parallels [9, Proposition 2.5]; however, the argument is complicated by the presence of (possibly discontinuous) isotropy.

**Proposition 2.8.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy, and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Then, if points are closed in  $(C^*(\mathcal{G}, \lambda))^\wedge$ ,  $[u]$  is closed for each  $u \in \mathcal{G}^{(0)}$ .*

*Proof.* By Lemma 2.5, the map  $u \mapsto L^u = \text{Ind}(u, A_u, 1)$  defines an injection  $\Psi : \mathcal{G}^{(0)}/\mathcal{G} \rightarrow (C^*(\mathcal{G}, \lambda))^\wedge$ . Thus it suffices to show that  $\Psi$  is continuous. In particular, it suffices to show that if  $L^u(f) \neq 0$  and if  $\{u_n\}$  converges to  $u$ , then, eventually,  $L^{u_n}(f) \neq 0$ . If the assertion fails, then it fails for some  $f \in C_c(\mathcal{G})$ . Since  $\Sigma_{\mathcal{A}}^{(0)}$  is compact, we can, by passing to a subsequence and relabeling, assume that  $A_{u_n}$  converges to  $C$ , and that  $L^{u_n}(f) = 0$  for all  $n$ . Note that  $C \subseteq A_u$ .

It follows from our choice of Haar measures that, for each  $g, h \in C_c(\mathcal{G})$ ,

$$(2.13) \quad (L^{u_n}(f)(g)|h)_{(1, u_n)} \text{ converges to } (\text{Ind}(u, C, 1)(f)(g)|h)_{(1, u)}.$$

It follows that  $\text{Ind}(u, C, 1)(f) = 0$ . Therefore Lemma 2.7(1) implies that  $\text{Ind}(u, A_u, \text{Ind}_C^{A_u}(1))(f) = 0$ . Since  $\text{Ind}_C^{A_u}(1)$  is weakly contained in the trivial representation of  $A_u$ , Lemma 2.7(2) implies that  $L^u(f) = 0$  as desired.  $\square$

*Remark 2.9.* It follows from Proposition 2.8 that, in order to prove Theorem 1.1, we may assume that  $\mathcal{G}$  has closed orbits. On the one hand, if  $C^*(\mathcal{G}, \lambda)$  has continuous trace, then it has Hausdorff spectrum and the proposition applies. On the other hand, if we assume that the  $\mathcal{R}$ -action on  $\mathcal{G}^{(0)}$  is proper, then  $\mathcal{R}$ , and hence  $\mathcal{G}$ , orbits are easily seen to be closed.

The next lemma is a well-known consequence of Renault’s disintegration theorem [15, 7, 12]. It is stated without proof in [16, Remark 4.10], and it is, unfortunately, used without comment in [9, 10]. We include the statement and proof here for the reader’s convenience.



**Lemma 2.10.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ , and that  $U$  is a  $\mathcal{G}$ -invariant open subset of  $\mathcal{G}^{(0)}$  with  $F = \mathcal{G}^{(0)} \setminus U$ . Then there is an exact sequence*

$$0 \longrightarrow C^*(\mathcal{G}|_U, \lambda) \xrightarrow{j} C^*(\mathcal{G}, \lambda) \xrightarrow{p} C^*(\mathcal{G}|_F, \lambda) \longrightarrow 0.$$

The map  $j$  and  $p$  are determined on continuous functions by extension by 0 and restriction, respectively.

*Proof.* First recall that every representation of  $C^*(\mathcal{G}, \lambda)$  may be disintegrated [15]. This means that given a representation  $\pi$ , there is an essentially uniquely determined triple  $(\mu, \mathcal{U}, \mathcal{H} * \mathcal{G}^{(0)})$  consisting of a quasi-invariant measure  $\mu$  on  $\mathcal{G}^{(0)}$ , a (Borel) Hilbert bundle  $\mathcal{H} * \mathcal{G}^{(0)}$ , and a Borel homomorphism  $\mathcal{U}$  from  $\mathcal{G}$  into the isomorphism groupoid of  $\mathcal{H} * \mathcal{G}^{(0)}$  such that Hilbert space of  $\pi$  may be realized as the direct integral  $\int_{\mathcal{G}^{(0)}}^{\oplus} \mathcal{H}(x) d\mu(x)$  and  $\pi$  may be expressed through the formula

$$(\pi(f)\xi, \eta) = \int f(\gamma) (\mathcal{U}(\gamma)\xi \circ s(\gamma), \eta \circ r(\gamma)) d\nu_0(\gamma),$$

where  $f \in C_c(\mathcal{G})$ ,  $\xi$  and  $\eta$  lie in  $\int_{\mathcal{G}^{(0)}}^{\oplus} \mathcal{H}(x) d\mu(x)$ , and  $\nu_0$  is the symmetric measure on  $\mathcal{G}$  determined by  $\mu$  and the Haar system (see [14, p. 52]). Depending on context  $\mathcal{U}$  or the entire triple  $(\mu, \mathcal{U}, \mathcal{H} * \mathcal{G}^{(0)})$  is called a *representation* of  $\mathcal{G}$ . Several things should be kept in mind. First, because  $\mu$  is quasi-invariant, its support is an invariant Borel subset of  $\mathcal{G}^{(0)}$ , say  $E$ . Second, we may assume that  $\mathcal{H}(u) = 0$ , for all  $u \notin E$ . And third, we may assume that  $\mathcal{U}(\gamma) = 0$ , unless both  $r(\gamma)$  and  $s(\gamma)$  lie in  $E$ . We shall therefore refer to  $E$  as the *support* of  $(\mu, \mathcal{U}, \mathcal{H} * \mathcal{G}^{(0)})$  or of  $\pi$ . It is perhaps worthwhile, too, to point out that in [14], representations of  $\mathcal{G}$  were “less than homomorphisms.” Thanks, however, to [13] we may take them to be true homomorphisms and we may make the assertions on their supports that we just made.

Suppose that  $f$  lies in  $C_c(\mathcal{G}|_U)$  and view  $f$  as lying also in  $C_c(\mathcal{G})$  by extending it to be zero on  $\mathcal{G}|_F$ . (Remember that because  $U$  and  $F$  are complementary invariant sets in  $\mathcal{G}^{(0)}$ ,  $\mathcal{G} = \mathcal{G}|_U \cup \mathcal{G}|_F$ .) The norm of  $f$  calculated in  $C^*(\mathcal{G}|_U, \lambda)$  is the supremum of  $\|\pi(f)\|$  as  $\pi$  runs over all representations of  $C_c(\mathcal{G}|_U, \lambda)$ . Each of these may be disintegrated in terms of a representation  $(\mu, \mathcal{U}, \mathcal{H} * U)$  of  $\mathcal{G}|_U$ . This means, in particular, that the support of such a representation is a subset of  $U$ . But a representation of  $\mathcal{G}|_U$  obviously extends to a representation of  $\mathcal{G}$ . Just set  $\mathcal{H}(u) = 0$ , for  $u \in F$ , and set  $\mathcal{U}(\gamma) = 0$  for  $\gamma \in \mathcal{G}|_F$ . This shows that the norm of  $f$  as an element of  $C^*(\mathcal{G}|_U, \lambda)$  is dominated by its norm in  $C^*(\mathcal{G}, \lambda)$ . On the other hand, every representation of  $C^*(\mathcal{G}, \lambda)$  yields, by restriction, a representation of  $C^*(\mathcal{G}|_U, \lambda)$ , so the norm of  $f$  is independent of the algebra in which it is calculated. This shows that  $j$  is injective at the level of  $C^*$ -algebras. It is also evident that the range of  $j$ ,  $j(C^*(\mathcal{G}|_U, \lambda))$ , is an ideal in  $C^*(\mathcal{G}, \lambda)$ . A similar analysis, using the disintegration theorem, shows that  $p$  is surjective at the  $C^*$ -algebra level. So the only thing that remains to be proved is that the kernel of  $p$  is the range of  $j$ . Of course the range of  $j$  is contained in the kernel of  $p$ . For the reverse inclusion, simply observe that a representation of  $C^*(\mathcal{G}, \lambda)$  annihilates  $j(C^*(\mathcal{G}|_U, \lambda))$  precisely when its support is disjoint from  $U$ , i.e., is contained in  $F$ . But since the representations of  $\mathcal{G}$  supported in  $F$  coincide (by extension) with the representations of  $\mathcal{G}|_F$ , we conclude that the quotient  $C^*(\mathcal{G}, \lambda) / (j(C^*(\mathcal{G}|_U, \lambda)))$  is isomorphic to  $C^*(\mathcal{G}|_F, \lambda)$ , i.e., that  $\ker(p) = j(C^*(\mathcal{G}|_U, \lambda))$ .  $\square$

Note that if orbits are closed then by a standard argument using the above lemma (see, for example, the last part of the proof of [9, Proposition 2.5]) every irreducible representation of  $C^*(\mathcal{G}, \lambda)$  factors through  $C^*(\mathcal{G}|_{[u]}, \lambda)$  for some  $u \in \mathcal{G}^{(0)}$ . Since  $\mathcal{G}|_{[u]}$  is equivalent to  $A_u$  [8, Example 2.2], every irreducible representation of  $C^*(\mathcal{G}|_{[u]}, \lambda)$  is induced from a character  $\chi$  of  $A_u$  [8, Theorem 2.8]. It is straightforward to see that the corresponding representation of  $C^*(\mathcal{G}, \lambda)$  is  $\text{Ind}(u, A_u, \chi)$ . Consequently, we have the following result.

**Lemma 2.11.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy, closed orbits, and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Then if  $u \in \mathcal{G}^{(0)}$  and  $\chi \in \widehat{A}_u$ ,  $\text{Ind}(u, A_u, \chi)$  is irreducible, and every irreducible representation of  $C^*(\mathcal{G}, \lambda)$  is of this form. Furthermore, if  $\chi \neq \psi$  in  $\widehat{A}_u$ , then  $\text{Ind}(u, A_u, \chi)$  is not equivalent to  $\text{Ind}(u, A_u, \psi)$ .*

### 3. THE DUAL ISOTROPY GROUPOID

**Proposition 3.1.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . If  $(C^*(\mathcal{G}, \lambda))^\wedge$  is Hausdorff, then the map  $u \mapsto A_u$  is continuous from  $\mathcal{G}^{(0)}$  to  $\Sigma^{(0)}$ .*

*Proof.* If the proposition were false, then there would be a sequence  $\{u_n\}$  converging to  $u$  in  $\mathcal{G}^{(0)}$ , but with  $A_{u_n} \rightarrow C$  with  $C$  strictly contained in  $A_u$ . Clearly, in view of Proposition 2.8 and Lemma 2.11, it will suffice to show that

$$\text{Ind}(u_n, A_{u_n}, 1) \text{ converges to } \text{Ind}(u, A_u, \chi)$$

for any  $\chi \in C^\perp = \{\chi \in \widehat{A}_u : \chi(c) = 1 \text{ for all } c \in C\}$ . But if  $\chi \in C^\perp$ , then  $\chi$  is weakly contained in  $\text{Ind}_C^{A_u}(1)$ . Therefore the claim is proved exactly as in the proof of Proposition 2.8.  $\square$

*Remark 3.2.* In the situation of Proposition 3.1,  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  is a Haar system for  $\mathcal{A}$ . It is useful to keep in mind that the existence of a Haar system on  $\mathcal{A}$  is equivalent to the continuity of the map  $u \mapsto A_u$  [16, Lemmas 1.1 and 1.3].

When  $\mathcal{A}$  has a Haar system  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  (note that we can identify  $\mathcal{A}^{(0)}$  with  $\mathcal{G}^{(0)}$ ), then  $C^*(\mathcal{A}, \beta)$  is a separable abelian  $C^*$ -algebra. In particular,  $\widehat{\mathcal{A}} = (C^*(\mathcal{A}, \beta))^\wedge$  is a second countable locally compact Hausdorff space. We will view elements of  $\widehat{\mathcal{A}}$  as continuous homomorphisms  $h : C^*(\mathcal{A}, \beta) \rightarrow \mathbb{C}$ , and  $h_n \rightarrow h$  in  $\widehat{\mathcal{A}}$  if and only if  $h_n(f) \rightarrow h(f)$  for every  $f \in C_c(\mathcal{A})$ .

Recall that there is a homomorphism  $V : C_0(\mathcal{G}^{(0)}) \rightarrow \mathcal{M}(C^*(\mathcal{A}, \beta))$  characterized by the formula  $V(\phi)f(\gamma) = \phi(r(\gamma))f(\gamma)$  [14, 2.1.14]. It follows (cf., e.g., [5, Proposition 9]) that there is a continuous map  $p : \widehat{\mathcal{A}} \rightarrow \mathcal{G}^{(0)}$  characterized by the equation  $h(V(\phi)f) = \phi(p(h))h(f)$  for each  $h \in \widehat{\mathcal{A}}$ ,  $\phi \in C_0(\mathcal{G}^{(0)})$ , and  $f \in C^*(\mathcal{A}, \beta)$ . Using Lemma 2.10<sup>3</sup>, each  $h \in \widehat{\mathcal{A}}$  is uniquely of the form  $h = (\chi, u)$  with  $u \in \mathcal{G}^{(0)}$ ,  $\chi \in \widehat{A}_u$ , and

$$(\chi, u)(f) = \int_{A_u} \chi(a)f(a) d\beta^u(a).$$

<sup>3</sup>Here we don't actually need the full power of the disintegration theorem. It suffices to notice that  $C^*(\mathcal{A}, \beta)$  is the enveloping  $C^*$ -algebra of the section algebra of a Banach algebra bundle. Thus, every irreducible representation lives on a fibre.

It is clear that  $\widehat{\mathcal{A}}$  has the structure of a group bundle (with  $r$  and  $s$  equal to  $p$ ). However it is not immediate that  $\widehat{\mathcal{A}}$  is a topological groupoid—the groupoid operations are not clearly continuous. This situation will be remedied by the next result.

**Proposition 3.3.** *Let  $\mathcal{A}$  be a second countable locally compact abelian group bundle with Haar system  $\{\beta^u\}_{u \in \mathcal{A}^{(0)}}$ . Then a sequence  $\{(\chi_n, u_n)\}$  in  $\widehat{\mathcal{A}}$  converges to  $(\chi_0, u_0)$  in  $\widehat{\mathcal{A}}$  if and only if*

- (1)  $\{u_n\}$  converges to  $u_0$  in  $\mathcal{A}^{(0)}$ , and
- (2) if  $a_n \in A_{u_n}$  and  $\{a_n\}$  converges to  $a_0$  in  $\mathcal{A}$ , then  $\{\chi_n(a_n)\}$  converges to  $\chi_0(a_0)$ .

*Proof.* First, suppose that  $h_n = (\chi_n, u_n)$  converges to  $h = (\chi_0, u_0)$ . The continuity of  $p$  implies that  $u_n \rightarrow u_0$ . If condition (2) fails, then there are  $a_n \in A_{u_n}$  converging to  $a_0$  yet with  $\chi_n(a_n)$  not converging to  $\chi_0(a_0)$ . Clearly, we may assume that no subsequence converges to  $\chi_0(a_0)$  either. Next we observe that we may assume that  $u_n \neq u_0$  for all  $n$ ; otherwise we obtain an immediate contradiction by passing to a subsequence and relabeling so that  $u_n = u_0$  for all  $n$ , and  $\chi_n \rightarrow \chi_0$  in  $\widehat{A}_{u_0}$ . Furthermore, again passing to a subsequence and relabeling if necessary, we can assume that  $u_n \neq u_m$  if  $n \neq m$ . In particular, we can define an integer valued function on  $S = r^{-1}(\{u_n\}_{n=0}^\infty)$  by  $\iota(b) = n$  when  $r(b) = u_n$ . Now fix  $f \in C_c(\mathcal{A})$  with  $h(f) = 1$ . Notice that  $S$  is closed, and  $g_0 : S \rightarrow \mathbb{C}$ , defined by

$$g_0(b) = f(a_{\iota(b)}^{-1}b) \quad (b \in S),$$

is continuous and compactly supported. The Tietze Extension Theorem implies that there is a  $g \in C_c(\mathcal{A})$  extending  $g_0$ . But

$$h_n(g) = \chi_n(a_n)h_n(f) \quad \text{for } n = 0, 1, 2, \dots$$

We obtain the desired contradiction by noting that  $h_n(f) \rightarrow 1$  and  $h_n(g) \rightarrow \chi_0(a_0)$ .

Conversely, now assume that  $h_n = (\chi_n, u_n)$  satisfies conditions (1) and (2) with respect to  $h_0 = (\chi_0, u_0)$ . Suppose that there is a  $f \in C_c(\mathcal{A})$  such that  $h_n(f)$  fails to converge to  $h_0(f)$ . As above we can reduce to the case that  $u_n \neq u_m$  if  $n \neq m$ . This time we define  $g_0 : S \rightarrow \mathbb{C}$  by

$$g_0(b) = \chi_{\iota(b)}(b)f(b) \quad (b \in S).$$

Again a few moments of reflection reveal that  $g_0$  is continuous and compactly supported so that there is a  $g \in C_c(\mathcal{A})$  extending  $g_0$ . The continuity of the Haar system on  $\mathcal{A}$  implies that  $(1, u_n)(g)$  converges to  $(1, u_0)(g)$ . Since  $(1, u_n)(g) = h_n(f)$ , we obtain the necessary contradiction.  $\square$

**Corollary 3.4.** *Let  $\mathcal{A}$  be a second countable locally compact abelian group bundle with Haar system  $\{\beta^u\}_{u \in \mathcal{A}^{(0)}}$ . Then  $\widehat{\mathcal{A}}$ , equipped with the Gelfand topology, is a locally compact group bundle.*

In the sequel we will need to see that  $p : \widehat{\mathcal{A}} \rightarrow \mathcal{G}^{(0)}$  is an open map. As we have noted above, Renault has shown that this is equivalent to the existence of a Haar system [15, Lemma 1.3]. Unfortunately, it does not seem straightforward to see this directly, and it takes a bit of work to see that the “obvious” choice does the job.

Our situation is covered by the following set-up. Suppose that  $r : \mathcal{A} \rightarrow X$  and  $p : \widehat{\mathcal{A}} \rightarrow X$  are locally compact abelian group bundle. We will say  $\widehat{\mathcal{A}}$  is a dual bundle for  $\mathcal{A}$  if there is a continuous map  $\langle \cdot, \cdot \rangle$  from  $\mathcal{A} * \widehat{\mathcal{A}} = \{(a, \chi) \in \mathcal{A} \times \widehat{\mathcal{A}} : r(a) = p(\chi)\}$

to  $\mathbb{C}$  such that  $\chi \mapsto \langle \cdot, \chi \rangle$  is an isomorphism of  $\widehat{\mathcal{A}}_u$  with the dual of  $\mathcal{A}_u$ . (Of course the example we have in mind is the maximal ideal space  $\widehat{\mathcal{A}}$  of the  $C^*$ -algebra of a locally compact abelian group bundle.) If  $\{\beta^u\}_{u \in X}$  is a Haar system for  $\mathcal{A}$ , then there is a natural choice of Haar measures on  $\widehat{\mathcal{A}}$ : namely each measure  $\widehat{\beta}^u$  is chosen so that the Fourier transform is an isometry from  $L^2(\mathcal{A}_u, \beta^u)$  onto  $L^2(\widehat{\mathcal{A}}_u, \widehat{\beta}^u)$ . We call  $\{\widehat{\beta}^u\}_{u \in X}$  the dual Haar system.

If  $f \in C_c(\mathcal{A})$ , then we can define a function  $\hat{f}$  on  $\widehat{\mathcal{A}}$  by taking the Fourier Transform in the appropriate fibre:

$$(3.1) \quad \hat{f}(\chi) = \int_{\mathcal{A}_{p(\chi)}} \langle a, \chi \rangle f(a) d\beta^{p(\chi)}(a).$$

**Lemma 3.5.** *The function  $\hat{f}$  defined in (3.1) is continuous on  $\widehat{\mathcal{A}}$ .*

*Proof.* Fix  $\sigma \in \widehat{\mathcal{A}}$ . Let  $K$  be a compact neighborhood of  $\sigma$ . Let  $F$  be the continuous function on the closed set  $\mathcal{A} * K = \{(a, \chi) : r(a) = p(\chi)\}$  defined by  $F(\chi, a) = \langle a, \chi \rangle f(a)$ . By the Tietze Extension Theorem, there is a  $\widetilde{F} \in C_c(\mathcal{A} \times K)$  extending  $F$ . Then

$$\phi(\chi, u) = \int_{\mathcal{A}_u} \widetilde{F}(a, \chi) d\beta^u(a)$$

is continuous on  $K \times X$ . (It suffices to consider  $\widetilde{F}$  of the form  $\widetilde{F}(a, \chi) = h(a)k(\chi)$  for  $h \in C_c(\mathcal{A})$  and  $k \in C(K)$ . Then the assertion is immediate since  $\{\beta^u\}_{u \in X}$  is a Haar system.) This suffices since  $\hat{f}(\chi) = \phi(\chi, p(\chi))$  for all  $\chi \in K$ .  $\square$

**Proposition 3.6.** *Suppose that  $r : \mathcal{A} \rightarrow X$  is an abelian group bundle, and that  $p : \widehat{\mathcal{A}} \rightarrow X$  is a dual bundle to  $\mathcal{A}$ . Then if  $\{\beta^u\}_{u \in X}$  is a Haar system for  $\mathcal{A}$ , the dual Haar system  $\{\widehat{\beta}^u\}_{u \in X}$  is a Haar system for  $\widehat{\mathcal{A}}$ .*

*Remark 3.7.* The proof is based on [4, p. 908] and [16, Lemma 1.3].

*Proof.* Suppose that  $K$  is compact in  $\widehat{\mathcal{A}}$ . We claim that  $u \mapsto \widehat{\beta}^u(K)$  is bounded on  $X$ . Of course, it suffices to consider only  $u \in p(K)$ . Let  $\rho \in C_c(\mathcal{A})$  be a non-negative function such that

$$(3.2) \quad \int_{\mathcal{A}_u} \rho(a)^2 d\beta^u(a) = 1 \quad \text{for all } u \in p(K).$$

Since  $p(K)$  is compact, there is an  $\epsilon > 0$  so that

$$(3.3) \quad \int_{\mathcal{A}_u} \rho(a) d\beta^u(a) > \epsilon \quad \text{for all } u \in p(K).$$

By definition, (3.2) implies that

$$(3.4) \quad \int_{\widehat{\mathcal{A}}_u} |\hat{\rho}(\chi)|^2 d\widehat{\beta}^u(\chi) = 1 \quad \text{for all } u \in p(K).$$

Moreover the continuity of  $\hat{\rho}$  and (3.3) imply that  $U = \{\chi \in \widehat{\mathcal{A}} : |\hat{\rho}(\chi)|^2 > \epsilon^2\}$  is an open neighborhood of  $p(K)$ .

If  $\chi \in K$ , then  $\chi^{-1}\chi \in p(K)$ . The continuity of multiplication implies that there is a neighborhood  $V$  of  $\chi$  such that  $V^{-1}V \subseteq U$ . Therefore there is a cover  $V_1, \dots, V_m$  of  $K$  such that  $V_j^{-1}V_j \subseteq U$  for each  $1 \leq j \leq m$ . In view of (3.4),

$\widehat{\beta}^u(U) \leq \epsilon^{-2}$  if  $u \in p(K)$ . Furthermore, if  $\widehat{\beta}^u(V_j) \neq 0$ , then there is a  $\chi \in V_j$  with  $p(\chi) = u$ . Then

$$\widehat{\beta}^u(V_j) = \widehat{\beta}^u(\chi\chi^{-1}V_j) \leq \widehat{\beta}^u(\chi V_j^{-1}V_j) \leq \widehat{\beta}^u(\chi U) = \widehat{\beta}^u(U) \leq 1/\epsilon^2.$$

It follows that  $\widehat{\beta}^u(K) \leq m/\epsilon^2$  for all  $u \in X$ . This proves the claim.

Now let  $\{u_i\}_{i \in I}$  be a net in  $X$  converging to  $u \in X$ . If  $\phi \in C_c(\widehat{\mathcal{A}})$ , then let  $\widehat{\beta}(\phi)(v) = \int_{\widehat{\mathcal{A}}_v} \phi(\chi) d\widehat{\beta}^v(\chi)$ . The above argument implies that  $\{\widehat{\beta}(\phi)(u_i)\}_{i \in I}$  is bounded. Thus if  $\omega$  is a generalized limit<sup>4</sup> on  $\ell^\infty(I)$ , then we obtain a positive linear functional  $\mu$  on  $C_c(\widehat{\mathcal{A}})$  by  $\mu(\phi) = \omega(\{\widehat{\beta}(\phi)(u_i)\})$ . Suppose that  $\phi, \psi \in C_c(\widehat{\mathcal{A}})$  agree on  $\widehat{\mathcal{A}}_u$ . Then if  $K$  is a compact set containing the supports of  $\phi$  and  $\psi$ ,

$$\begin{aligned} |\widehat{\beta}(\phi)(u_i) - \widehat{\beta}(\psi)(u_i)| &\leq \int_{\widehat{\mathcal{A}}_{u_i}} |\phi(\chi) - \psi(\chi)| d\widehat{\beta}^{u_i}(\chi) \\ &\leq \sup_{\chi \in \widehat{\mathcal{A}}_{u_i}} |\phi(\chi) - \psi(\chi)| \widehat{\beta}_{u_i}(K). \end{aligned}$$

Since  $\sup_{\chi \in \widehat{\mathcal{A}}_v} |\phi(\chi) - \psi(\chi)|$  tends to zero as  $v$  tends to  $u$  and since  $v \mapsto \widehat{\beta}^v(K)$  is bounded, it follows that  $\mu(\phi) = \mu(\psi)$ . Since every function in  $C_c(\widehat{\mathcal{A}}_u)$  has an extension to an element of  $C_c(\widehat{\mathcal{A}})$ , we can view  $\mu$  as a Radon measure on  $\widehat{\mathcal{A}}_u$ .

However, if  $f \in C_c(\widehat{\mathcal{A}})$ ,

$$\widehat{\beta}(|\widehat{f}|^2)(u_i) = \beta(|f|^2)(u_i)$$

which converges to  $\beta(|f|^2)(u) = \widehat{\beta}(|\widehat{f}|^2)(u)$ . It follows that

$$(3.5) \quad \widehat{\beta}(|\widehat{f}|^2)(u) = \mu(|\widehat{f}|^2) \quad \text{for all } f \in C_c(\mathcal{A}_u).$$

By density, (3.5) holds for all  $\widehat{f}$  in  $L^2(\widehat{\mathcal{A}}_u, \widehat{\beta}^u)$ . In particular,  $\mu = \widehat{\beta}^u$  on  $C_c(\widehat{\mathcal{A}}_u)$ .

We have shown that if  $\{u_i\}$  is any net converging to  $u$  in  $X$ , then  $\omega(\{\widehat{\beta}(\phi)(u_i)\}) = \widehat{\beta}(\phi)(u)$ . Therefore  $\lim_i \widehat{\beta}(\phi)(u_i) = \widehat{\beta}(\phi)(u)$ , and it follows that  $\{\widehat{\beta}^u\}$  is a Haar system.  $\square$

In view of the above, there is a groupoid action of  $\mathcal{R}$  (and hence  $\mathcal{G}$ ) on the right of  $\widehat{\mathcal{A}}$ :  $(\chi, u) \cdot \gamma = (\chi \cdot \gamma, u \cdot \gamma)$  (when  $u = r(\gamma)$ ). Since  $\mathcal{R}$  has a Haar system, and therefore open range and source maps, the quotient map from  $\mathcal{R}$  to  $\widehat{\mathcal{A}}/\mathcal{R}$  is always continuous and open [11, Lemma 1.2]. Notice also that if  $\mathcal{R}$  acts properly on  $\mathcal{G}^{(0)}$ , then  $\mathcal{R}$  must also act properly on  $\widehat{\mathcal{A}}$ . In that case,  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff and locally compact. Our next result shows that for the purposes of proving Theorem 1.1, we may always assume that  $\widehat{\mathcal{A}}/\mathcal{R}$  is locally compact Hausdorff.

**Proposition 3.8.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy, and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . If  $(C^*(\mathcal{G}, \lambda))^\wedge$  is Hausdorff, then  $\widehat{\mathcal{A}}$  can be defined as above and  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff.*

*Proof.* Since Lemma 2.5, Proposition 2.8, and Lemma 2.11 imply that the map  $(\chi, u) \mapsto \text{Ind}(u, A_u, \chi)$  defines a bijection of  $\widehat{\mathcal{A}}/\mathcal{R}$  and  $(C^*(\mathcal{G}, \lambda))^\wedge$ , the argument is identical to that of Proposition 2.8. One only has to observe, with our choice of Haar measures and the topology on  $\widehat{\mathcal{A}}$  given in Proposition 3.3, that in analogy with (2.13),  $\{(\text{Ind}(u_n, A_{u_n}, \chi_n)(f)(g), h)_{(\chi_n, u_n)}\}$  converges to

<sup>4</sup>That is,  $\omega$  is a norm one extension of the ordinary limit functional on the subspace of  $\ell^\infty(I)$  consisting of those nets  $\{a_i\}$  such that  $\lim_i a_i$  exists.

$\langle \text{Ind}(u, C, \chi|_C)(f)(g), h \rangle_{(\chi, u)}$  provided  $\{(\chi_n, u_n)\}$  converges to  $(\chi, u)$  in  $\widehat{\mathcal{A}}$ , and that  $\text{Ind}_C^{A_u}(\chi|_C)$  is weakly contained in  $\chi$ .  $\square$

#### 4. THE ASSOCIATED T-GROUPOID

Throughout this section, we will assume that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy, closed orbits, and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . We will also assume that  $u \mapsto A_u$  is continuous so that there is a Haar system  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  for the isotropy groupoid  $\mathcal{A}$ .

We let  $\widehat{\mathcal{A}} \rtimes \mathcal{R} = \{(\chi, u, \dot{\gamma}) \in \widehat{\mathcal{A}} \times \mathcal{R} : u = r(\gamma)\}$ . Note that  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  is closed in  $\widehat{\mathcal{A}} \times \mathcal{R}$  and therefore locally compact. For convenience, we will hereafter write elements of  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  as pairs  $(\chi, \dot{\gamma})$  with  $\chi \in A_{r(\gamma)}$ . Note that  $(\chi_\alpha, \dot{\gamma}_\alpha) \rightarrow (\chi, \dot{\gamma})$  if and only if  $\dot{\gamma}_\alpha \rightarrow \dot{\gamma}$  in  $\mathcal{R}$ , and if  $a_\alpha \in A_{r(\gamma_\alpha)} \rightarrow a$ , then  $\chi_\alpha(a_\alpha) \rightarrow \chi(a)$ .

We have chosen the notation  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  (rather than  $\widehat{\mathcal{A}} * \mathcal{R}$ ) to emphasize that  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  is a locally compact groupoid in a natural way. Recall that if  $\chi \in \widehat{A}_{r(\gamma)}$ , then  $\chi \cdot \gamma$  denotes the element of  $\widehat{A}_{s(\gamma)}$  given by  $a \mapsto \chi(\gamma a \gamma^{-1})$ , and that  $\chi \cdot \gamma$  depends only on the class  $\dot{\gamma} = j(\gamma)$  of  $\gamma$  in  $\mathcal{R}$ . The inverse operation

$$(4.1) \quad (\chi, \dot{\gamma})^{-1} = (\chi \cdot \gamma, \dot{\gamma}^{-1})$$

is easily seen to be a homeomorphism. Two elements  $(\chi, \dot{\gamma})$  and  $(\chi', \dot{\gamma}')$  are composable exactly when  $\chi' = \chi \cdot \gamma$ , and then

$$(\chi, \dot{\gamma})(\chi \cdot \gamma, \dot{\gamma}') = (\chi, \dot{\gamma}\dot{\gamma}').$$

The range and source maps, given by

$$r_{\widehat{\mathcal{A}} \rtimes \mathcal{R}}(\chi, \dot{\gamma}) = (\chi, r(\gamma)) \quad \text{and} \quad s_{\widehat{\mathcal{A}} \rtimes \mathcal{R}}(\chi, \dot{\gamma}) = (\chi \cdot \gamma, s(\gamma)),$$

define continuous surjections on  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  onto a closed subset which can be identified with  $\widehat{\mathcal{A}}$ . (We will normally write simply  $r$ , rather than  $r_{\widehat{\mathcal{A}} \rtimes \mathcal{R}}$ , when it is clear from context with which groupoid  $r$  is associated.) It is not hard to verify that  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  is a principal locally compact groupoid with unit space  $\widehat{\mathcal{A}}$ .

There is an analogous groupoid structure on  $\widehat{\mathcal{A}} \rtimes \mathcal{G}$  making it a locally compact groupoid with unit space  $\widehat{\mathcal{A}}$ . (The operations are exactly those above, but “without the dots.”)

Notice that, having fixed a Haar measure  $\beta^u$  on each  $A_u$ , there is no reason to suspect that (2.12) holds. We do, however have the following.

**Lemma 4.1.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Suppose also that  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  is a Haar system for the isotropy subgroupoid  $\mathcal{A}$ . Then there is a continuous  $\mathcal{A}$ -invariant homomorphism  $\omega$  from  $\mathcal{G}$  to  $\mathbb{R}^+$  such that*

$$\int_{A_{r(\gamma)}} f(a) d\beta^{r(\gamma)}(a) = \omega(\gamma) \int_{A_{s(\gamma)}} f(\gamma a \gamma^{-1}) d\beta^{s(\gamma)}(a) \quad \text{for all } f \in C_c(\mathcal{A}).$$

*Proof.* We show only that  $\omega$  is continuous. Suppose, to the contrary, that  $\gamma_n \rightarrow \gamma_0$  and  $|\omega(\gamma_n) - \omega(\gamma_0)| \geq \epsilon > 0$ . We can certainly choose  $f$  such that  $\int_{A_{r(\gamma_0)}} f(a) d\beta^{r(\gamma_0)}(a) = 1$ . Therefore  $\int_{A_{r(\gamma_n)}} f(a) d\beta^{r(\gamma_n)}(a)$  is eventually nonzero as well. We claim that we may as well assume that  $s(\gamma_n) \neq s(\gamma_0)$  for all  $n > 0$ . Otherwise we can pass to a subsequence, relabel, and assume that  $s(\gamma_n) = s(\gamma_0) = u$

for all  $n$ . Then  $f(\gamma_n \cdot \gamma_n^{-1})$  converges to  $f(\gamma_0 \cdot \gamma_0^{-1})$  in the inductive limit topology on  $C_c(A_u)$ . Then

$$\int_{A_u} f(\gamma_n a \gamma_n^{-1}) d\beta^u(a) \rightarrow \int_{A_u} f(\gamma_0 a \gamma_0^{-1}) d\beta^u(a),$$

and

$$(4.2) \quad \omega(\gamma_n)^{-1} = \left( \int_{A_{r(\gamma_n)}} f(a) d\beta^{r(\gamma_n)}(a) \right)^{-1} \left( \int_{A_u} f(\gamma_n a \gamma_n^{-1}) d\beta^{s(\gamma_n)}(a) \right) \rightarrow \omega(\gamma_0)^{-1},$$

which is a contradiction.

But if we assume that  $s(\gamma_n) \neq s(\gamma_0)$  for all  $n > 0$ , then passing to a subsequence and relabeling, we can assume that  $s(\gamma_n) \neq s(\gamma_m)$  for all  $n \neq m$ . Then  $S = s_{\mathcal{A}}^{-1}(\{s(\gamma_n)\}_{n=0}^\infty)$  is closed in  $\mathcal{A}$ , and we can define  $\iota$  on  $S$  by  $\iota(a) = n$  when  $s(a) = s(\gamma_n)$ . Then the function

$$F_0(a) = f(\gamma_{\iota(a)} a \gamma_{\iota(a)}^{-1})$$

is continuous and compactly supported on  $S$  and therefore has an extension  $F \in C_c(\mathcal{A})$ . Then, too,

$$\int_{A_{s(\gamma_n)}} f(\gamma_n a \gamma_n^{-1}) d\beta^{s(\gamma_n)}(a) = \int_{A_{s(\gamma_n)}} F(a) d\beta^{s(\gamma_n)}(a)$$

converges to

$$\int_{A_{s(\gamma_0)}} F(a) d\beta^{s(\gamma_0)}(a) = \int_{A_{s(\gamma_0)}} f(\gamma_0 a \gamma_0^{-1}) d\beta^{s(\gamma_0)}(a).$$

We now obtain a contradiction just as in (4.2). □

The presence of continuously varying isotropy allows a much improved version of Lemma 2.1.

**Lemma 4.2.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Suppose also that  $\{\beta^u\}_{u \in \mathcal{G}^{(0)}}$  is a Haar system for the isotropy subgroupoid  $\mathcal{A}$ .*

(1) *The formula*

$$\mathcal{Q}(f)(\dot{\gamma}) = \int_{A_u} f(\gamma a) d\beta^{s(\gamma)}(a)$$

*defines a surjection from  $C_c(\mathcal{G})$  onto  $C_c(\mathcal{R})$ .*

(2) *There is a Bruhat approximate cross-section for  $\mathcal{G}$  over  $\mathcal{R}$ ; that is, there is a non-negative, bounded, continuous function  $b$  on  $\mathcal{G}$  such that for any compact set  $K$  in  $\mathcal{G}$  the support of  $b$  and  $K\mathcal{A}$  have compact intersection, and*

$$\int_{\mathcal{A}} b(\gamma a) d\beta^{s(\gamma)}(a) = 1 \quad \text{for all } \gamma \in \mathcal{G}.$$

(3) *There is a Haar system  $\{\alpha^u\}_{u \in \mathcal{G}^{(0)}}$  for  $\mathcal{R}$  satisfying*

$$\int_{\mathcal{G}} f(\gamma) d\lambda^u(\gamma) = \int_{\mathcal{R}} \int_{\mathcal{A}} f(\gamma a) d\beta^{s(\gamma)}(a) d\alpha^u(\dot{\gamma}).$$

*Proof.* The argument closely parallels that of Lemma 2.1. To see that  $\mathcal{Q}(f) \in C_c(\mathcal{R})$ , notice that  $u \mapsto \beta^u$  is (part of) a Haar system for the canonical group bundle  $\Sigma = \{(C, \gamma) : C \in \Sigma^{(0)} \text{ and } \gamma \in C\}$  ([16, Corollary 1.4]). Part (2) is [14, Lemma II.2.2]. The existence of  $\alpha^u$  follows readily once we show that if

$$(4.3) \quad \int_{\mathcal{A}} f(\gamma a) d\beta^{s(\gamma)}(a) = 0 \quad \text{for all } \gamma \in \mathcal{G}^u,$$

then

$$\int_{\mathcal{G}} f(\gamma) d\lambda^u = 0.$$

So if (4.3) holds, then for any  $h \in C_c(\mathcal{G})$ ,

$$(4.4) \quad \begin{aligned} \int_{\mathcal{G}} f * h(a) d\beta^u(a) &= \int_{\mathcal{A}} \int_{\mathcal{G}} f(a\gamma)h(\gamma^{-1}) d\lambda^u(\gamma) d\beta^u(a) \\ &= \int_{\mathcal{G}} \left( \omega(\gamma) \int_{\mathcal{A}} f(\gamma a) d\beta^{s(\gamma)}(a) \right) h(\gamma^{-1}) d\lambda^u(\gamma) = 0. \end{aligned}$$

But if  $h$  is an appropriately cut down Bruhat approximate section (as in part (2) above), then (4.4) equals  $\int_{\mathcal{G}} f(\gamma) d\lambda^u(\gamma)$ .

If  $F \in C_c(\mathcal{R})$ , then using  $b$  from part (2),

$$\int_{\mathcal{R}} F(\dot{\gamma}) d\alpha^u(\dot{\gamma}) = \int_{\mathcal{G}} F(\dot{\gamma})b(\gamma) d\lambda^u(\gamma).$$

It now follows easily that the right-hand side is continuous in  $u$ . Since the left-invariance is easily checked,  $\{\alpha^u\}_{u \in \mathcal{G}^{(0)}}$  is a Haar system for  $\mathcal{R}$ . □

The next step is to construct a  $\mathbb{T}$ -groupoid over  $\widehat{\mathcal{A}} \times \mathcal{R}$  (see [10, §2]). For this it will be convenient to write  $\widehat{\mathcal{A}} * \mathcal{G} \times \mathbb{T}$  as triples

$$\{(\chi, z, \gamma) : \chi \in \widehat{A}_{r(\gamma)}, z \in \mathbb{T}, \text{ and } \gamma \in \mathcal{G}\}.$$

Our twist is the quotient  $\mathcal{D}$  obtained by identifying  $(\chi, \chi(a)z, \gamma)$  and  $(\chi, z, a \cdot \gamma)$ . Alternatively,  $\mathcal{D}$  is the quotient groupoid of  $\widehat{\mathcal{A}} * \mathcal{G} \times \mathbb{T}$  by the *closed* subgroupoid  $B = \{(\chi, \overline{\chi(a)}, a) \in \widehat{\mathcal{A}} * \mathcal{G} \times \mathbb{T} : a \in \mathcal{A}\}$ . Therefore  $\mathcal{D}$  is a Hausdorff locally compact groupoid with unit space  $\widehat{\mathcal{A}}$ . Note that

$$\begin{aligned} [\chi, z, \gamma]^{-1} &= [\chi \cdot \gamma, \bar{z}, \gamma^{-1}], \\ r([\chi, z, \gamma]) &= (\chi, r(\gamma)), \quad s([\chi, z, \gamma]) = (\chi \cdot \gamma, s(\gamma)), \\ [\chi, z, \gamma][\chi \cdot \gamma, z', \gamma'] &= [\chi, zz', \gamma\gamma']. \end{aligned}$$

We can identify  $\widehat{\mathcal{A}} \times \mathbb{T}$  with a subgroupoid of  $\mathcal{D}$ :

$$i(u, \chi, z) = [\chi, z, u],$$

and we can define  $j_{\mathcal{D}} : \mathcal{D} \rightarrow \widehat{\mathcal{A}} \times \mathcal{R}$  by

$$j_{\mathcal{D}}([\chi, z, \gamma]) = (\chi, \dot{\gamma}).$$

Evidently,  $j_{\mathcal{D}}$  is a continuous surjection with kernel  $\widehat{\mathcal{A}} \times \mathbb{T}$  (really  $i(\widehat{\mathcal{A}} \times \mathbb{T})$ ). We claim  $j_{\mathcal{D}}$  is open. Indeed if  $j_{\mathcal{D}}([\chi_n, z_n, \gamma_n])$  converges to  $j_{\mathcal{D}}([\chi, x, \gamma])$ , then  $\dot{\gamma}_n$  converges to  $\dot{\gamma}$ . Since  $j : \mathcal{G} \rightarrow \mathcal{R}$  is open (recall  $\dot{\gamma} = j(\gamma)$ ), we may assume that there are  $a_n \in \mathcal{A}$  such that  $a_n \gamma_n$  converges to  $\gamma$  in  $\mathcal{G}$ . Then  $(\chi_n, z, a_n \gamma_n)$  converges to  $(\chi, z, \gamma)$  in  $\widehat{\mathcal{A}} * \mathcal{G} \times \mathbb{T}$ . Since  $j_{\mathcal{D}}([\chi_n, z, a_n \gamma_n]) = j_{\mathcal{D}}([\chi_n, z_n, \gamma_n])$ , this shows that  $j_{\mathcal{D}}$  is open. Thus we have proved the following proposition.



**Proposition 4.3.** *The groupoid  $\mathcal{D}$  constructed above is a  $\mathbb{T}$ -groupoid over  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$ .*

*Remark 4.4.* Our construction of  $\mathcal{D}$  is motivated by the usual “push-out” construction for abelian groups (cf. [18, Lemma 10.10]):

$$\begin{array}{ccccc} \widehat{\mathcal{A}} * \mathcal{A} & \hookrightarrow & \widehat{\mathcal{A}} \rtimes \mathcal{G} & \twoheadrightarrow & \widehat{\mathcal{A}} \rtimes \mathcal{R} \\ \downarrow & & \downarrow & & \parallel \\ \widehat{\mathcal{A}} \rtimes \mathbb{T} & \hookrightarrow & \mathcal{D} & \dashrightarrow & \widehat{\mathcal{A}} \rtimes \mathcal{R} \end{array}$$

Recall that  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  is the completion, as described in [15, §3] and [10, §3], of  $C_c(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D})$  which, in turn, consists of functions  $\widetilde{F} \in C_c(\mathcal{D})$  satisfying

$$t\widetilde{F}([\chi, z, \gamma]) = \widetilde{F}(t \cdot [\chi, z, \gamma]) = \widetilde{F}([\chi, t \cdot z, \gamma]).$$

We will make use of the fact that such  $\widetilde{F}$  are determined by their values on classes of the form  $[\chi, 1, \gamma]$ , and we will identify  $C_c(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D})$  with the collection  $C_c^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  of continuous functions  $F$  on  $\widehat{\mathcal{A}} \rtimes \mathcal{G}$  that satisfy  $F(\chi, a\gamma) = \chi(a)F(\chi, \gamma)$  for all  $a \in \mathcal{A}$  and  $\gamma \in \mathcal{G}$ , and that are such that the support of  $F$  has compact image in  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$ .<sup>5</sup> We'll write  $C_{\infty}^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  for the collection of functions on  $\widehat{\mathcal{A}} \rtimes \mathcal{G}$  which satisfy the same functional equation and which are such that (1)  $(\chi, \dot{\gamma}) \mapsto |F(\chi, \gamma)|$  vanishes at infinity on  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  and (2) there is a compact set  $K \subset \mathcal{R}$  such that  $F(\chi, \gamma) = 0$  for all  $\dot{\gamma} \notin K$ . Then there is a  $*$ -algebra structure on  $C_{\infty}^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  given by

$$F * G(\chi, \gamma) = \int_{\mathcal{R}} F(\chi, \gamma\eta)G(\chi \cdot \gamma\eta, \eta^{-1}) d\alpha^{s(\gamma)}(\dot{\eta}), \text{ and} \\ F^*(\chi, \gamma) = \overline{F(\chi \cdot \gamma, \gamma^{-1})},$$

such that  $C_c^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  may be viewed as a  $*$ -subalgebra.

If  $f \in C_c(\mathcal{G})$  and  $(\chi, \gamma) \in \widehat{\mathcal{A}} \rtimes \mathcal{G}$ , then we can define

$$(4.5) \quad \Phi(f)(\chi, \gamma) = \omega(\gamma)^{-1/2} \int_{A_r(\gamma)} \overline{\chi(a)} f(a\gamma) d\beta^{r(\gamma)}(a).$$

**Proposition 4.5.** *Suppose that  $\mathcal{G}$  is a second countable locally compact groupoid with abelian isotropy and Haar system  $\{\lambda^u\}_{u \in \mathcal{G}^{(0)}}$ . Then  $\Phi$  is a  $*$ -homomorphism of  $C_c(\mathcal{G})$  into  $C_{\infty}^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ . If  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff, then  $\Phi$  extends to an isomorphism of  $C^*(\mathcal{G}, \lambda)$  with  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$ .*

The proof of Proposition 4.5 is somewhat complicated, and we will require several preliminary results.

**Lemma 4.6.** *For each  $f \in C_c(\mathcal{G})$ ,  $\Phi(f) \in C_{\infty}^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ .*

<sup>5</sup>Suppose  $F$  and  $\widetilde{F}$  are related by  $F(\chi, \gamma) = \widetilde{F}([\chi, 1, \gamma])$  and  $\widetilde{F}([\chi, z, \gamma]) = zF(\chi, \gamma)$ . If  $\widetilde{F}$  has compact support in  $\mathcal{D}$ , and  $F(\chi_n, \gamma_n) \neq 0$  for all  $n$ , then we may assume that  $[\chi_n, 1, \gamma_n] \rightarrow [\chi, z, \gamma]$  in  $\mathcal{D}$ . Then there are  $a_n$  such that  $(\chi_n, \chi_n(a_n), a_n\gamma_n) \rightarrow (\chi, z, \gamma)$ . Thus  $(\chi_n, \dot{\gamma}_n) \rightarrow (\chi, \dot{\gamma})$ , and the support of  $F$  has compact image in  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$ .

Conversely, suppose that the support of  $F$  has compact image in  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$ , and that  $\widetilde{F}([\chi_n, z_n, \gamma_n]) \neq 0$  for all  $n$ . Then  $F(\chi_n, \gamma_n) \neq 0$ , and we can assume that  $a_n\gamma_n \rightarrow \gamma$  and  $\chi_n \rightarrow \chi$ . But  $[\chi_n, z_n, \gamma_n] = [\chi_n, \chi_n(a_n)z_n, \alpha_n\gamma_n]$ . Since  $\chi_n(a_n)z_n \in \mathbb{T}$ , we can assume that  $\chi_n(a_n)z_n \rightarrow \omega$  so that  $(\chi_n, \chi_n(a_n)z_n, \alpha_n\gamma_n) \rightarrow (\chi, \omega, \gamma)$ . Thus  $\{[\chi_n, z_n, \gamma_n]\}$  has a convergent subsequence, and  $\widetilde{F}$  must have compact support too.

*Proof.* First we show that  $\Phi(f)$  is continuous. Towards this end, suppose that  $(\chi_n, \gamma_n) \rightarrow (\chi_0, \gamma_0)$  in  $\widehat{\mathcal{A}} \rtimes \mathcal{G}$ . Let  $u_n = r(\gamma_n)$ . If  $u_n = u_0$  for all  $n$ , then  $f_n = \omega(\gamma_n)^{-1}f(\cdot\gamma_n)$  converges to  $f_0$  in the inductive limit topology on  $C_c(A_{u_0})$  and  $\chi_n$  converges to  $\chi$  in  $\widehat{A}_{u_0}$ . Consequently,  $\Phi(f)(\chi_n, \gamma_n) = \chi_n(f_n)$  converges to  $\Phi(f)(\chi_0, \gamma_0) = \chi_0(f_0)$ . Otherwise, we may assume that  $u_n \neq u_m$  for all  $n \neq m$ . Then  $S = r_{\mathcal{A}}^{-1}(\{u_n\}_{n=0}^\infty)$  is closed and, defining  $\iota(a) = n$  when  $a \in A_{u_n}$ ,

$$F_0(a) = f(a\gamma_{\iota(a)})$$

is continuous with compact support on  $S$ . Thus if  $F$  is any extension in  $C_c(\mathcal{A})$ ,  $\Phi(f)(\chi_n, \gamma_n) = (\chi_n, u_n)(F)$ , and the desired conclusion follows.

This leaves only the question of support. Since  $f$  has compact support in  $\mathcal{G}$ , the question of support in the second variable is clear and we need only see that  $|\Phi(f)|$  vanishes at infinity on  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$ . But if  $|\Phi(\chi_n, \gamma_n)| \geq \epsilon > 0$  for all  $n = 1, 2, \dots$ , then we can assume that  $\dot{\gamma}_n$  converges to some  $\dot{\gamma}_0$  in  $\mathcal{R}$ . Then replacing  $\gamma_n$  by  $a_n\gamma_n$  for appropriate  $a_n \in \mathcal{A}$ , we may as well assume from the onset that  $\gamma_n$  converges to  $\gamma_0$  in  $\mathcal{G}$ . As above, we may pass to a subsequence so that either  $r(\gamma_n) = u$  for all  $n$ , or such that  $r(\gamma_n) = u_n \neq u_m = r(\gamma_m)$  whenever  $n \neq m$ . In the first case,  $f_n = \omega(\gamma_n)^{-1}f(\cdot\gamma_n)$  converges to  $f_0$  in the inductive limit topology. Consequently, we can assume that  $\|f_n - f_0\|_{L^1(A_{u_n})} < \epsilon/2$ . Then  $\chi_n(f_0) \geq \epsilon/2$  and  $\{\chi_n\}$  must have a convergent subsequence in  $\widehat{A}_u$ . But then  $(\chi_n, \gamma_n)$  has a convergent subsequence in  $\widehat{\mathcal{A}} \rtimes \mathcal{G}$  as required.

In the second case, we can, as above, produce a  $F \in C_c(\mathcal{A})$  so that  $\Phi(f)(\chi_n, \gamma_n) = (\chi_n, u_n)(F)$ . Therefore  $\{(\chi_n, u_n)\}$  has a convergent subsequence in  $\widehat{\mathcal{A}}$ , and the result follows.  $\square$

Recall from the proof of [10, Proposition 3.3], that if points are closed in  $\widehat{\mathcal{A}}/\mathcal{R}$ , then every irreducible representation of  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  is of the form  $\widetilde{L}^{(x,u)}$  as defined in the beginning of [10, §3]. Let  $h = (x, u)$  and recall that  $\widetilde{L}^h$  acts on the completion  $\mathcal{W}_h$  of the collection  $\mathcal{W}_h^0$  of bounded compactly supported Borel functions on  $\mathcal{D}_h$  such that  $\xi([\chi \cdot \gamma^{-1}, z, \gamma]) = z\xi([\chi \cdot \gamma^{-1}, 1, \gamma])$  with respect to the inner product

$$(4.6) \quad \langle \xi, \eta \rangle_h = \int_{\mathcal{R}} \xi([\chi \cdot \gamma^{-1}, z, \gamma]) \overline{\eta([\chi \cdot \gamma^{-1}, z, \gamma])} d\alpha_u(\dot{\gamma}).$$

As above, we may view elements of  $\mathcal{W}_h^0$  as functions on  $(\widehat{\mathcal{A}} \rtimes \mathcal{G})_h$  such that

$$\xi(\chi \cdot \gamma^{-1}, a\gamma) = \chi(\gamma^{-1}a\gamma)\xi(\chi \cdot \gamma^{-1}, \gamma),$$

and we can replace (4.6) by

$$\langle \xi, \eta \rangle_h = \int_{\mathcal{R}} \xi(\chi \cdot \gamma^{-1}, \gamma) \overline{\eta(\chi \cdot \gamma^{-1}, \gamma)} d\alpha_u(\dot{\gamma}).$$

As observed in [10], we can restrict attention to  $\xi, \eta \in C_c(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D})$ , and therefore to  $C_c^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ . If  $f, \xi \in C_c^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ , then

$$\begin{aligned} \widetilde{L}^h(f)\xi(\chi \cdot \gamma^{-1}, \gamma) &= f * \xi(\chi \cdot \gamma^{-1}, \gamma) \\ &= \int_{\mathcal{R}} f(\chi \cdot \gamma^{-1}, \gamma\eta)\xi(\chi \cdot \eta, \eta^{-1}) d\alpha^u(\dot{\eta}). \end{aligned}$$

Recall that  $\text{Ind}(u, A_u, \chi)$  is equivalent to the representation  $T^{(x,u)}$  on  $\mathcal{H}_{(x,u)}$  given by (2.10).

**Lemma 4.7.** *Suppose that  $h \in \widehat{\mathcal{A}}$  and that  $f \in C_c(\mathcal{G})$ . The map  $S$  defined by the formula  $S(\xi)(\gamma) = \omega(\gamma)^{-1/2}\xi(\chi \cdot \gamma^{-1}, \gamma)$  is a unitary map of  $\mathcal{W}_h$  onto  $\mathcal{H}_h$  such that for all  $f \in C_c(\mathcal{G})$ ,*

$$T^h(f) = S\widetilde{L}^h(\Phi(f))S^*.$$

*Proof.* Using Lemmas 2.1 and 4.2, it is not hard<sup>6</sup> to see that the Radon-Nikodym derivative of  $\alpha_u$  with respect to  $\sigma^u$  at  $\dot{\gamma}$  is  $\omega(\gamma)^{-1}$ . It then follows easily that  $S$  is unitary. Now we compute

$$\begin{aligned} S\widetilde{L}^h(\Phi(f))\xi(\gamma) &= \omega(\gamma)^{-1/2}\widetilde{L}^h(\Phi(f))\xi(\chi \cdot \gamma^{-1}, \gamma) \\ &= \omega(\gamma)^{-1/2} \int_{\mathcal{R}} \Phi(f)(\chi \cdot \gamma^{-1}, \gamma)\xi(\chi \cdot \eta, \eta^{-1}) d\alpha^u(\dot{\eta}) \\ &= \omega(\gamma)^{-1} \int_{\mathcal{R}} \omega(\eta)^{-1/2} \int_{A_r(\gamma)} \overline{\chi(\gamma^{-1}a\gamma)} f(a\eta)\xi(\chi \cdot \eta, \eta^{-1}) d\beta^{r(\gamma)}(a) \alpha^u(\dot{\eta}) \\ &= \int_{\mathcal{R}} \omega(\eta)^{-1/2} \int_{A_u} \overline{\chi(a)} f(\gamma a\eta)\xi(\chi \cdot \eta, \eta^{-1}) d\beta^u(a) d\alpha^u(\dot{\eta}) \\ &= \int_{\mathcal{R}} \omega(\eta)^{1/2} \int_{A_s(\eta)} f(\gamma\eta a)\xi(\chi \cdot \eta, a^{-1}\eta^{-1}) d\beta^{s(\eta)}(a) d\alpha^u(\dot{\eta}) \\ &= \int_{\mathcal{G}} f(\gamma\eta)\omega(\eta)^{1/2}\xi(\chi \cdot \eta, \eta^{-1}) d\lambda^u(\eta) \\ &= T^h(f)S(\xi)(\gamma). \quad \square \end{aligned}$$

**Lemma 4.8.** *Suppose that  $F \in C_\infty^D(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ ,  $h \in \widehat{\mathcal{A}}$ , and  $\xi, \zeta \in \mathcal{W}_h^0$ . Then there is a compact set  $K$  in  $\mathcal{R}$  so that  $F(\chi, \gamma) = 0$  if  $\dot{\gamma} \notin K$ . Furthermore, there is a  $M \in \mathbb{R}^+$ , depending only on  $K$ , such that*

$$|\langle \widetilde{L}(F)\xi, \zeta \rangle_h| \leq M\|F\|_\infty\|\xi\|_{\mathcal{W}_h}\|\zeta\|_{\mathcal{W}_h}.$$

*Proof.* Fix a compact set  $K$  in  $\mathcal{R}$  so that  $F(\chi, \gamma) = 0$  if  $\dot{\gamma} \notin K$ . Since  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff, it follows that  $[u]$  is closed. Thus  $\mathcal{R}|_{[u]}$  is a transitive second countable locally compact groupoid. It follows from [8, Theorem 2.2B] that  $\mathcal{R}$  is homeomorphic to  $\mathcal{R}_u \times \mathcal{R}^u$  via the obvious map:  $(\gamma, \eta) \mapsto \gamma\eta$ . In particular, there is a compact set  $C$  in  $\mathcal{R}$  such that  $C = C^{-1}$ , and such that  $\gamma\eta \in K$  implies that both  $\gamma, \eta \in C$ . We let  $M = \sup_{u \in \mathcal{G}^{(0)}} \alpha_u(C)$ . Then  $M < \infty$  and depends only on  $F$ .

<sup>6</sup>Just compute

$$\begin{aligned} \int_{\mathcal{G}_u/A_u} \omega(\gamma)^{-1}F(\dot{\gamma})d\sigma^u(\dot{\gamma}) &= \int_{\mathcal{G}} \omega(\gamma)^{-1}F(\dot{\gamma})b(\gamma)d\lambda_u(\gamma) = \int_{\mathcal{G}} \omega(\gamma)F(\dot{\gamma}^{-1})b(\gamma^{-1})d\lambda^u(\gamma) \\ &= \int_{\mathcal{R}} \int_{A_s(\gamma)} \omega(\gamma)F(\dot{\gamma}^{-1})b(a\gamma^{-1})d\beta^{s(\gamma)}(a) d\alpha^u(\dot{\gamma}) \\ &= \int_{\mathcal{R}} \int_{A_u} F(\dot{\gamma}^{-1})b(\gamma^{-1}a)d\beta^u(a) d\alpha^u(\dot{\gamma}) = \int_{\mathcal{G}_u/A_u} F(\dot{\gamma})d\alpha_u(\dot{\gamma}). \end{aligned}$$

$$\begin{aligned} |\langle \tilde{L}(F)\xi, \zeta \rangle_h| &= \left| \int_{\mathcal{G}} \int_{\mathcal{G}} F(\chi \cdot \gamma^{-1}, \gamma\eta) \xi(\chi \cdot \eta, \eta^{-1}) \overline{\zeta(\chi \cdot \gamma^{-1}, \gamma)} d\alpha^u(\dot{\eta}) d\alpha_u(\dot{\gamma}) \right| \\ &\leq \|F\|_\infty \int_C \int_C |\xi(\chi \cdot \eta, \eta^{-1}) \overline{\zeta(\chi \cdot \gamma^{-1}, \gamma)}| d\alpha^u(\dot{\eta}) d\alpha_u(\dot{\gamma}) \\ &= \|F\|_\infty \left( \int_C |\xi(\chi \cdot \eta^{-1}, \eta)| d\alpha_u(\dot{\eta}) \right) \left( \int_C |\zeta(\chi \cdot \gamma^{-1}, \gamma)| \alpha_u(\dot{\gamma}) \right) \\ &\leq \|F\|_\infty M^{1/2} \|\xi\|_{\mathcal{W}_h} M^{1/2} \|\zeta\|_{\mathcal{W}_h}, \end{aligned}$$

where the last inequality follows from the Hölder inequality. □

*Proof of Proposition 4.5.* It now follows from Lemma 4.6 and a routine computation that  $\Phi$  is a  $*$ -homomorphism from  $C_c(\mathcal{G})$  into  $C_\infty^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$ . As a consequence of Lemma 4.8, we see that  $C_c^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  is dense in  $C_\infty^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  in the pre- $C^*$ -norm induced by the  $\{\tilde{L}^h\}_{h \in \widehat{\mathcal{A}}}$ . Therefore we may view  $C_\infty^{\mathcal{D}}(\widehat{\mathcal{A}} \rtimes \mathcal{G})$  as a dense subalgebra of  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$ , and  $\Phi$  defines an injection of  $C^*(\mathcal{G}, \lambda)$  into  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$ .

Since  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff, this implies that  $\mathcal{G}^{(0)}/\mathcal{R}$  is Hausdorff, and hence that orbits are closed in  $\mathcal{G}^{(0)}$ . As remarked in the discussion preceding Lemma 2.11, each irreducible representation  $\text{Ind}(u, A_u, \chi)$  factors through  $C^*(\mathcal{G}|_{[u]}, \lambda)$  for some orbit  $[u]$ . Since  $C^*(\mathcal{G}|_{[u]}, \lambda)$  is Morita equivalent to  $C^*(A_u)$ , it follows that  $T^{(\chi, u)}$ , and hence  $\tilde{L}^{(\chi, u)}$ , is a CCR representation for each  $(\chi, u) \in \widehat{\mathcal{A}}$ . It follows that  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  is a CCR algebra, and it is fairly clear that  $\Phi(C^*(\mathcal{G}, \lambda))$  is a rich subalgebra of  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  (as defined in [2, Definition 11.1.1]). Thus,  $\Phi$  is surjective [2, Proposition 11.1.6]. □

Now Theorem 1.1 follows almost immediately.

*Proof of Theorem 1.1.* It is straightforward to check that  $\mathcal{R}$  acts properly on  $\mathcal{G}^{(0)}$  if and only if  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  acts properly on  $\widehat{\mathcal{A}}$ . If  $C^*(\mathcal{G}, \lambda)$  has continuous trace, then  $(C^*(\mathcal{G}, \lambda))^\wedge$  is Hausdorff. We have already noted that this implies that the stabilizer map is continuous (Proposition 3.1). Furthermore Proposition 3.8 implies that  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff, and Proposition 4.5 applies. We may therefore conclude that  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  has continuous trace. It follows from [10, Theorem 4.3] that  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  acts properly on  $\widehat{\mathcal{A}}$ , and, as pointed out above, this implies that  $\mathcal{R}$  acts properly on  $\mathcal{G}^{(0)}$ . This proves the necessity of conditions (1) and (2).

If, on the other hand, conditions (1) and (2) are satisfied, then as pointed about above,  $\widehat{\mathcal{A}} \rtimes \mathcal{R}$  acts properly on  $\widehat{\mathcal{A}}$  so that  $\widehat{\mathcal{A}}/\mathcal{R}$  is Hausdorff. Again Proposition 4.5 applies, and it suffices to observe that  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$  has continuous trace. But that follows from [10, Theorem 4.2]. □

*Remark 4.9.* It is a consequence of the above proof that if  $\mathcal{G}$  has abelian isotropy and if  $C^*(\mathcal{G}, \lambda)$  has continuous trace, then  $C^*(\mathcal{G}, \lambda)$  is isomorphic to the restricted groupoid  $C^*$ -algebra of a  $\mathbb{T}$ -groupoid over an equivalence relation: namely,  $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{R}; \mathcal{D}, \alpha)$ .

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