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CONTINUOUS TRACE GROUPOID C*-ALGEBRAS

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§1. Introduction.

A *groupoid* is a small category in which every morphism is invertible. If there is at most one morphism from any given object to another, then the groupoid is called a *principal* groupoid. Thus a principal groupoid is essentially an equivalence relation on its set of objects: two objects are equivalent if, and only if, there is a morphism from one to the other. If one is given a finite principal groupoid, then, thinking of it as an equivalence relation on the finite set $\{1, 2, \ldots, n\}$, say, it indexes the matrix units of a finite dimensional C*-subalgebra A of the $n \times n$ matrices. The algebra A is the direct sum of full matrix algebras, the number of summands equaling the number of equivalence classes and the sizes of the summands equaling the sizes of the equivalence classes. Our objective in this note is to investigate an infinite dimensional generalization of this observation.

We follow the notation and terminology of Renault [9] except that we write s for the map he denotes by d. Throughout this paper, $\mathfrak{G}$ will denote a locally compact groupoid that admits a Haar system, which will be fixed and denoted by $\{\lambda^u\}_{u \in \mathfrak{G}^0}$. Unless otherwise indicated, we shall assume that $\mathfrak{G}$ is principal and we shall also assume that $\mathfrak{G}$ is second countable. While a number of our arguments work without this second assumption, the main results appear to use it in essential ways. Consequently, we have made no systematic effort to achieve the greatest possible generality in our ancillary lemmas and propositions. The space of objects in $\mathfrak{G}$, $\mathfrak{G}^0$, is called the *unit space* of $\mathfrak{G}$. The groupoid is principal precisely when the map π from $\mathfrak{G}$ to $\mathfrak{G}^0 \times \mathfrak{G}^0$ defined by the formula $\pi(\gamma) = (r(\gamma), s(\gamma))$, $\gamma \in \mathfrak{G}$, where $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$, is one to one. The image of π is an equivalence relation on $\mathfrak{G}^0 \times \mathfrak{G}^0$, denoted by R. While π is always continuous, it need not be a homeomorphism onto its range. However, we shall also want to require that R be closed in $\mathfrak{G}^0 \times \mathfrak{G}^0$ – we call such an equivalence relation *proper*. When π is a homomorphism onto a proper equivalence relation, then $\mathfrak{G}$ is called a *proper* principal groupoid.

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If R is transitive, i.e., if $R = \mathfrak{G}^0 \times \mathfrak{G}^0$, then as is proved in [9], the C*-algebra of the groupoid (and Haar system), $C^*(\mathfrak{G}, \lambda)$, is (isomorphic to) the algebra of all compact operators on a separable Hilbert space, i.e., $C^*(\mathfrak{G}, \lambda)$ is an elementary C*-algebra. On the other hand, by [6], if R is proper, then $C^*(\mathfrak{G}, \lambda)$ is isomorphic to the C*-algebra of a continuous field of elementary C*-algebras defined by a continuous field of Hilbert spaces over $\mathfrak{G}^0/R$ (See Proposition 2.2, below.) So, if $\mathfrak{G}$ is a proper principal groupoid, then an exact analogy exists between this setting and what happens when the groupoid is finite. Our primary goal here is prove the converse, i.e., we prove that if $C^*(\mathfrak{G}, \lambda)$ is the C*-algebra associated with such a continuous field of C*-algebras, then $\mathfrak{G}$ is proper. Actually we prove a slightly stronger statement than this in Theorem 2.3, which is our main theorem. It asserts: *If $\mathfrak{G}$ is second countable, locally compact, principal groupoid with Haar system $\{\lambda^u\}_{u \in \mathfrak{G}^0}$, then $C^*(\mathfrak{G}, \lambda)$ has continuous trace if and only if $\mathfrak{G}$ is proper.* Thus, while a priori $C^*(\mathfrak{G}, \lambda)$ might have non-trivial Dixmier-Douady invariant, our theorem (and Proposition 2.2) shows that it can't.

Part of our motivation for the present investigation comes from Green's paper [4] in which he proved that if G is a second countable, locally compact group acting freely on a second countable, locally compact, Hausdorff space, then the transformation group C*-algebra $C^*(G, X)$ has continuous trace if, and only if, G acts properly on X. Our result contains his because under his hypotheses $X \times G$ has the structure of a principal groupoid which is proper in our sense if, and only if, G acts properly on X. While portions of our line of reasoning follow his arguments, we feel that groupoid methods put more clearly into evidence the roles played by the various hypotheses. In addition to yielding more general results, groupoid arguments appear cleaner and more natural than arguments tied to transformation groups.

§2 Equivalence Relations.

Recall that $\mathfrak{G}$ acts on $\mathfrak{G}^0$ on the right: let

$$\mathfrak{G}^0 * \mathfrak{G} = \{(u, \gamma) \in \mathfrak{G}^0 \times \mathfrak{G} : r(\gamma) = u\},$$

and for each $(u, \gamma) \in \mathfrak{G}^0 * \mathfrak{G}$, let $u \cdot \gamma = s(\gamma)$. We say that $\mathfrak{G}^0$ is a *proper* right $\mathfrak{G}$-space when the map $\Phi : \mathfrak{G}^0 * \mathfrak{G} \to \mathfrak{G}^0 \times \mathfrak{G}^0$, defined by $\Phi(u, \gamma) = (u, s(\gamma))$, is proper. Notice that $\Phi(u, \gamma) = \pi(\gamma)$. The next lemma is almost an immediate consequence of the definitions.

LEMMA 2.1. *If $\mathfrak{G}$ is a principal groupoid, then the following are equivalent.*

(i) *π is a closed map.*

(ii) *π is a homeomorphism onto a closed subset of $\mathfrak{G}^0 \times \mathfrak{G}^0$. (i.e., $\mathfrak{G}$ is a proper principal groupoid).*

(iii) *π is a proper map.*

(iv) *Φ is a closed map.*
(v) *Φ is a homeomorphism onto a closed subset of $\mathfrak{G}^0 \times \mathfrak{G}^0$.*
(vi) *Φ is a proper map* (i.e., *$\mathfrak{G}^0$ is a proper right $\mathfrak{G}$-space).*
(vii) *Given $K \subseteq \mathfrak{G}^0$ compact,*

$$\mathfrak{G}(K) = \{\gamma \in \mathfrak{G} : K \cdot \gamma \cap K \neq \emptyset\}$$

is compact in $\mathfrak{G}$.
(viii) *Given $K \subseteq \mathfrak{G}^0$ compact, $\mathfrak{G}(K)$ is relatively compact in $\mathfrak{G}$.*

PROOF. It follows from [1; I.10.1 Proposition 2] that (i), (ii), and (iii) are equivalent, as are (iv), (v), and (vi). Since π and Φ have identical ranges, (i)–(vi) are equivalent.

On the other hand, if $K \subseteq \mathfrak{G}^0$ is compact, and if Φ is a proper map, then $\Phi^{-1}(K \times K)$ is compact in $\mathfrak{G}^0 * \mathfrak{G}$. Since the projection pr_2 of $\mathfrak{G}^0 * \mathfrak{G}$ onto $\mathfrak{G}$ is continuous, it follows that $\mathfrak{G}(K) = \mathrm{pr}_2(\Phi^{-1}(K \times K))$ is compact. Thus, (vi) implies (vii).

Of course, (vii) implies (viii). Notice that if $\mathfrak{G}(K)$ is relatively compact, then since $\Phi^{-1}(K \times K) \subseteq K * \mathfrak{G}(K)$, $\Phi^{-1}(K \times K)$ is relatively compact and closed. Therefore, the latter is compact and Φ is a proper map. In short, (viii) implies (vi).

If $\mathfrak{G}$ is a proper principal groupoid, then it is easy to see that the orbit space $\mathfrak{G}^0/\mathfrak{G}$ is Hausdorff.

PROPOSITION 2.2. *If $\mathfrak{G}$ is a second countable proper principal gropoid with left Haar system $\{\lambda^u\}_{u \in \mathfrak{G}^0}$, then $C^*(\mathfrak{G}, \lambda)$ is strongly Morita equivalent to $C_0(\mathfrak{G}_0/\mathfrak{G})$. In particular, $C^*(\mathfrak{G}, \lambda)$ has continuous trace with trivial Dismier-Douady invariant.*

PROOF. Let $Z = \mathfrak{G}^0$. Then Z is a $(\mathfrak{G}, \mathfrak{G}^0/\mathfrak{G})$-equivalence ([6; Definition 2.1 and Example 2.5]). The proposition now follows from [6; Theorem 2.8] and [8; Proposition C1].

Our object in this article is to prove the following theorem which is a strengthened converse of Proposition 2.2.

THEOREM 2.3. *Suppose that $\mathfrak{B}$ is a second countable locally compact principal groupoid with left Haar system $\{\lambda^u\}_{u \in \mathfrak{G}^0}$. Then $C^*(\mathfrak{G}, \lambda)$ has continuous trace if and only if $\mathfrak{G}$ is a proper principal groupoid.*

Our proof of Theorem 2.3 will parallel Green's proof of Theorem 17 in [4]. In fact, the example at the end of [4] shows that even when $\mathfrak{G}$ is the groupoid of a locally compact transformation group, $C^*(\mathfrak{G}, \lambda)$ need not have continuous trace when one merely assmes that $\mathfrak{G}^0/\mathfrak{G}$ is Hausdorff. This contrasts with the foliation case: it follows from [3; Théorème 3.2] that if (V, F) is a smooth foliation without holonomy, then $C^*(V, F)$ has continuous trace if and only if V/F is Hausdorff.

We begin by recalling the definition of the representation L^u induced from the point mass δ_u at $u \in \mathfrak{G}^0$ (see [9] pages 81–82). In the case of a principal groupoid, L^u acts on $L^2(\mathfrak{G}, \lambda_u)$ where

$$\lambda_u(E) = \lambda^u(E^{-1}),$$

and $E^{-1} = \{\gamma^{-1} : \gamma \in E\}$. Recall that $\operatorname{supp}(\lambda_u) = \mathfrak{G}_u = s^{-1}(u)$. Then, if $f \in C_c(\mathfrak{G})$ and $\xi \in L^2(\mathfrak{G}, \lambda_u)$,

$$L^u(f)\xi(\gamma) = \int_{\mathfrak{G}} f(\gamma\alpha)\xi(\alpha^{-1})d\lambda^u(\alpha).$$

Our next lemma is certainly well known, but we know of no proof in the literature. We give a brief sketch of the proof for the reader's convenience.

LEMMA 2.4. *If $\mathfrak{G}$ is a second countable locally compact principal groupoid with left Haar system $\{\lambda^u\}_{u \in \mathfrak{G}^0}$, then the representation L^u is irreducible for each $u \in \mathfrak{G}^0$. Furthermore, if $[u] = [v]$, then L^u is unitarily equivalent to L^v.*

PROOF. The last statement is straightforward to check. For the first assertion, we first observe that L^u is equivalent to the representation R^u on $L^2([u], \mu_{[u]})$ defined by

$$R^u(f)\xi(\gamma \cdot u) = \int_{\mathfrak{G}} f(\gamma\alpha)\xi(\alpha^{-1} \cdot u)\, d\lambda^u(\alpha),$$

where $\gamma \cdot u = r(\gamma)$ and $\mu_{[u]}$ is the measure on $[u]$ defined by

$$\int \phi(v)\, d\mu_{[u]}(v) = \int_{\mathfrak{G}} \phi(s(\gamma))\, d\lambda^u(\gamma),$$

for $\phi \in C_c(\mathfrak{G}^0)$ (i.e., $\mu_{[u]} = s_*(\lambda^u)$).

On the other hand, any projection commuting with $R^u(C^*(\mathfrak{G}, \lambda))$ must also commute with $N_u(C_0(\mathfrak{G}^0))''$ where N_u is the representation of $C_0(\mathfrak{G}^0)$ on $L^2([u], \mu_{[u]})$ defined by

$$(N_u(\phi)\xi)(v) = \phi(v)\xi(v)$$

for $v \in [u]$, $\phi \in C_0(\mathfrak{G}^0)$, and $\xi \in L^2([u])$. Since $C_c(\mathfrak{G}^0)|_{[u]}$ separates points of $[u]$, $N_u(C_0(\mathfrak{G}))''$ is a maximal abelian subalgebra of operators on $L^2([u])$. Hence any projection commuting with $R_u(C^*(\mathfrak{G}, \lambda))$ must be of the form $N_u(\phi)$ for $\phi = \chi_E$ with $E \subseteq [u]$. Notice that since $N_u(\phi)$ commutes with every $R^u(f)$ we have

$$\phi(v) \int_{\mathfrak{G}} f(\alpha)\xi(\alpha^{-1} \cdot v)\, d\lambda^v(\alpha) = \int_{\mathfrak{G}} f(\alpha)\phi(\alpha^{-1} \cdot v)\xi(\alpha^{-1} \cdot v)\, d\lambda^v(\alpha)$$

for $\mu_{[u]}$-almost every v, all $\xi \in L^2$, and all $f \in C_c(\mathfrak{G})$. Therefore, for some $v \in [u]$,

$$\phi(v) = \phi(\alpha^{-1} \cdot v)$$

for λ^v-almost every α. It follows that ϕ is constant (a.e.) on $[u]$; this suffices.

PROPOSITION 2.5. *Suppose that* $\mathfrak{G}$ *is a second countable locally compact principal groupoid with left Haar system* $\{\lambda^u\}_{u\in\mathfrak{G}^0}$, *and that* $C^*(\mathfrak{G}, \lambda)$ *has continuous trace. Then* $u \mapsto [L^u]$ *defines a continuous open surjection of* $C^*(\mathfrak{G}, \lambda)^\wedge$ *which is constant on* $\mathfrak{G}$*-orbits. In particular,* $\mathfrak{G}^0/\mathfrak{G}$ *is homeomrphic to* $C^*(\mathfrak{G}, \lambda)^\wedge$.

PROOF. Now if f, g, and h are in $C_c(\mathfrak{G})$, the continuity of the Haar system implies that

$$u \mapsto \langle L^u(f)g, h\rangle_{L^2(\mathfrak{G},\lambda_u)}$$

is continuous. It follows from this and the preceding lemma that $u \mapsto L^u$ induces a continuous map Ψ of $\mathfrak{G}^0/\mathfrak{G}$ into $C^*(\mathfrak{G}, \lambda)^\wedge$.

On the other hand, following [9; 2.1.4], there is a homomorphism

$$V: C_0(\mathfrak{G}^0) \to \mathcal{M}(C^*(\mathfrak{G}, \lambda))$$

so that, given $\phi \in C_0(\mathfrak{G}^0)$ and $f \in C_c(\mathfrak{G})$ we have

$$(V(\phi)f)(\gamma) = \phi(r(\gamma))f(\gamma), \text{ and}$$

$$(fV(\phi))(\gamma) = \phi(s(\gamma))f(\gamma).$$

In particular, fixing $u \in \mathfrak{G}^0$, there is a representation M_u of $C_0(\mathfrak{G}^0)$ on $L^2(\mathfrak{G}, \lambda_u)$ such that

$$L^u(V(\phi)f) = M_u(\phi)L^u(f)$$

([9; 2.1.13]). Of course, $\ker(M_u)$ is an ideal J_{F_u} in $C_0(\mathfrak{G}^0)$ of functions which vanish on a closed set $F_u \subseteq \mathfrak{G}^0$. Using the fact that Ψ factors through $\mathfrak{G}^0/\mathfrak{G}$, it follows that F_u is $\mathfrak{G}$-invariant (i.e., saturated with respect to the $\mathfrak{G}$-action). Since each L^u is irreducible, it follows that F_u can't be the union of two closed $\mathfrak{G}$-invariant sets. Since the map $\delta: \mathfrak{G}^0 \to \mathfrak{G}^0/\mathfrak{G}$ is continuous and open, [5; Lemma on page 222] implies that F_u is an orbit closure.

The point here is that if $L^u \cong L^v$, then $M_u \cong M_v$. But M_u and M_v are equivalent to representations N_u and N_v on $L^2([u], \mu_{[u]})$, respectively, defined by

$$N_u(\phi)\xi(t) = \phi(t)\xi(t), \text{ and}$$

$$N_v(\phi)\eta(t) = \phi(t)\eta(t),$$

where $\phi \in C_0(\mathfrak{G}^0)$, $\xi \in L^2([u])$, and $\eta \in L^2([v])$. In particular, $N_u \cong N_v$. If $[u] \cap [v] = \emptyset$, then we can apply [10; Lemma 4.15] to conclude that $N_u \ncong N_v$. (Here the i and j of the lemma are the inclusion maps of the orbits into $\mathfrak{G}^0$). It follows that Ψ is an injection on $\mathfrak{G}^0/\mathfrak{G}$. Since $C^*(\mathfrak{G},\lambda)^\wedge$ is Hausdorff, it follows that $\mathfrak{G}^0/\mathfrak{G}$ is Hausdorff; and in particular, it follows that orbits are closed.

Now it is clear from the definition of L^u that, if $\phi \in C_0(\mathfrak{G}^0)$ and $\phi(v) = 0$ for all $v \in [u]$, then $M_u(\phi) = 0$. Thus, $F_u \subseteq \overline{[u]} = [u]$. Therefore, $F_u = [u]$.

On the other hand, if L is any irreducible representation of $C^*(\mathfrak{G},\lambda)$ and M is the associated representation of $C_0(\mathfrak{G}^0)$, then $\ker(M) = J_{[u]}$ for some $u \in \mathfrak{G}^0$. Thus, L factors through $C^*(\mathfrak{G}|_{[u]},\lambda)$. Now $\mathfrak{G}|_{[u]}$ is also a transitive groupoid; it follows from [6; Theorem 3.1] that L is equivalent to L^u. Therefore Ψ is surjective.

Finally, if $L^{u_n} \to L^u$ in $C^*(\mathfrak{G},\lambda)^\wedge$, then $M_{u_n} \to M_u$ as representations of $C_0(\mathfrak{G}^0)$. By [10; Lemma 2.4], we may, passing to a subsequence and relabeling if necessary, assume that there are $v_n \in \mathfrak{G}^0$ and $\gamma_n \in \mathfrak{G}$ such that $r(\gamma_n) = u_n$ and $s(\gamma_n) = v_n$ (i.e., $v_n \sim u_n$) and $v_n \to u$. Thus, $\dot{v}_n = \dot{u}_n$ converges to $\dot{u}$ in $\mathfrak{G}^0/\mathfrak{G}$. In sum, Ψ is open and defines a homeomorphism of $\mathfrak{G}^0/\mathfrak{G}$ onto $C^*(\mathfrak{G},\lambda)^\wedge$ as desired.

LEMMA 2.6. *Suppose that $\mathfrak{G}$ is a second countable principal groupoid which is not proper. Then there is a $z \in \mathfrak{G}^0$ and a sequence $\{\gamma_n\} \subseteq \mathfrak{G}$ such that*

(i) $r(\gamma_n) \to z$,

(ii) $s(\gamma_n) \to z$, *and*

(iii) *given $C \subseteq \mathfrak{G}$ compact, there is an integer N_C such that $n \geqq N_C$ implies that $\gamma_n \notin C$.*

PROOF. In view of Lemma 2.1, we may assume that there is a compact set $K \in \mathfrak{G}^0$ such that

$$\mathfrak{G}(K) = \{\gamma \in \mathfrak{G} : K \cdot \gamma \cap K \neq \emptyset\}$$

fails to have compact closure. Now let $\{C_n\}$ be a sequence of compact sets such that $\mathfrak{G} = \bigcup_{n=1}^{\infty} C_n$ and $C_n \subseteq \operatorname{Int}(C_{n+1})$. By assumption, we can choose $\gamma'_n \in \mathfrak{B} \setminus C_n$ such that both $r(\gamma'_n)$ and $s(\gamma'_n)$ are in K. We can then pass to a subsequence (which we still denote by $\{\gamma'_n\}$), and assume that $r(\gamma'_n) \to z$ and $s(\gamma'_n) \to y$. Since $\mathfrak{G}^0/\mathfrak{G}$ is Hausdorff, we have $[z] = [y]$, and hence, $y = z \cdot \gamma$ for some $\gamma \in \mathfrak{B}$. But s: $\mathfrak{G} \to \mathfrak{G}^0$ is an open map ([9; 1.2.4]), and passing to yet another sequence (and relabeling again), we can find $\beta_n \to \gamma$ such that $s(\beta_n) = s(\gamma_n)$. Put $\gamma_n = \gamma'_n \beta_n^{-1}$. Then (i) and (ii) are satisfied by construction. Suppose that $C \subseteq \mathfrak{G}$ is compact and that N is a compact set containing the β_n's. If $\gamma_n \in C$, then $\gamma'_n \in CN$. However, there is a N_C such that if $k \geqq N_C$, then $CN \subseteq C_k$. Hence, if $n \geqq N_C$, then $\gamma_n \notin C$.

For the remainder of this article, we assume that $\mathfrak{G}$ is a second countable principal groupoid which is not proper. We will show that $C^*(\mathfrak{G},\lambda)$ cannot have continuous trace; this will complete the proof of Theorem 2.3. The strategy is to assume that $C^*(\mathfrak{G},\lambda)$ has continuous trace, and derive a contradiction by producing an element $c \in C^*(\mathfrak{G},\lambda)$ in the Pedersen ideal ([7]) which is not in the ideal of continuous trace elements ([2]). Specifically, if $\{\gamma_n\} \subseteq \mathfrak{G}$ and $z \in \mathfrak{G}^0$ are as specified in Lemma 2.6, we will show that $u \mapsto \operatorname{Tr}(L^u(c))$ fails to be continuous at z.

To this end we fix once and for all a function $g \in C_c(\mathfrak{G}^0)$ which is identically one on a neighborhood U of z. Let $N = \text{supp}(g)$. Put

$$F_z = \mathfrak{G}_z \bigcap r^{-1}([z] \cap N), \text{ and}$$

$$F^z = \mathfrak{G}^z \bigcap s^{-1}([z] \cap N).$$

Since each orbit $[z]$ is closed, $\mathfrak{G}|_{[z]}$ is a locally compact transitive groupoid in the relative topology. Furthermore, topologically $\mathfrak{G}_z$ and $\mathfrak{G}^z$ may be viewed as subsets of $\mathfrak{G}|_{[z]}$. The point is that we may apply [6; Theorem 2.2A and Theorem 2.2B] to conclude that the natural maps of $\mathfrak{G}_z$ and $\mathfrak{G}^z$ onto $[z]$ are homeomorphisms, and that the map $(\alpha, \beta) \mapsto \alpha\beta$ is a homeomorphism of $\mathfrak{G}_z \times \mathfrak{G}^z$ onto $\mathfrak{G}|_{[z]}$. Thus, F_z and F^z are compact, and if $\gamma \in \mathfrak{G}|_{[z]}$ and either $g(r(\gamma)) \neq 0$ or $g(s(\gamma)) \neq 0$, then $\gamma \in F_z F^z$.

Recall that a neighborhood W *of* $\mathfrak{G}^0$ in $\mathfrak{G}$ is called *conditionally compact* if VW and WV are relatively compact in $\mathfrak{G}$ whenever V is relatively compact in $\mathfrak{G}$.

LEMMA 2.7. *If* $\mathfrak{G}$ *is a second countable groupoid then* $\mathfrak{G}^0$ *has a fundamental system of symmetric open conditionally compact neighborhoods. In fact, if* W_1 *is any neighborhood of* $\mathfrak{G}^0$, *then there is an open symmetric conditionally compact set* W_0 *such that*

$$\mathfrak{G}^0 \subseteq W_o \subseteq \overline{W}_0 \subseteq W_1.$$

PROOF. As in the proof of [9; 2.1.9], we see that $\mathfrak{G}^0$ has a fundamental system of open s-relatively compact neighborhoods. That is, a fundamental system of neighborhoods U such that $U \cap s^{-1}(K)$ is relatively compact set $K \subseteq \mathfrak{G}^0$. (Thus, UL is relatively compact if L is.) Of course, U^{-1} is r-relatively compact, and $W = U \cap U^{-1}$ is conditionally compact. The final assertion follows from the fact that $\mathfrak{G}$ is a normal topological space.

Using the above lemma, we can choose symmetric conditionally compact neighborhoods W_0 and W_1 such that $\overline{W_1}$ is conditionally compact and such that $W_0 \subseteq W_1$. Furthermore, since the union of a conditionally compact set with a relatively compact set is still conditionally compact, we may assume that $F_z F^z \subseteq W_0 z W_0$. By construction,

$$\overline{W}_1^7 z \setminus W_0 z \subseteq r^{-1}(\mathfrak{G}^0 \setminus N).$$

Using a straightforward compactness argument, we can find symmetric neighborhood V_0 and V_1 of z in $\mathfrak{G}$ such that $\overline{V}_0 \subseteq V_1$ and

$$\overline{W}_1^7 \overline{V}_1 \setminus W_0 V_0 \subseteq r^{-1}(\mathfrak{G}^0 \setminus N).$$

Of course, we may assume that $V_0 \subseteq W_0$. Now we have *a fortiori* that

$$\overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \setminus W_0 V_0 W_0 \subseteq r^{-1}(\mathfrak{G}^0 \setminus N).$$

We define

$$g^{(1)}(\gamma) = \begin{cases} g(r(\gamma)) & \text{if } \gamma \in \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7, \\ 0 & \text{if } \gamma \notin W_0 V_0 W_0. \end{cases}$$

The point of all this is that, because of our choices of W_1, V_0, etc., $g^{(1)}$ is continuous with compact support on $\mathfrak{G}$.

Furthermore, by construction,

$$W_0 V_0 W_0^2 V_0 W_0 \subseteq W_0^4 V_0 W_0^4 \subseteq \overline{W}_0^4 \overline{V}_0 \overline{W}_0^4 \subseteq W_1^4 V_1 W_1^4.$$

In particular, there is a function $b \in C_c^+(\mathfrak{G})$ such that $0 \leqq b \leqq 1$; b is identically one on $W_0 V_0 W_0^2 V_0 W_0$, and b vanishes off of $\overline{W}_1^4 \overline{V}_1 \overline{W}_1^4$. By replacing b with $(b + b^*)/2$ we can assume that $b = b^*$.

LEMMA 2.8. *With our choices above,*

$$g(r(\gamma))g(r(\alpha))b(\gamma\alpha^{-1})g^{(1)}(\alpha) = g^{(1)}(\gamma)g(r(\alpha))g^{(1)}(\alpha),$$

for all $\gamma, \alpha \in \mathfrak{G}$.

PROOF. If $\alpha \notin W_0 V_0 W_0$, then both sides are zero; so we assume throughout that $\alpha \in W_0 V_0 W_0$.

If $\gamma \in W_0 V_0 W_0$, then $g^{(1)}(\gamma) = g(r(\gamma))$. Furthermore, $\gamma\alpha^{-1} \in W_0 V_0 W_0^2 V_0 W_0$, so that $b(\gamma\alpha^{-1}) = 1$, and both sides agree.

If $\gamma \in \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7 \setminus W_0 V_0 W_0$, then both sides are zero again.

Finally, if $\gamma \notin \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7$, then the right hand side is zero. On the other hand, if $\gamma\alpha^{-1} \in \overline{W}_1^4 \overline{V}_1 \overline{W}_1^4$, then $\gamma \in \overline{W}_1^4 \overline{V}_1 \overline{W}_1^7$. Hence, $\gamma \notin \overline{W}_1^7 \overline{V}_1 \overline{W}_1^7$ implies that the left hand side equals zero as well.

Recall that g is identically one on the neighborhood U.

LEMMA 2.9. *With the choices made above, there is a neighborhood $V_2 \subseteq V_0$ and a conditionally compact neighborhood Y of $\mathfrak{G}^0$ such that if $v \in V_2$, then $r(Yv) \subseteq U$.*

PROOF. Let $\{V_n\}$ be a neighborhood basis of z in $\mathfrak{G}^0$ consisting of relatively compact sets. Also let $\{Y_n\}$ be a fundamental system of conditionally compact open neighborhoods of $\mathfrak{G}^0$, satisfying $Y_{n+1} \subseteq Y_n$ for all n. Suppose that for each n, there is a $z_n \in V_n$ and a $\gamma_n \in Y_n$ such that $r(\gamma_n z_n) \notin U$. Now $z_n \to z$ and the $z_n \gamma_n$ are contained in the relatively compact set $V_1 Y_1$. Thus, we may assume that $\gamma_n \to \gamma$. If $\gamma \notin \mathfrak{G}^0$, then eventually we have $\gamma \notin Y_N$ for some $N \in \mathbf{Z}$. Since $\mathfrak{G}$ is a normal topological space, there is a neighborhood Q of γ disjoint from Y_N. Hence the γ_n are eventually in Q and disjoint from Y_N. This is nonsense. But if $\gamma \in \mathfrak{G}^0$, then $\gamma_n z_n \to \gamma z = z \in U$; again, this is silly, and the result follows.

We begin our search for the appropriate element of the Pedersen ideal by introducing

$$f(\gamma) = g(r(\gamma))g(s(\gamma))b(\gamma).$$

Notice that f is self adjoint. Of course $f \in C_c(\mathfrak{G})$, and

$$(\dagger) \qquad L^u(f)\xi(\gamma) = \int_{\mathfrak{G}} f(\gamma\alpha)\xi(\alpha^{-1})\,d\lambda^u(\alpha)$$
$$= \int_{\mathfrak{G}} g(r(\gamma))g(s(\alpha))b(\gamma\alpha)\xi(\alpha^{-1})\,d\lambda^u(\alpha)$$
$$= g(r(\gamma)) \int_{\mathfrak{G}} g(r(\alpha))b(\gamma\alpha^{-1})\xi(\alpha)\,d\lambda_u(\alpha).$$

Now if $u = z$, then we may assume that $\gamma \in \mathfrak{G}_z$ and $\alpha \in \mathfrak{G}_z$. In particular, the choice of b implies $L^u(f)\xi(\gamma)$ is equal to

$$g(r(\gamma)) \int_{\mathfrak{G}} g(r(\alpha))\xi(\alpha)\,d\lambda_z(\alpha).$$

Thus, $L^z(f)$ is a positive rank one operator with eigenvalue

$$\mu_z^1 = \int_{\mathfrak{G}} g(r(\alpha))^2\,d\lambda_z(\alpha).$$

On the other hand,

$$L^u(f)g^{(1)}(\gamma) = g(r(\gamma)) \int_{\mathfrak{G}} g(r(\alpha))b(\gamma\alpha^{-1})g^{(1)}(\alpha)\,d\lambda_u(\alpha),$$

which by Lemma 2.8 is equal to

$$g^{(1)}(\gamma) \int_{\mathfrak{G}} g(r(\alpha))g^{(1)}(\alpha)\,d\lambda_u(\alpha).$$

It follows that $g^{(1)}$ is an eigenvector for $L^u(f)$ with eigenvalue

$$\mu_u^1 = \int_{\mathfrak{G}} g(r(\alpha))g^{(1)}(\alpha)\,d\lambda_u(\alpha).$$

Since $g^{(1)}$ is continuous with compact support, μ_u^1 converges to μ_z^1 as u converges to z.

Now we can choose Y and V_2 as in Lemma 2.9. We can also assume that $Y \subseteq W_0$, and that $V_2 \subseteq V_0$. Thus, if $\{\gamma_n\}$ is the sequence from Lemma 2.6 and if K is the compact set $\overline{W}_1^2\overline{V}_1\overline{W}_1^2$, then we can find an N such that $n \geqq N \geqq N_K$ guarantees that whenever γ is in the compact set $Y\gamma_n$ then, $\gamma \notin W_0V_0W_0$ and $r(\gamma), s(\gamma) \in U$. Note that the characteristic function $\xi_n = \chi_{Y\gamma_n}$ is in the L^2-orthogonal complement of $g^{(1)}$ viewed as an element of $L^2(\mathfrak{G}, \lambda_{s(\gamma_n)})$. Also, when $\alpha \in Y\gamma_n$, we have $\gamma\alpha^{-1} \in Y\gamma_n\gamma_n^{-1}Y \subseteq YV_0Y \subseteq W_0V_0W_0$. Therefore, $b(\gamma\alpha^{-1}) = 1$ in this case. Thus, when $\gamma \in Y\gamma_n$, we may use (†) to obtain the equation

$$L^{s(\gamma_n)}(f)(\xi_n)(\gamma) = \langle \xi_n, \xi_n \rangle_{L^2(\mathfrak{G}, \lambda_{s(\gamma_n)})}.$$

In particular,

$$\langle L^{s(\gamma_n)}(f)(\xi_n), \xi_n \rangle = \langle \xi_n, \xi_n \rangle \lambda_{s(\gamma_n)}(Y\gamma_n).$$

Since $L^{s(\gamma_n)}(f)$ is obviously a self-adjoint compact operator, it follows that as an operator on the orthogonal complement of $g^{(1)}$, the positive part of $L^{s(\gamma_n)}(f)$ has norm at least

$$\lambda_{s(\gamma_n)}(Y\gamma_n) = \lambda_{r(\gamma_n)}(Y).$$

Thus, $L^{s(\gamma_n)}(f)$ has a second eigenvalue $\mu^2_{s(\gamma_n)}$ such that

$$\mu^2_{s(\gamma_n)} \geqq \lambda_{r(\gamma_n)}(Y).$$

Now if $k \in C_c^+(\mathfrak{G})$ is such that $0 \leqq k \leqq 1$, $k(z) = 1$, and k vanishes off Y, then

$$\lambda_{r(\gamma_n)}(Y) \geqq \int_{\mathfrak{G}} k(\gamma)\, d(\gamma)\, d\lambda_{r(\gamma_n)}.$$

Since the right hand side converges to

$$\int_{\mathfrak{G}} k(\gamma)\, d\lambda_z(\gamma) > 0,$$

there is an $a > 0$ such that

(1) $\mu^1_z \geqq 3a$,

(2) $\mu^2_{s(\gamma_n)} \geqq 2a$.

(3) $\mu^1_{s(\gamma_n)} > \mu^1_z - a$.

Accordingly, we define $q\colon (-\infty, \infty) \to (0, \infty)$ by

$$q(t) = \begin{array}{ll} 0 & \text{if } t \leqq a, \\ 2(t-a) & \text{if } a \leqq t \leqq 2a, \\ t & \text{if } t \geqq 2a. \end{array}$$

By [7; page 134], $q(f)$ is positive and in the Pedersen ideal of $C^*(\mathfrak{G}, \lambda)$. Furthermore, $L^u(q(f)) = q(L^u(f))$ and $L^z(q(f)) = L^z(f)$. The last equality is a consequence of the fact that $L^z(f)$ is a rank one operator with eigenvalue $\mu^1_z \geqq 2a$. Finally, for sufficiently large n,

$$\operatorname{Tr}(q(L^{s(\gamma_n)}(f))) \geqq \mu^1_{s(\gamma_n)} + \mu^2_{s(\gamma_n)} \geqq \mu^1_z + a.$$

In summary, $\mathrm{Tr}\,(L^{s(\gamma_n)}(q(f)))$ doesn't converge to $\mathrm{Tr}\,(L^z(q(f)))$, and $C^*(\mathfrak{G}, \lambda)$ cannot have continuous trace. This completes the proof of Theorem 2.3.

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