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Topological Invariants Associated with the Spectrum of Crossed Product C^*-Algebras

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A separable C^*-dynamical system (A, G, α) in which A is a continuous-trace C^*-algebra and G is Abelian is called N-principal if N is a closed subgroup of G such that α restricted to N is locally unitary and the action of G on $\hat{A}$ defines a principal bundle $p(\alpha)\colon \hat{A} \to \hat{A}/G$. In this event, it is known that the spectrum of $A \rtimes_\alpha G$ is a principal $\hat{N}$-bundle $q(\alpha)$ over $\hat{A}/G$. In this article we show that a pair $([p], [q])$, where $p\colon X \to Z$ is a principal G/N-bundle and $q\colon Y \to Z$ is principal $\hat{N}$-bundle, determines a class in $H^4(Z)$ which vanishes if and only if there is a continuous-trace C^*-algebra A with spectrum X and a N-principal system (A, G, α) with $[p(\alpha)] = [p]$. More generally, given A, G, and $[p]$ as above, we consider the question of when systems (A, G, α) with $[p(\alpha)] = [p]$ exist. © 1993 Academic Press, Inc.

It has become clear over the past few years that crossed product C^*-algebras $A \rtimes_\alpha G$ in which G is Abelian and A is non-commutative can be much more complicated than they are when A is commutative. Thus, for example, the criteria for simplicity of $C_0(X) \rtimes G$ do not have good analogues (e.g. [7–9, 11]), and the description of Prim $C_0(X) \rtimes G$ given in [24] does not extend even to crossed products by actions of $\mathbb{R}$ on non-

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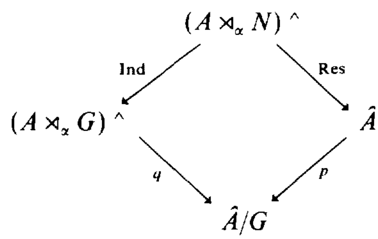
commutative A [17]. The extra complications occur even when A is a continuous-trace algebra, which is the very nicest kind of non-commutative C^*-algebra, and then a detailed analysis of $A \rtimes_\alpha G$ has been possible only if the orbit space for the action of G on the spectrum $\hat{A}$ is Hausdorff.

The two limiting situations, in which G acts trivially on $\hat{A}$, or $\hat{A} \to \hat{A}/G$ is a principal G-bundle, are now well understood: in both cases we can describe the spectrum and identify $A \rtimes_\alpha G$ up to stable isomorphism by specifing its Dixmier–Douady class (see, respectively, [12, 15] and [17, 19]). The most interesting results obtained so far, however, concern systems (A, G, α) for which the action of G on $\hat{A}$ has constant isotropy subgroup N and $\hat{A} \to \hat{A}/G$ is a principal G/N-bundle. If in addition $\alpha|_N$ is locally unitary in the sense of [15] (which is automatic if $N = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{T}^k$), then the spectrum of $A \rtimes_\alpha G$ is a principal $\hat{N}$-bundle over $\hat{A}/G$, which fits into a commutative diamond of principal bundles [17, Theorem 2.2].



The bundle maps in this diamond are all familiar from representation theory—Ind is induction of representations, Res is restriction, and q sends an irreducible representation of $A \rtimes_\alpha G$ to the (quasi-) orbit on which it lives—and it is the possibility that q could be a non-trivial bundle which is a striking departure from the case of commutative A. It was further proved in [17, Sect. 3(a)] that any principal $\hat{N}$-bundle could be realized as q, by taking A to be a suitable induced C^*-algebra, and this settled several open problems (cf., [17, Sect. 4; 12, Sect. 3(b)]). In the key examples, the nontriviality of q arose when A was a continuous trace algebra with non-zero Dixmier–Douady class, confirming that actions on these algebras are likely to be particulary interesting.

Here we continue analysis of these crossed products of continuous-trace algebras by actions with constant isotropy, and we look closely at the relationship between the different topological invariants associated with the diamond. The main problem we discuss is that of identifying those diamonds which can arise for particular choices of G, N, and A. More precisely, suppose A is a continuous-trace C^*-algebra, whose spectrum is a principal G/N-bundle, and $q\colon Y \to \hat{A}/G$ is a principal $\hat{N}$-bundle. When is there an action α of G on A, including the given action on $\hat{A}$, such that $(A \rtimes_\alpha G)^\wedge$ is isomorphic to Y as an $\hat{N}$-bundle over $\hat{A}/G$?

We answer this question by proving the following theorem. In it, $H^p(Z, \mathscr{G})$ denotes the sheaf cohomology of Z with coefficients in the sheaf $\mathscr{G}$ of germs of continuous G-valued functions on Z, and we identify $H^1(Z, \mathscr{G})$ with (isomorphism classes of) locally trivial principal G-bundles over Z.

THEOREM. *Suppose that G is a locally compact Abelian group of the form $\mathbb{R}^m \times \mathbb{T}^n \times \mathbb{Z}^p \times F$ with F finite, N is a closed subgroup of G, and $p: X \to Z$ is a principal G/N-bundle.*

(1) *There is a natural pairing*

$$\langle \cdot, \cdot \rangle_G : H^1(Z, \mathscr{G}/\mathscr{N}) \times H^1(Z, \hat{\mathscr{N}}) \to \check{H}^4(Z; \mathbb{Z})$$

such that $\langle [p], [q] \rangle_G = 0$ if and only if there is a continuous-trace algebra A with spectrum X and an action $\alpha: G \to \operatorname{Aut}(A)$ which is locally unitary on N, induces the given action of G/N on $X = \hat{A}$, and has $(A \rtimes_\alpha G)^\wedge$ isomorphic to q as an $\hat{N}$-bundle over $Z = \hat{A}/G$.

(2) *There is a homomorphism*

$$d_p: \{ [q] \in H^1(Z, \hat{\mathscr{N}}) : \langle [p], [q] \rangle_G = 0 \} \to \check{H}^3(X; \mathbb{Z})/p^*(\check{H}^3(Z; \mathbb{Z}))$$

with the following property: if A is a stable continuous-trace C^*-algebra with spectrum X, and q satisfies $\langle [p], [q] \rangle_G = 0$, then there is an action α of G on A as in (1) if and only if $\delta(A) + p^*(\check{H}^3(Z; \mathbb{Z})) = d_p([q])$.

The appearance of an obstruction in $\check{H}^4(Z; \mathbb{Z})$ is at first sight surprising, but seems more reasonable when we recall the analysis of the case $G = \mathbb{R}$ and $N = \mathbb{Z}$ in [17, Sect. 4]. Here $\hat{N} \cong \mathbb{T} \cong G/N$, so that both $[p]$ and $[q]$ lie in $H^1(Z, \mathscr{S}) \cong \check{H}^2(Z; \mathbb{Z})$, and the pairing $\langle [p], [q] \rangle_{\mathbb{R}}$ turns out to be the usual cup product. From the Gysin exact sequence

$$\cdots \longrightarrow \check{H}^3(Z; \mathbb{Z}) \xrightarrow{p^*} \check{H}^3(X; \mathbb{Z}) \xrightarrow{p_!} \check{H}^2(Z; \mathbb{Z}) \xrightarrow{\cup [p]} \check{H}^4(Z; \mathbb{Z}) \longrightarrow \cdots$$

we deduce that $[q]$ is realizable as in (1) if and only if $[q]$ is the image of a class in $\check{H}^3(X; \mathbb{Z})$. The exactness at $\check{H}^3(X)$ says this class is determined modulo the image of p^*, just as in part (2) of the theorem.

There are two main ingredients in the proof of the theorem. First, we have to analyze the local structure of one of these *N-principal systems* (A, G, α), and for this we use the characterization of N-principal systems with $p: X \to Z$ trivial as induced systems; this was first established in [20], but we can now give a more direct and elementary proof using recent work of Echterhoff [4, 5]. This analysis leads to a pair (ν, λ) of cocycles, in which $\nu \in Z^2(X, \mathscr{S})$ represents $\delta(A)$, and λ comes from the action of G. The second main ingredient is the direct construction from [18] of a continuous-trace algebra $A(\nu)$ from a cocycle $\nu \in Z^2(X, \mathscr{S})$; the auxiliary data λ is preseicely what we need to construct an N-principal action of G on $A(\nu)$. Our problems thus reduce to ones involving cocycles, which we solve in Proposition 3.3 and Theorem 4.1. This construction using [18] is quite different in nature from previous ones based on crossed products [15, Sect. 3; 17, Sect. 3; 16, Proposition 3.5], but so far this is the only way we have been able to produce all the required examples. It could be very interesting to find a more crossed-product oriented collection of examples.

We begin with a short section on preliminaries, in which we discuss the pairing $\langle \cdot, \cdot \rangle_G$ used in our theorem, and recal some backgroup material on continuous-trace and induced C^*-algebras. In Section 2, we prove our local structure theorem for an N-principal system (A, G, α) (Theorem 2.1), and show how to compute the class of bundle $q: (A \rtimes_\alpha G)^\wedge \to \hat{A}/G$ in terms of local Green twisting maps for α on N (Proposition 2.6). Although the results of this section were all motivated by an earlier work [20], we have tried to give direct arguments here. Sections 3 and 4 contain, respectively, the proofs of parts (1) and (2) of our theorem.

Our final section contains some examples and applications. We first discuss the cases $G = \mathbb{R}$, $N = \mathbb{Z}$ and $G = \mathbb{T}$, $N = \mathbb{Z}_n$, where the pairing, and hence our results, can be expressed in terms of the usual cup product in Čech cohomology. For $G = \mathbb{R}$, we already have the information coming from the Gysin sequence, and comparing this with our present results suggested to us that our arguments should constitute part of a proof of exactness of a generalized Gysin sequence. We plan to discuss this elsewhere, but in Section 5(a) we do outline how the results of [17, Sect. 4] in the case $G = \mathbb{R}$ can be recovered from our theorem. We have already stressed that our constructions are quite different from the crossed-product methods previously used, and in Section 5(c) we show how the most general of these, namely that of [16, Proposition 3.5], fits in with our present results. We close by considering briefly another promising construction, based on the pull-backs of [19], which turns out to be the Takai dual of that considered in Section 5(c).

1. Preliminaries

Let G be a locally compact group and N a closed normal subgroup. Since we shall be concerned throughout with locally trivial bundles, we shall frequently want to assume that G itself is a (locally trivial) principal N-bundle over G/N. As the action of N on G by left multiplication is always free and proper, this happens if and only if the quotient map from G to G/N has local sections: if $c: G/N \to G$ is a section, then the map

$s \mapsto (sN, sc(sN)^{-1})$ is a bundle isomorphism of G onto $G/N \times N$. Thus we shall often say simply that "$G \to G/N$ has local sections." Most of the time G will be Abelian, in which case we can identify the dual $\hat{N}$ of N with $\hat{G}/N^{\perp}$, and we also insist that $\hat{G} \to \hat{N}$ has local sections. These assumptions are automatic for reasonable groups, and in particular if G is an elementary Abelian group in the sense of Weil [23]: i.e., if $G = \mathbb{R}^p \times \mathbb{T}^q \times \mathbb{Z}^r \times F$ for some finite Abelian group F (cf., e.g., [10, 13]).

When G is Abelian, the (isomorphism classes of) principal G-bundles over a fixed paracompact base Z are parametrized by the first sheaf cohomology group $H^1(Z, \mathscr{G})$ of Z with coefficients in the sheaf $\mathscr{G}$ of germs of continuous G-valued functions on Z: if $p: X \to Z$ is a principal G-bundle, the transition functions of p form a cocycle representing $[p] \in H^1(Z, \mathscr{G})$. Our main results concern a natural pairing of $\hat{N}$-bundles with G/N-bundles, which we view as a bilinear map

$$\langle \cdot, \cdot \rangle_G : H^1(Z, \mathscr{G}/\mathscr{N}) \times H^1(Z, \hat{\mathscr{N}}) \to \check{H}^4(Z; \mathbb{Z})$$

with values in the Čech cohomology group $\check{H}^4$; although we define this pairing using the cup product in sheaf cohomology, we immediately describe it using cocycles and use this description throughout.

If $\mathscr{U}$ and $\mathscr{V}$ are sheaves over Z, then the cup product

$$H^p(Z, \mathscr{U}) \times H^q(Z, \mathscr{V}) \to H^{p+q}(Z, \mathscr{U} \otimes \mathscr{V})$$

is defined on cocylces by

$$(\mu \cup v)_{i_0 \cdots i_{p+q}} = \mu_{i_0 \cdots i_p} \otimes v_{i_p i_{p+1} \cdots i_{p+q}}$$

(cf. [6, p. 245]); it is straightforward to check that the class of $\mu \cup v$ depends only on the classes $[\mu]$ and $[v]$, and the resulting pairing has the standard functorial properties. If $\mathscr{U} = \mathscr{G}$ and $\mathscr{V} = \hat{\mathscr{G}}$, then evaluation gives a natural homomorphism of $\mathscr{G} \otimes \hat{\mathscr{G}}$ onto the sheaf $\mathscr{S}$ of $\mathbb{T}$-valued functions, and we denote the image of $\mu \cup v$ by $[\mu] \cup [v]$. If $G \to G/N$ has local sections, then

$$1 \to \mathscr{N} \to \mathscr{G} \to \mathscr{G}/\mathscr{N} \to 1$$

is a short exact sequence of sheaves, and there is a corresponding long exact sequence in sheaf cohomology:

$$\cdots \longrightarrow H^p(Z, \mathscr{G}) \longrightarrow H^p(Z, \mathscr{G}/\mathscr{N}) \xrightarrow{\partial_G} H^{p+1}(Z, \mathscr{N}) \longrightarrow H^{p+1}(Z, \mathscr{G}) \longrightarrow \cdots.$$

In particular, we denote by ∂ the connecting homomorphism $\partial_{\mathbb{R}}: H^p(Z, \mathscr{S}) \to \check{H}^{p+1}(Z; \mathbb{Z})$, which is an isomorphism because the sheaf $\mathscr{R}$ is fine and hence $H^p(Z, \mathscr{R}) = 0$ for $p \geqslant 1$.

DEFINITION 1.1. Let G be a locally compact Abelian group, and N a closed subgroup of G. Then for $[p] \in H^1(Z, \mathcal{G}/\mathcal{N})$ and $[q] \in H^1(Z, \hat{\mathcal{N}})$ we define

$$\langle [p], [q] \rangle_G = \partial(\partial_G([p]) \cup [q]) \qquad \text{in} \quad \check{H}^4(Z, \mathbb{Z}).$$

We have included the subscript G to stress that this pairing depends crucially on the extension G of N by G/N: if the extension splits, for example, so that $G = N \times G/N$, then $\partial_G = 0$ and the pairing is trivial. More generally, if there is a bundle representing $[p]$ which is a quotient E/N of a principal G-bundle E, then $\partial_G([p]) = 0$ and $\langle [p], [q] \rangle_G = 0$ for any $[q]$. However, as we see in Lemma 1.3, the pairing is not always trivial.

We now give the promised explicit construction of a cocycle representing $\langle [p], [q] \rangle_G$. Suppose that $\gamma_{ij}: N_{ij} \to \hat{N}$ and $t_{ij}: N_{ij} \to G/N$ are cocycles representing $[q]$ and $[p]$, defined on the same cover $\{N_i\}$. Since $\pi: G \to G/N$ has local sections, we can locally lift any function $t: Z \to G/N$ to a function $s: Z \to G$, and by the argument of [3, Lemma 10.7.11] we can refine the cover to ensure that there are continuous functions $s_{ij}: N_{ij} \to G$ such that $t_{ij} = \pi \circ s_{ij}$. Because $\{t_{ij}\}$ is a cocycle, there are continuous functions $n_{ijk}: N_{ijk} \to N$ such that

$$s_{ij}(z)\, s_{jk}(z) = s_{ik}(z)\, n_{ijk}(z) \qquad \text{for} \quad z \in N_{ijk},$$

and $\{n_{ijk}\}$ is a cocycle representing $\partial_G([p])$. Then a cocycle $\{\nu_{ijkl}\}$ representing the class $\langle [p], [q] \rangle_G$ in $H^3(Z, \mathcal{S}) \cong \check{H}^4(Z; \mathbb{Z})$ is given by

$$\nu_{ijkl}(z) = \gamma_{ij}(z)(n_{jkl}(z)). \tag{1.1}$$

(Strictly speaking, $\{\nu_{ijkl}\}$ represents $\partial^{-1}(\langle [p], [q] \rangle_G)$, but we usually suppress the isomorphism ∂.)

If the map $\hat{G} \to \hat{N}$ also has local sections, we can also form the cup product $[p] \cup \partial_{\hat{G}}([q])$ as follows. We refine $\{N_i\}$ to ensure that there are $\tilde{\gamma}_{ij}: N_{ij} \to \hat{G}$ with $\gamma_{ij}(z) = \tilde{\gamma}_{ij}(z)|_N$, define $\chi_{ijk}: N_{ijk} \to N^{\perp} = (G/N)^{\wedge}$ by $\tilde{\gamma}_{ij}\tilde{\gamma}_{jk} = \tilde{\gamma}_{jk}\chi_{ijk}$, and let

$$\mu_{ijkl}(z) = \chi_{ijk}(z)(t_{kl}(z)). \tag{1.2}$$

LEMMA 1.2. *The cocycles μ and ν defined in Eqs.* (1.1) *and* (1.2) *define equivalent classes in $\check{H}^4(Z; \mathbb{Z})$.*

Proof. Retaining the notation of the previous two paragraphs, we define $\lambda_{ijk}: N_{ijk} \to \mathbb{T}$ by $\lambda_{ijk}(z) = \tilde{\gamma}_{ij}(z)(s_{jk}(z))$, and then verify that $(\partial\lambda)\,\nu = \mu$. ∎

LEMMA 1.3. *Let* $G = \mathbb{R}$ *and* $N = \mathbb{Z}$. *Then* $G/N \cong \mathbb{T} \cong \hat{N}$, *and if* $[p]$, $[q] \in H^1(Z, \mathcal{S})$, *then* $\langle [p], [q] \rangle_G$ *is the usual cup product of the classes* $\partial_{\mathbb{R}}([p])$ *and* $\partial_{\mathbb{R}}([q])$ *in the cohomology ring* $H^*(Z, \mathcal{S})$.

Proof. Suppose q and p have transition functions $\gamma_{ij} = \exp(2\pi i \rho_{ij})$ and $t_{ij} = \exp(2\pi i r_{ij})$, where ρ_{ij} and r_{ij} functions from N_{ij} to $\mathbb{R}$. On the one hand, $\partial([q])$ and $\partial([p])$ are the classes of $\{m_{ijk}\}$ and $\{n_{ijk}\}$, where

$$\rho_{ij} + \rho_{jk} = m_{ijk} + \rho_{ik} \qquad \text{and} \qquad r_{ij} + r_{jk} = n_{ijk} + r_{ik}.$$

Thus, $\partial([q]) \cup \partial([p])$ is represented by the cocycle $v_{ijklm} = m_{ijk} n_{klm}$. On the other hand, for $w \in \mathbb{T} = G/N$ we have $\chi_{ijk}(z)(w) = w^{m_{ijk}(z)}$, and hence Eq. (1.2) becomes

$$\mu_{ijkl}(z) = (\exp(2\pi i r_{kl}(z)))^{m_{ijk}(z)} = \exp(2\pi i s_{ijkl}(z)),$$

where $s_{ijkl}(z) = m_{ijk}(z)\, r_{kl}(z)$. Now a calculation shows that

$$\partial(\mu)_{ijklm} = s_{jklm} - s_{iklm} + s_{ijlm} - s_{ijkm} + s_{ijkl} = m_{ijk} n_{klm},$$

and the lemma follows. ∎

Since we are interested in topological invariants, it is natural for us to work with seperable continuous-trace C^*-algebras, which are classified up to stable isomorphism by their Dixmier–Douady invariants. (Unfortunately, there is not one good reference for this; it can be deduced from, for example, [2, Théorème 2; 14, Lemma 1.11; 3, Theorem 10.8.4].) For such algebras, it is covenient to use bundle notation, even though we do not make any serious use of C^*-bundle theory. Thus, if A is a C^*-algebra with Hausdorff spectrum X, we write $A|_K$ for the quotient of A whose spectrum is the closed subset K of X, and we write $a \mapsto a(x)$ or ε_x for an irreducible representation corresponding to $x \in X$. This last practice is fine so long as we remember that this representation is only specified up to unitary equivalence, and thus, for example, if $\alpha: G \to \mathrm{Aut}(A)$ induces the action $(s, x) \mapsto s \cdot x$ of G on $X = \hat{A}$, the representation $\varepsilon_{s \cdot x}$ will only be equivalent to $s \cdot \varepsilon_x = \varepsilon_x \circ \alpha_s^{-1}$. (This problem arises because X is really a quotient of the space of irreducible representations, and it is not always possible to choose a continuous section for the quotient map; strictly speaking, $a(x)$ is canonically defined as the image of a in the quotient $A/\ker x \cong \mathcal{K}(\mathcal{H}_x)$ of A by the common kernel of the representations in x).

As we saw in [17, Sect. 3a; 20], important examples of the systems we study involve induced C^*-algebras. If N is closed subgroup of G and β is an action of N on a C^*-algebra B, the *induced C^*-algebra* is the subalgebra $\mathrm{Ind}_N^G(B, \beta)$ of $C_b(G, B)$ consisting of functions f such that $f(sn) = \beta_n^{-1}(f(s))$ for $s \in G$ and $n \in N$, and such that $sN \mapsto \|f(s)\|$ vanishes at infinity on G/N. The *induced action* Ind β of G on $\mathrm{Ind}_N^G(B, \beta)$ is defined by $(\mathrm{Ind}\,\beta)_t(f)(s) =$

$f(t^{-1}s)$ for $s, t \in G$. Every irreducible representation of $\mathrm{Ind}(B, \beta)$ is equivalent to one of the form $M(\rho, s): f \mapsto \rho(f(s))$ for some $s \in G$, $\rho \in \hat{B}$, and if N acts on $\hat{B} \times G$ via

$$n \cdot (\rho, s) = (n \cdot \rho, sn^{-1}) = (\rho \circ \beta_n^{-1}, sn^{-1}),$$

then M induces a homeomorphism of $(\hat{B} \times G)/N$ onto $(\mathrm{Ind}(B, \beta))^\wedge$ [19, Proposition 3.1]. For us β is normally locally unitary [15] and therefore acts trivially on $\hat{B}$, giving $(\mathrm{Ind}(B, \beta))^\wedge \cong \hat{B} \times G/N$.

2. A Structure Theorem for N-Principal Systems

Let A be a separable continuous-trace C^*-algebra with spectrum X, N a closed subgroup of a second countable locally compact Abelian group G, and $p: X \to Z$ a locally trivial principal G/N-bundle. We shall say that a dynamical system (A, G, α) is *N-principal with spectrum $p: X \to Z$* if $\alpha|_N$ is locally unitary and $\hat{A} \to \hat{A}/G$ is a principal G/N-bundle isomorphic to $p: X \to Z$. (Note that these assumptions imply that the stabilizer of each point $x \in X$ is precisely N.) Such systems exist in abundance: we can take $A = C_0(X, \mathscr{K})$, and let α be the action τ of G by translation on X, so that $\alpha|_N$ is trivial. Our structure theorem says that every stable N-principal system is locally isomorphic to this system, and describes the data required to reconstruct the original system from this local information.

THEOREM 2.1. *Suppose that A is a stable separable continuous-trace C^*-algebra with spectrum X, that G is a second countable locally compact group, and that N is a closed subgroup of G such that $G \to G/N$ and $\hat{G} \to \hat{N}$ have local sections. If (A, G, α) is an N-principal system with spectrum $p: X \to Z$, then there is a locally finite cover $\{N_i\}_{i \in I}$ of Z by relatively compact open sets such that*

(1) *for each $i \in I$, there is a $C_0(p^{-1}(\bar{N}_i))$-isomorphism Φ_i of $A|_{p^{-1}(\bar{N}_i)}$ onto $C_0(p^{-1}(\bar{N}_i), \mathscr{K})$ which carries α to an action exterior equivalent to τ, and*

(2) *for each pair $i, j \in I$, there is a unitary $v_{ij} \in \mathscr{M}(C_0(p^{-1}(\bar{N}_{ij}), \mathscr{K}))$ such that $\Phi_i \circ \Phi_j^{-1} = \mathrm{Ad}\, v_{ij}$.*

The first part follows from the local triviality of the bundle $p: X \to Z$ and our knowledge of systems with p trivial: up to isomorphism, they are all induced from systems (D, N, β), where $\hat{D} = Z$ and β is locally unitary (Proposition 2.2). A system induced from a unitary action $\beta = \mathrm{Ad}\, u$ is Morita equivalent to translation on $C_0(G/N, D)$, and, since D is locally stably isomorphic to $C_0(Z, \mathscr{K})$, part 1 follows from the equivariant stabilization theorem of Combes [1] and a little bookkeeping (Proposition 2.4). To get part 2, we have to show that we can refine the cover to ensure that $\Phi_i \circ \Phi_j^{-1} \in \mathrm{Aut}_{C(p^{-1}(\bar{N}_{ij}))} C_0(p^{-1}(\bar{N}_{ij}), \mathscr{K})$ are inner. For this, we have to exploit our special circumstances, because the fibre G/N of p could easily have non-vanishing second Čech cohomology, in which case $C_0(p^{-1}(\bar{N}_{ij}), \mathscr{K})$ has outer center-fixing automorphisms [14].

PROPOSITION 2.2. *Suppose that (A, G, α) is an N-principal system with spectrum $p: X \to Z$, and suppose that p is trivial as a G/N-bundle. Let $c: Z \to X$ be a continuous section for p, and let $D = A/I$ be the quotient of A with spectrum $c(Z)$. Then $\hat{D}$ is homeomorphic to Z, the action $\beta: N \to \mathrm{Aut}(D)$ defined by $\beta_n(a + I) = \alpha_n(a) + I$ is locally unitary, and the homomorphism $\Phi: A \to \mathrm{Ind}(D)$ defined by $\Phi(a)(s) = \alpha_s(a) + I$ is an equivariant isomorphism of (A, G, α) onto $(\mathrm{Ind}_N^G(D, \beta), G, \mathrm{Ind}\,\beta)$.*

Proof. The triviality of p is equivalent to the existence of a continuous G-equivariant map $w: X \to G/N$—given a global section c, define w by $w(x) = sN$ where $s \cdot c(x) = x$. Then $c(Z) = w^{-1}(N)$ is closed and, identifying $X = \hat{A}$ with $\mathrm{Prim}(A)$, we have $I = \bigcap \{J: w(J) = N\}$. Therefore it follows from [5] that the map Φ is an equivariant isomorphism, and since c is a homeomorphism of Z onto $c(Z)$ with inverse $p|_{c(Z)}$, we only have to check that β is locally unitary. However, if $q: A \to D$ is the quotient map, $\hat{q}: \pi \to \pi \circ q$ is the continuous embedding of $\hat{D}$ as a subset of $\hat{A}$, and if $u: N \to \mathscr{U}\mathscr{M}(A)$ implements $\alpha|_N$ over the open set W, then $q \circ u$ implements $\beta|_N$ over $\hat{q}^{-1}(W)$. ∎

LEMMA 2.3. *Let G be a locally compact group and N a closed normal subgroup such that $G \to G/N$ has local sections. Suppose that D is a stable separable C^*-algebra with Hausdorff spectrum Z, and $u: N \to \mathscr{U}\mathscr{M}(D)$ is a strictly continuous homomorphism. Then there is a $C_0(Z \times G/N)$-linear isomorphism of $\mathrm{Ind}_N^G(D, \mathrm{Ad}\,u)$ onto $D \otimes C_0(G/N)$ which carries the action $\mathrm{Ind}(\mathrm{Ad}\,u)$ of G onto one exterior equivalent to $\iota \otimes \tau$.*

Proof. Let $B = \mathrm{Ind}_N^G(D, \mathrm{Ad}\,u)$, $C = C_0(G/N, D)$, and

$$Y = \left\{ f \in C_b(G, D) \;\middle|\; \begin{array}{l} f(sn) = u_n^*(f(s)) \text{ for } s \in G,\, n \in N, \text{ and} \\ sN \mapsto \|f(s)\| \text{ vanishes at infinity on } G/N \end{array} \right\}$$

We define actions of B and C on Y by

$$(b \cdot f)(s) = b(s) f(s), \qquad f \cdot c(s) = f(s)\, c(sN),$$

and B- and C-valued inner products on Y by

$$\langle f, g \rangle_B (s) = f(s)\, g(s)^*, \qquad \langle f, g \rangle_C (sN) = f(s)^*\, g(s).$$

As in the proof of [19, Theorem 3.2(2)] (where $\mathrm{Ind}_N^G(A)$ appears as $GC(X, A)^\alpha$), Y is a B–C-imprimitivity bimodule.

Now define an action v of G on Y by the formula $v_t(f)(s) = f(t^{-1}s)$. It is apparent that v defines a strongly continuous action on the imprimitivity bimodule Y, which is complete in the imprimitivity norm (the sup-norm). Furthermore,

$$\langle v_t(f), v_t(g)\rangle_B = \sigma_t(\langle f, g\rangle_B) \qquad \text{and} \qquad \langle v_t(f), v_t(g)\rangle_C = \tau_t(\langle f, g\rangle_C),$$

so (B, σ) and (C, τ) are Morita equivalent in the sense of Definition 1 in Section 3 of [1]. Also note that, since we are assuming that $G \to G/N$ has local sections, B is locally isomorphic to $C_0(Z \times G/N, \mathscr{K})$. It follows from [14, Proposition 1.12] that B is stable. Since B and C are separable, the Proposition in Section 9 of [1] implies that σ and τ are outer conjugate; that is, there is a $*$-isomorphism Φ if B onto C such that $\Phi \circ \sigma \circ \Phi^{-1}$ is exterior equivalent to τ.

To see that Φ is $C_0(Z \times G/N)$-linear, we first observe that the action of $C_0(Z \times G/N) = C_0(G/N, C_0(Z))$ by pointwise multiplication on Y is compatible with the natural actions on B and C, in the sense that

$$(\phi \cdot b) \cdot f \cdot c = b \cdot (\phi \cdot f) \cdot c = b \cdot f \cdot (\phi \cdot c) \qquad \text{for} \quad \phi \in C_0(Z \times G/N).$$

Thus we can define an action of $C_0(Z \times G/N)$ on the linking algebra $L = \left(\begin{smallmatrix} B & Y \\ \tilde{Y} & C \end{smallmatrix}\right)$ by

$$\phi \cdot \begin{pmatrix} b & f \\ \tilde{g} & c \end{pmatrix} = \begin{pmatrix} \phi \cdot b & \phi \cdot f \\ (\bar{\phi} \cdot g)^\sim & \phi \cdot c \end{pmatrix}.$$

Since the isomorphism Φ is given by

$$b \mapsto \left[\text{bottom right-hand corner of } w^* \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} w\right]$$

for a suitable partial isometry w in $\mathcal{M}(L)$, it follows that Φ is $C_0(Z \times G/N)$-linear. ∎

PROPOSITION 2.4. *Suppose that (A, G, α) is a stable N-principal system with spectrum $p: X \to Z$. Then for each point $z \in Z$ there is a compact neighborhood W of z and a $C_0(p^{-1}(W))$-linear isomorphism Φ of $A|_{p^{-1}(W)}$ onto $C_0(p^{-1}(W), \mathscr{K})$ which carries (the action on the quotient induced by) α onto an action exterior equivalent to τ.*

Proof. Since the problem is local in Z, we may suppose that $X = Z \times G/N$. Then by Proposition 2.2, there is an isomorphism Φ of (A, G, α) onto $(\mathrm{Ind}(D, \beta), G, \mathrm{Ind}(\beta))$, where D is a quotient of A and β is locally unitary. For any compact subset W of Z, $D|_W$ is the quotient of $A|_{p^{-1}(W)}$ with spectrum $\{(z, N): z \in W\}$, and the isomorphism Φ induces an equivariant isomorphism Φ_W of $A|_{p^{-1}(W)}$ onto $\mathrm{Ind}(D|_W, \beta)$. Since A is stable, so is its quotient D, and hence D is locally $C(Z)$-isomorphic to $C_0(Z, \mathcal{K})$. Thus we can take W to be a neighborhood of z such that $D|_W$ is $C(W)$-isomorphic to $C(W, \mathcal{K})$, and such that there is a strictly continuous homomorphism $u: N \to \mathcal{U}\mathcal{M}(D|_W)$ implementing $\beta|_W$. Now Proposition 2.4 gives us a $C_0(W \times G/N)$-isomorphism Ψ of $\mathrm{Ind}(D|_W)$ onto $C_0(W \times G/N, \mathcal{K})$ carrying α into an action exterior equivalent to τ, and we just have to check that the composition $\Psi \circ \Phi_W$ is $C_0(W \times G/N)$-linear.

Since $\hat{D}$ is the closed subset $Z \times \{N\}$ of $Z \times G/N$, which we identify with Z, for any $x = (z, sN)$ in $Z \times G/N = X$ the representation ε_x is equivalent to $M(\varepsilon_z, s) \circ \Phi = \varepsilon_{z, N} \circ \alpha_s^{-1}$. Therefore, for any ϕ in $C_0(W \times G/N)$, we have

$$M(\varepsilon_z, s) \circ \Phi(\phi \cdot a) = \phi(z, sN)\, M(\varepsilon_z, s) \circ \Phi(a). \tag{2.1}$$

But the formula $M(\varepsilon_z, s)(\phi \cdot b) = \phi(z, sN)\, M(\varepsilon_z, s)(b)$ characterizes the canonical action of $C_0(W \times G/N)$ in $\mathrm{Ind}(D)$, and hence Eq. (2.1) says that Φ_W is $C_0(W \times G/N)$-linear. ∎

PROPOSITION 2.5. *Let G be a locally Abelian group and let N be a closed subgroup such that $\hat{G} \to \hat{N}$ has local sections. Suppose that $p: X \to Z$ is a principal G/N-bundle, that $\alpha, \beta: G \to \mathrm{Aut}\, C_0(X, \mathcal{K})$ are both exterior equivalent to the action τ of G by translation, and that $\Psi \in \mathrm{Aut}_{C(X)}\, C_0(X, \mathcal{K})$ satisfies $\Psi \circ \alpha_t \circ \Psi^{-1} = \beta_t$. Then each point $z_0 \in Z$ has a compact neighborhood W for which there is a unitary $u \in \mathcal{M}(C_0(p^{-1}(W), \mathcal{K}))$ satisfying*

$$\Psi(a)|_{p^{-1}(W)} = u(a|_{p^{-1}(W)})\, u^* \qquad \textit{for} \quad a \in C_0(X, \mathcal{K}).$$

Proof. Again we may suppose that $X = Z \times G/N$. We begin by producing a neighborhood V and a function $h: V \times G \to \mathcal{U}(\mathcal{H})$ such that

$$\Psi(a)(z, sN) = h(z, s)\, a(z, sN)\, h(z, s)^* \tag{2.2}$$

for $(z, s) \in V \times G$ and $a \in C_0(X, \mathcal{K})$. By assumption, there are τ 1-cocycles w, v such that

$$\alpha_s(a) = w_s \tau_s(a)\, w_s^* \qquad \text{and} \qquad \beta_s(a) = v_s \tau_s(a)\, v_s^*$$

for $a \in C_0(X, \mathcal{K})$. Further, since Ψ must be locally inner [14, Proposition 2.9], we can choose a strictly continuous function y from a neighborhood V of z_0 to $\mathcal{U}(\mathcal{H})$ such that

$$\Psi(a)(z, N) = y(z)\, a(z, N)\, y(z)^*$$

for all $z \in V$. Using $\Psi \circ \alpha = \beta \circ \Psi$, and the formula $w_s(z, tN)^* = w_{s^{-1}}(z, s^{-1}tN)$, we compute

$$\begin{aligned}\Psi(z, sN) &= \beta_s(\Psi(\alpha_s^{-1}(a)))(z, sN) \\ &= v_s(z, sN)\, \Psi(\alpha_s^{-1}(a))(z, N)\, v_s(z, sN)^* \\ &= v_s(z, sN)\, y(z)\, \alpha_{s^{-1}}(a)(z, N)\, y(z)^*\, v_s(z, sN)^* \\ &= v_s(z, sN)\, y(z)\, w_s(z, sN)^*\, a(z, sN)\, w_s(z, sN)\, y(z)^*\, v(z, sN)^*.\end{aligned}$$

Therefore we may define

$$h(z, s) = v_s(z, sN)\, y(z)\, w_s(z, sN)^*,$$

and above shows that Eq. (2.2) holds.

Observe that if $n \in N$, then

$$\begin{aligned}\Psi(\alpha_n(a))(z, N) &= y(z)\, \alpha_n(a)(z, N)\, y(z)^* \\ &= y(z)\, w_n(z, N)\, a(z, N)\, w_n(z, N)^*\, y(z)^*,\end{aligned}$$

while, because $\Psi \circ \alpha = \beta \circ \Psi$, the left-hand side also equals

$$\begin{aligned}\beta_n(\Psi(a))(z, N) &= v_n(z, N)\, \Psi(a)(z, N)\, v_n(z, N)^* \\ &= v_n(z, N)\, y(z)\, a(z, N)\, y(z)^*\, v_n(z, N)^*.\end{aligned}$$

It follows that there is a continuous function $\kappa \colon V \times N \to \mathbb{T}$ such that

$$\kappa(z, n)\, y(z)\, w_n(z, N) = v_n(z, N)\, y(z)$$

for all $z \in V$ and $n \in N$. Since we must have $\kappa(z, nm) = \kappa(z, n)\, \kappa(z, m)$, the function $z \mapsto \kappa(z, \cdot)$ is continuous from V to $\hat{N}$. Because $G \to G/N$ has local sections, we may choose a neighborhood W of z_0 contained in V and a continuous function $\varepsilon \colon W \to \hat{G}$ such that $\varepsilon(z, n) = \overline{\kappa(z, n)}$ for all $z \in V$ and $n \in N$.

Next note that

$$\begin{aligned}h(z, sn) &= v_{sn}(z, sN)\, y(z)\, w_{sn}(z, sN)^* \\ &= v_s(z, sN)\, v_n(z, N)\, y(z)(w_s(z, sN)\, w_n(z, N))^* \\ &= \kappa(z, n)\, v_s(z, sN)\, y(z)\, w_n(z, N)\, w_n(z, N)^*\, w_s(z, sN)^* \\ &= \kappa(z, n)\, h(z, s).\end{aligned}$$

Now define $u(z, s) = \varepsilon(z, s)\, h(z, s)$; since $u(z, sn) = u(z, s)$ for all $z \in W$, $s \in G$, and $n \in N$, u has the required properties. ∎

Proof of Theorem 2.1. Using Proposition 2.4, we can assume that there is a open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X/G and $C_0(p^{-1}(\bar{U}_\lambda))$-isomorphisms Ψ_λ of $A|_{p^{-1}(\bar{U}_\lambda)}$ onto $C_0(p^{-1}(\bar{U}_\lambda), \mathscr{K})$ which carry α to actions exterior equivalent to τ. The issue then is the existence of v_{ij} on overlaps. However, if $z\in U_\beta\cap U_\lambda$, then by Proposition 2.5 there is a compact neighborhood $Z\subseteq U_\beta\cap U_\lambda$ of z and a unitary $v\in\mathscr{M}(C_0(p^{-1}(Z), \mathscr{K}))$ which satisfies $\Psi_\beta\circ\Psi_\lambda^{-1}=\operatorname{Ad} v$. The argument now proceeds by selecting an appropriate refinement of the $\{U_\lambda\}$ exactly as in [3, Lemma 10.7.11]. ∎

The main point we made in [20] was that, if (A, G, α) is an N-principal system, then the class of the $\hat{N}$-bundle q is the obstruction to implementing α by a Green twist on N—that is, by a homomorphism $u: N\to\mathscr{U}\mathscr{M}(A)$ such that $\alpha_s(u_n)=u_n$ for $s\in G$ and $n\in N$ [20, Theorem 7.2]. We now want to show how we can compute a cocycle representing $[q]$ from a family of local Green twists. (The local triviality of q as an $\hat{N}$-bundle implies, via [20], that there is such a family, but this also follows directly from Proposition 2.2; indeed, if $u: G\to\mathscr{U}\mathscr{M}(A)$ implements an exterior equivalence between α and any action of G/N, then $u|_N$ is a Green twisting map for α on N.)

PROPOSITION 2.6. *Suppose that (A, G, α) is an N-principal with spectrum $p: X\to Z$. Suppose that $\{N_i\}$ is an open cover of Z and that there are strictly continuous homomorphisms $u^i: N\to\mathscr{U}\mathscr{M}(A|_{p^{-1}(\bar{N}_i)})$ such that*

$$(1)\quad \alpha_n(a)|_{p^{-1}(\bar{N}_i)}=\operatorname{Ad} u_n^i(a|_{p^{-1}(\bar{N}_i)})\qquad \textit{for}\quad a\in A,\quad n\in N;$$

$$(2)\quad \alpha_s(u_n^i)=u_n^i\quad \textit{in } \mathscr{U}\mathscr{M}(A|_{p^{-1}(\bar{N}_i)})\qquad \textit{for}\quad s\in G,\quad n\in N.$$

Then the cocycle $\gamma_{ij}: N_{ij}\to\hat{N}$ defined by $u_n^i(x)=\gamma_{ij}(p(x))\,u_n^j(x)$ is a representative for $q: (A\rtimes_\alpha G)^\wedge\to Z$.

LEMMA 2.7. *Suppose that (A, G, α) is N-principal, and that there is a strictly continuous homomorphism $u: N\to\mathscr{U}\mathscr{M}(A)$ such that $\alpha|_N=\operatorname{Ad} u$ and $\alpha_s(u_n)=u_n$ for $s\in G$, $n\in N$. Then the map*

$$(\pi, \gamma)\mapsto \operatorname{Ind}_N^G(\pi\times\gamma(\pi\circ u))$$

induces an $\hat{N}$-equivariant homeomorphism of $\hat{A}/G\times\hat{N}$ onto $(A\rtimes_\alpha G)^\wedge$.

Proof. Since $\alpha|_N=\operatorname{Ad} u$, $A\rtimes_\alpha N$ is isomorphic to $A\otimes C^*(N)$ and the map $(\pi,\gamma)\mapsto\pi\times\gamma(\pi\circ u)$ is an $\hat{N}$-equivariant homeomorphism of $\hat{A}\times\hat{N}$ onto $(A\rtimes_\alpha N)^\wedge$. As in [17, Proposition 2.1], induction induces an equivariant homeomorphism of $(A\rtimes_\alpha N)^\wedge/G$ onto $(A\rtimes_\alpha G)^\wedge$, so it remains to identify the G-action on $\hat{A}\times\hat{N}$ which is carried onto the action of G on $(A \rtimes_\alpha N)^\wedge$ coming from the action $\beta: G \to \mathrm{Aut}(A \rtimes_\alpha N)$ given by $\beta_s(z)(n) = \alpha_s(z(n))$ for $z \in C_c(N, A)$. But

$$\begin{aligned} s \cdot (\pi \times \gamma(\pi \circ u)) &= (\pi \times \gamma(\pi \circ u)) \circ \beta_s^{-1} \\ &= (\pi \circ \alpha_s^{-1}) \times \gamma(\pi \circ u) \\ &= (\pi \circ \alpha_s^{-1})\, \gamma((\pi \circ \alpha_s^{-1}) \circ u) \\ &= (s \cdot \pi) \times \gamma((s \cdot \pi) \circ u), \end{aligned}$$

which gives the lemma. ∎

Proof of Proposition 2.6. Recall that the map $q: (A \rtimes_\alpha G)^\wedge \to A/G$ is defined as follows: for any $\pi \times U \in (A \rtimes_\alpha G)^\wedge$, $\ker \pi = \bigcap \{\ker s \cdot \rho : s \in G\}$ for some $\rho \in \hat{A}$, and $q(\pi \times U)$ is the orbit $G \cdot \rho \in \hat{A}/G$ (we say that $\pi \times U$ lives on $G \cdot \rho$). Now if F is a closed subset of $\hat{A}/G$, $p^{-1}(F)$ is a closed G-invariant subset of $\hat{A}$, and $A|_{p^{-1}(F)} \rtimes_\alpha G$ is the quotient of $A \rtimes_\alpha G$ whose irreducible representations are those of $A \rtimes_\alpha G$ living on orbits contained in $p^{-1}(F)$—in other words, we can naturally identify $(A|_{p^{-1}(F)} \rtimes_\alpha G)^\wedge$ with $q^{-1}(F)$. Thus, for each i, the above lemma says that

$$(\pi, \gamma) \mapsto \mathrm{Ind}_N^G(\pi \times \gamma(\pi \circ u^i))$$

is a homeomorphism h_i of $\bar{N}_i \times \hat{N}$ onto $q^{-1}(\bar{N}_i)$. Now just have to recall that the dual action of $\gamma \in \hat{N}$ on $(A \rtimes_\alpha G)^\wedge$ is given by

$$\chi \cdot (\mathrm{Ind}_N^G(\pi \times U)) = \mathrm{Ind}_N^G(\pi \times \chi^U)$$

[17, Lemma 2.3], to realize that

$$\begin{aligned} h_i(G \cdot \pi, \gamma) &= \mathrm{Ind}_N^G(\pi \times \gamma(\pi \circ u^i)) \\ &= \mathrm{Ind}_N^G(\pi \times (\gamma\gamma_{ij}(p(\pi))(\pi \circ u^j))) \\ &= \gamma_{ij}(p(\pi))\, h_j(G \cdot \pi, \gamma), \end{aligned}$$

as claimed. ∎

3. Realizing Bundles as the Spectra of Crossed Products

Our object in this section is the following theorem.

THEOREM 3.1. *Let G be a second countable locally compact Abelian group, and let N be a closed subgroup such that $G \to G/N$ and $\hat{G} \to \hat{N}$ have local sections. Suppose that $p: X \to Z$ is a principal G/N-bundle and that $q: Y \to Z$ is a principal $\hat{N}$-bundle. Then there is an N-principal system*

(A, G, α) with spectrum $p: X \to Z$ and with $(A \rtimes_\alpha G)^\wedge$ $\hat{N}$-isomorphic to q if and only if $\langle [p], [q] \rangle_G = 0$ in $\check{H}^4(Z; \mathbb{Z})$.

We have broken the proof of this theorem into three propositions. In Proposition 3.2, we show how the structure theorem of Section 2 leads to a pair (ν, λ) of cocycles, in wich $\nu \in Z^2(Z, \mathcal{S})$ represents the Dixmier–Douady class of A, λ is determined by the action of G on A, and λ also contains information determining $[q]$. Our next proposition (3.3) shows that there can be such data (ν, λ) compatible with a given $[q]$ if and only if $\langle [p], [q] \rangle_G = 0$. Finally, in Proposition 3.6 we use the construction of [18] to produce an N-principal system from one of these pairs (ν, λ).

PROPOSITION 3.2. *Let (A, G, α) be an N-principal system with spectrum $p: X \to Z$, and suppose that both $G \to G/N$ and $\hat{G} \to \hat{N}$ have local sections. Then there are an open cover $\{N_i\}$ of Z, a cocycle $\nu = \{\nu_{ijk}\} \in Z^2(\{p^{-1}(N_i)\}, \mathcal{S})$ representing $\delta(A)$, and continuous maps $\lambda_{ij}: p^{-1}(N_{ij}) \times G \to \mathbb{T}$ such that*

$$\text{(a)} \quad \lambda_{ij}(x, st) = \lambda_{ij}(x, s)\, \lambda_{ij}(s^{-1} \cdot x, t),$$

$$\text{(b)} \quad \lambda_{ij}(x, t)\, \lambda_{jk}(x, t)\, \nu_{ijk}(x) = \lambda_{ik}(x, t)\, \nu_{ijk}(t^{-1} \cdot x),$$

and such that the bundle $q: (A \rtimes_\alpha G)^\wedge \to Z$ has transition functions $\gamma_{ij}: N_{ij} \to \hat{N}$ satisfying

$$\text{(c)} \quad \gamma_{ij}(p(x))(n) = \lambda_{ij}(x, n) \qquad \textit{for} \quad x \in p^{-1}(N_{ij}), \quad n \in N.$$

Proof. Since we may replace (A, G, α) by $(A \otimes \mathcal{K}, G, \alpha \otimes \mathrm{id})$, we may assume that A is stable, and invoke Theorem 2.1. We take $\{N_i\}_{i \in I}$, $\{\Phi_i\}_{i \in I}$, and $\{v_{ij}\}_{i, j \in J}$ as constructed there, and let $u^i: G \to \mathcal{U}\mathcal{M}(C_0(p^{-1}(\bar{N}_i), \mathcal{K}))$ be unitary τ 1-cocycles implementing the exterior equivalence of $\alpha^i = \Phi_i \circ \alpha \circ \Phi_i^{-1}$ and τ, so that $\alpha^i = \mathrm{Ad}\, u^i \circ \tau$. Note that, since G is Abelian, $u^i|_N$ is a Green twisting map for α^i on N.

On $A|_{p^{-1}(\bar{N}_{ij})}$, we have

$$\mathrm{Ad}\, \Phi_i^{-1} \circ \mathrm{Ad}\, u^i \circ \tau = \mathrm{Ad}\, \Phi_j^{-1} \circ \mathrm{Ad}\, u^j \circ \tau,$$

since both implement the restriction of α to $A|_{p^{-1}(\bar{N}_{ij})}$. Consequently

$$\Phi_i \circ \Phi_j^{-1} \circ \mathrm{Ad}\, u^j \circ \tau = \mathrm{Ad}\, u^i \circ \tau \circ \Phi_i \circ \Phi_j^{-1}$$

on $C_0(p^{-1}(\bar{N}_{ij}), \mathcal{K})$. Since $\Phi_i \circ \Phi_j^{-1} = \mathrm{Ad}\, v_{ij}$ by construction, there is a continuous map

$$\lambda_{ij}: p^{-1}(\bar{N}_{ij}) \times G \to \mathbb{T}$$

defined by

$$\lambda_{ij}(x, t)\, v_{ij}(x)\, u_t^j(x) = u_t^i(x)\, v_{ij}(t^{-1}\cdot x), \tag{3.1}$$

and a straightforward computation shows that

$$\lambda_{ij}(x, rt) = \lambda_{ij}(x, r)\, \lambda_{ij}(r^{-1}\cdot x, t) \tag{3.2}$$

for all $x \in p^{-1}(\bar{N}_{ij})$, and $r, t \in G$. Because G is Abelian, Eq. (3.2) implies that for all $n \in N$ and $t \in G$,

$$\lambda_{ij}(x, tn) = \lambda_{ij}(x, t)\, \lambda_{ij}(x, n), \qquad \text{and} \qquad \lambda_{ij}(x, n) = \lambda_{ij}(t^{-1}\cdot x, n). \tag{3.3}$$

In particular, if follows from the second equation that Eq. (c) gives a well-defined continuous function $\gamma_{ij}\colon \bar{N}_{ij} \to \hat{N}$.

Of course, since $(\Phi_i \circ \Phi_j^{-1}) \circ (\Phi_j \circ \Phi_k^{-1}) = \Phi_i \circ \Phi_k^{-1}$, we can define a continuous function $v_{ijk}\colon p^{-1}(\bar{N}_{ijk}) \to \mathbb{T}$ by

$$v_{ij}(x)\, v_{jk}(x) = v_{ijk}(x)\, v_{ik}(x), \tag{3.4}$$

and then $\{p^{-1}(N_i), v_{ijk}\}$ is a 2-cocycle in $Z^2(X, \mathscr{S})$ which represents the Dixmier–Douady class $\delta(A)$ of A [3, Section 10]. By comparing $v_{ijk}(x)\, v_{ik}(x)\, u_t^k(x)$ with $v_{ij}(x)\, v_{jk}(x)\, u_t^k(x)$, and using Eq. (3.1) repeatedly, we obtain Eq. (b). It follows, restricting t to lie in N, that $\{N_i, \gamma_{ij}\}$ defines a 1-cocycle in $Z^1(Z, \hat{\mathscr{N}})$. Since Φ_i extends to an isomorphism of $\mathscr{M}(A|_{p^{-1}(\bar{N}_i)})$ onto $\mathscr{M}(C_0(p^{-1}(\bar{N}_i), \mathscr{K}))$, we can define a strictly continuous homomorphism $w^i\colon N \to \mathscr{U}\mathscr{M}(A|_{p^{-1}(\bar{N}_i)})$ by $w_n^i = \Phi_i^{-1}(u_n^i)$. Note that the restriction of $\alpha|_N$ to $A|_{p^{-1}(\bar{N}_i)}$ satisfies $\alpha_n(a) = w_n^i a (w_n^i)^*$. (This exhibits $\alpha|_N$ explicitly as a locally unitary action on A.) Furthermore, considering w_n^i and w_n^j as elements of $\mathscr{U}\mathscr{M}(A|_{p^{-1}(\bar{N}_{ij})})$,

$$\begin{aligned} w_n^i &= \Phi_i^{-1}(u_n^i v_{ij} v_{ij}^*) \\ &= \lambda_{ij}(\cdot, n)\, \Phi_i^{-1}(v_{ij} u_n^j v_{ij}^*) \\ &= \gamma_{ij}(\cdot)(n)\, \Phi_i^{-1} \circ \Phi_i \circ \Phi_j^{-1}(u_n^j) = \gamma_{ij}(\cdot)(n)\, w_n^j. \end{aligned}$$

Thus, Proposition 2.6 implies that $\{\gamma_{ij}\}$ represents $[q]$, and the result is proved. ∎

PROPOSITION 3.3. *Suppose that G is a locally compact Abelian group, that N is a closed subgroup such that $G \to G/N$ has local sections, and that $p\colon X \to Z$ is a principal G/N-bundle.*

(1) *Suppose that* $\{p^{-1}(N_i), \nu_{ijk}\}$ *is a cocycle in* $Z^2(X, \mathscr{S})$ *defined on a G-invariant cover, and that* $\lambda_{ij}: p^{-1}(N_{ij}) \times G \to \mathbb{T}$ *satisfy*

$$\text{(a)} \quad \lambda_{ij}(x, st) = \lambda_{ij}(x, s)\, \lambda_{ij}(s^{-1} \cdot x, t),$$

$$\text{(b)} \quad \lambda_{ij}(x, t)\, \lambda_{jk}(x, t)\, \nu_{ijk}(x) = \lambda_{ik}(x, t)\, \nu_{ijk}(t^{-1} \cdot x).$$

Then we can define continuous maps $\gamma_{ij}: N_{ij} \to \hat{N}$ *by*

$$\text{(c)} \quad \gamma_{ij}(p(x))(n) = \lambda_{ij}(x, n),$$

and $q = \{\gamma_{ij}\}$ *is a cocycle in* $Z^1(\{N_i\}, \hat{\mathcal{N}})$ *such that* $\langle [p], [q] \rangle_G = 0$.

(2) *Conversely, suppose that* $\hat{G} \to \hat{N}$ *has local sections, and that* $\{\gamma_{ij}\} \in Z^1(\{N_i\}, \hat{\mathcal{N}})$ *is a cocycle such that* $\langle [p], [q] \rangle_G = 0$. *Then there is a cocycle* $\{\nu_{ijk}\} \in Z^2(\{p^{-1}(N_i)\}, \mathscr{S})$, *and a family* $\lambda_{ij}: p^{-1}(N_{ij}) \times G \to \mathbb{T}$ *of continuous maps satisfying conditions* a, b, *and* c.

Proof. Since N acts trivially on X and since G is Abelian, condition a implies that $\lambda_{ij}(x, \cdot)$ is multiplicative on N, and γ_{ij} is a well-defined continuous function on N_{ij} (see the argument following Equation (3.2) above). Condition b implies that $\{\gamma_{ij}\}$ is a cocycle, and allows us to define $\chi_{ijk}: p^{-1}(N_{ijk}) \times G/N \to \mathbb{T}$ by

$$\chi_{ijk}(x, sN) = (\partial\lambda(\cdot, s))_{ijk}\,(x);$$

note that by condition a, χ_{ijk} will also be a cocycle in the second variable. We may suppose that $p: X \to X/G$ has local sections on each N_i, and hence there are equivariant projections $w_i: p^{-1}(N_i) \to G/N$. In particular, $p(x) \mapsto w_i^{-1}(x)\, w_j(x)$ define transition functions for p, and since $G \to G/N$ has local sections, we assume these transition functions have the form $z \mapsto s_{ij}(z)\, N$ for continuous functions $s_{ij}: N_{ij} \to G$. Then we define n_{ijk} by $s_{ij} s_{jk} = s_{ik}\, n_{ijk}$. Next, we try to define a cochain $\{\mu_{ijk}\} \in C^p(\{N_i\}, \mathscr{S})$ by

$$\mu_{ijk}(p(x)) = \chi_{ijk}(x, w_k(x))\, \overline{\lambda_{ij}(w_j(x)^{-1} \cdot x, s_{jk}(p(x)))}\, \nu_{ijk}(x).$$

The right-hand side, $R(x)$ is obviously continuous on $p^{-1}(N_{ijk})$ and defines a continuous map μ_{ijk} on N_{ijk} if $R(t^{-1} \cdot x) = R(x)$. Well,

$$\begin{aligned} R(t^{-1} \cdot x) &= \chi_{ijk}(t^{-1} \cdot x, t^{-1} w_k(x))\, \overline{\lambda_{ij}(w_j(x)^{-1} \cdot x, s_{jk}(p(x)))}\, \nu_{ijk}(t^{-1} \cdot x) \\ &= \overline{\chi_{ijk}(x, tN)}\, \chi_{ijk}(x, w_k(x))\, \overline{\lambda_{ij}(w_j(x)^{-1} \cdot x, s_{jk}(p(x)))} \\ &\quad \cdot (\partial\lambda(\cdot, t))_{ijk}\,(x)\, \nu_{ijk}(x), \end{aligned}$$

which is just $R(x)$ by definition of χ_{ijk}.

We claim that $\partial\mu$ is the cocycle $\gamma_{ij}(n_{jkl})$ representing $\langle [\gamma], [p] \rangle$. To see this, we fix $x \in p^{-1}(z)$ and expand

$$(\partial\mu)_{ijkl}(z) = (\partial\chi(\cdot, s))_{ijkl}(x)|_{s=w_l(x)} \overline{\chi_{ijk}(x, w_k(x))}\, \chi_{ijk}(x, w_l(x))$$
$$\cdot \overline{(\partial\lambda(\cdot, s))_{ijk}(w_k(x)^{-1}\cdot x)|_{s=s_{kl}(z)}}\, \lambda_{ij}(w_k(x)^{-1}\cdot x, s_{kl}(z))$$
$$\cdot \overline{\lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jl}(z))}\, \lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jk}(z)) \cdot (\partial v)_{ijkl}(x).$$

Now $\partial v = 1 = \partial\chi(\cdot, s)$, so combining the second two χ's and the second and fourth λ's gives

$$(\partial\mu)_{ijkl}(z) = \chi_{ijk}(w_k(x)^{-1}\cdot x, w_k(x)^{-1} w_l(x))\, \lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jk}(z)\, s_{kl}(z))$$
$$\cdot \overline{\lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jl}(z))\, \chi_{ijk}(w_k(x)^{-1}\cdot x, s_{kl}(z))}$$
$$= \lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jl}(z)\, n_{jkl}(z))\, \overline{\lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jl}(z))}$$
$$= \gamma_{ij}(z)(n_{jkl}(z)), \tag{3.5}$$

as required. This proves part 1.

For part 2 we use the usual notation, so that $\langle [p], [\gamma] \rangle = 0$ means we can refine the cover to ensure that there is a cochain $\{\mu_{ijk}\} \in C^2(\{N_i\}, \mathscr{S})$ satisfying

$$(\partial\mu)_{ijkl}(z) = \gamma_{ij}(z)(n_{jkl}(z)). \tag{3.6}$$

We refine the cover again so that there are continuous functions $\tilde{\gamma}_{ij}: N_{ij} \to \hat{G}$ with $\gamma_{ij} = \tilde{\gamma}_{ij}|_N$, and define

$$\lambda_{ij}(x, s) = \tilde{\gamma}_{ij}(p(x))(s); \tag{3.7}$$

note that conditions a and c are then obviously true. Next, reversing the argument of part 1, we define

$$\chi_{ijk}(x, sN) = (\partial\lambda(\cdot, s))_{ijk}(x) = (\partial\tilde{\gamma})_{ijk}(p(x))(s), \tag{3.8}$$

and

$$v_{ijk}(x) = \overline{\chi_{ijk}(x, w_k(x))}\, \lambda_{ij}(w_j(x)^{-1}\cdot x, s_{jk}(p(x)))\, \mu_{ijk}(p(x)). \tag{3.9}$$

The computation (3.5) used only the identities $\partial\chi = 1$ and condition a, and can therefore be turned around to show that $\{v_{ijk}\}$ is cocycle. Condition b follows from the computation

$$\overline{\chi_{ijk}(t^{-1}\cdot x, w_k(t^{-1}\cdot x))} = \chi_{ijk}(x, tN)\, \overline{\chi_{ijk}(x, w_k(x))}$$
$$= (\partial\lambda(\cdot, t))_{ijk}(x)\, \overline{\chi_{ijk}(x, w_k(x))},$$

and the proposition is proved. ∎

At this point, we hit a minor but irritating technicality. We want to use the data λ_{ij} to build an action of G on the concrete C^*-algebra $A(v)$ with

Dixmier–Douady class $[\nu]$ constructed in [18, Theorem 1]. For this, however, it is necessary that the cocycle $\nu = \{\nu_{ijk}\}$ be alternating, in the sense that, if σ is a permutation of $\{i, j, k\}$, then

$$\nu_{\sigma(i)\,\sigma(j)\,\sigma(k)} = (\nu_{ijk})^{\operatorname{sgn}(\sigma)};$$

unless this is the case, the involution on $A(\nu)$ defined by $e_{jk}^* = e_{kj}$ will not satisfy $(ab)^* = b^* a^*$. (Unfortunately, this point was overlooked in [18].) Of course, it is well-known that every cocycle is equivalent to an alternating one, but we give the details to make it clear that the new one can still be defined relative to a G-invariant cover.

LEMMA 3.4. *Let X be a locally compact space, let $\mathscr{A} = \{N_i\}_{i \in I}$ be an open cover of X, and let $\nu = \{N_i, \nu_{ijk}\} \in Z^2(\mathscr{A}, \mathscr{S})$. Then there is a cochain $\rho \in C^1(\mathscr{A}, \mathscr{S})$ such that $(\partial\rho)\,\nu$ is alternating.*

Proof. We may assume that I is totally ordered (this is trivial if I is countable, and equivalent to the axiom of choice in general). Define $\rho = \{N_i, \rho_{ij}\} \in Z^1(\mathscr{A}, \mathscr{S})$ by

$$\rho_{ij}(z) = \begin{cases} \overline{\nu_{iji}(z)} & \text{if } i < j, \\ \overline{\nu_{jjj}(z)} & \text{if } i \geqslant j, \end{cases}$$

and let $\mu = (\partial\rho)\,\nu$. Note immediately that for any $i \in I$,

$$\mu_{iii}(z) = 1. \tag{3.10}$$

On the other hand, if $i < j$, then

$$\mu_{iji}(z) = \overline{\nu_{iji}(z)} \cdot \overline{\nu_{iii}(z)} \cdot \nu_{iii}(z) \cdot \nu_{iji}(z) = 1. \tag{3.11}$$

The remaining properties now follows from Eqs. (3.10) and (3.11) applied to the cocycle identities

$$\begin{aligned} 1 &= \mu_{iij}(z) \cdot \overline{\mu_{iij}(z)} \cdot \mu_{iij}(z) \cdot \overline{\mu_{iii}(z)}, \\ &= \mu_{iii}(z) \cdot \overline{\mu_{jii}(z)} \cdot \mu_{jii}(z) \cdot \overline{\mu_{jii}(z)}, \\ &= \mu_{jij}(z) \cdot \overline{\mu_{iij}(z)} \cdot \mu_{ijj}(z) \cdot \overline{\mu_{iji}(z)}, \\ &= \mu_{jik}(z) \cdot \overline{\mu_{iik}(z)} \cdot \mu_{ijk}(z) \cdot \overline{\mu_{iji}(z)}, \end{aligned}$$

and

$$= \mu_{iji}(z) \cdot \overline{\mu_{kji}(z)} \cdot \mu_{kii}(z) \cdot \overline{\mu_{kij}(z)}. \qquad \blacksquare$$

COROLLARY 3.5. *Suppose that $p: X \to Z$ is a principal G/N-bundle, $\{N_i, \gamma_{ij}\} \in Z^1(Z, \hat{\mathscr{N}})$, and that (ν, λ) is a pair satisfying the conditions* a, b, *and* c *in Proposition* 3.3. *Then there is a pair* (v', λ') *satisfying* a, b, *and* c, *in which* v' *is an alternating cocycle of the form* $(\partial\rho)\,v$.

Proof. Choose ρ as in the lemma, let $v' = (\partial\rho)\,v$, and define $\lambda'_{ij}(x, s)$: $p^{-1}(N_{ij}) \times G \to \mathbb{T}$ by

$$\lambda'_{ij}(x, s) = \overline{\rho_{ij}(x)}\,\rho_{ij}(s^{-1}\cdot x)\,\lambda_{ij}(x, s). \quad \blacksquare$$

PROPOSITION 3.6. *Let G be a locally compact group acting on a paracompact locally compact space X. Suppose there are an open cover* $\{M_i\}$ *of X by G-invariant sets, a cocycle* $v_{ijk}: M_{ijk} \to \mathbb{T}$, *and continuous maps* $\lambda_{ij}: M_{ij} \times G \to \mathbb{T}$ *satisfying conditions* a *and* b *of Proposition* 3.3.

(1) *There is a dynamical system* (A, G, α) *with spectrum G-homeomorphic to X and with* $\delta(A) = [v_{ijk}]$.

(2) *If G is Abelian and X is a principal G/N-bundle with orbit map p, then there is an N-principal system with spectrum* $p: X \to X/G$ *such that* $\delta(A) = [v_{ijk}]$ *and* $q: (A \times_\alpha G)^\wedge \to X/G$ *has transition functions* $\gamma_{ij}: p(M_{ij}) \to \hat{N}$ *defined by Proposition* 3.3.(c).

Proof. Let A be the algebra constructed in [18, Theorem 1], with λ_{ijk} there equal to our v_{ijk}; Condition 1 holds by [18]. For each $t \in G$, define α_t by

$$\alpha_t\Big(\sum_{i,j} \phi_{ij}\,e_{ij}\Big) = \sum_{i,j} \overline{\lambda_{ij}(\cdot, t)}\,\tau_t(\phi_{ij})\,e_{ij}:$$

note that $\tau_t(\phi_{ij}) \in C_0(M_{ij})$, since M_{ij} is G/N-invariant, and that $\alpha_t(\sum_{ij}\phi_{ij}\,e_{ij})$ is in A because each λ_{ij} is continuous and bounded on M_{ij}. Because v is alternating, condition b implies that $\overline{\lambda_{ij}} = \lambda_{ji}$, and α_t is $*$-preserving; also, condition b immediately implies that α_t is a homomorphism. Since condition a implies that $\alpha_{rt}(a) = \alpha_r(\alpha_t(a))$, it follows that each α_t has an inverse, and that α is a homomorphism from G to $\mathrm{Aut}(A)$. Furthermore, if $t_p \to t$ in G, then $\lambda_{ij}(\cdot, t_p) \to \lambda_{ij}(\cdot, t)$ uniformly on compact subsets of M_{ij}, and it follows easily that $\alpha_t(a)$ is continuous in t for fixed $a \in A$. Thus α is strongly continuous action of G on A, justifying (1).

Using condition a, we see that the formula

$$u^i_n(x) = \sum_j \lambda_{ij}(x, n)\,e_{jj}$$

for $x \in M_i$ and $n \in N$ defines a strictly continuous homomorphism u^i of N into $\mathcal{M}(A|_{M_i})$. It follows from condition b that

$$\alpha_n\Big(\sum_{j,k} \phi_{jk}\,e_{jk}\Big) = u^i_n(x)\Big(\sum_{j,k} \phi_{jk}(x)\,e_{jk}\Big)\,u^i_n(x)^*.$$

Since the irreducible representations of $A|_{M_i}$ are just the point evaluations, we have shown that u^i implements $\alpha|_N$ over M_i; hence, $\alpha|_N$ is locally unitary. Further, since $\lambda_{ij}=1$, $\alpha_s(u^i_n)=u^i_n$, so u^i is a local Green twist for α, and since condition b implies

$$u^i_n(x)=\lambda_{ij}(x,n)\,u^j_n(x)=\gamma_{ij}(p(x))(n)\,u^j_n(x) \qquad \text{for} \quad x\in M_{ij},$$

the conclusion follows from Proposition 2.6. ∎

4. Realizing Bundles as the Spectra of Crossed Products of a Given Algebra

Now that we know which bundles can arise as the spectrum of the crossed product of an N-principal system (A, G, α), it is natural to ask for which algebras A we can do this. Since continuous-trace algebras are determined up to stable isomophism by their Dixmier–Douady classes, the question can be reformulated as follows: given principal bundles $p: X\to Z$ and $q: Y\to Z$ satisfying $\langle[p],[q]\rangle_G=0$, then for which $\delta\in\check{H}^3(X;\mathbb{Z})$ is there an N-principal system (A, G, α) with spectrum $p: X\to Z$, such that $(A\rtimes_\alpha G)^\wedge$ is $\hat{N}$-isomorphic to Y, and such that $\delta(A)=\delta$? We have already shown how to construct one such class $\delta(q)$ in the proof of Proposition 3.3, and it is not difficult to adapt this argument to produce systems (A, G, α) in which $\delta(A)$ is any class of the form $\delta(q)+p^*\varepsilon$ for $\varepsilon\in\check{H}^3(Z;\mathbb{Z})$ (see the first part of the proof of Theorem 4.1 below). Our main result says that these are in fact the only classes for which this is possible.

We begin by recalling the construction of the class $\delta(q)$. Given $[q]$ such that $\langle[p],[q]\rangle_G=0$, we extend the transition functions γ_{ij} for q to $\tilde{\gamma}_{ij}: N_{ij}\to\hat{G}$, and define

$$\lambda_{ij}(x,s)=\tilde{\gamma}_{ij}(p(x))(s) \tag{4.1}$$

$$\begin{aligned}\chi_{ijk}(x,s)&=(\partial\lambda(\cdot,s))_{ijk}\,(x)\\ &=\lambda_{ij}(x,s)\,\lambda_{jk}(x,s)\,\overline{\lambda_{ik}(x,s)}.\end{aligned} \tag{4.2}$$

Next, we choose a cochain $\{\mu_{ijk}\}$ with

$$(\partial\mu)_{ijkl}(z)=\gamma_{ij}(z)(n_{jkl}(z)), \tag{4.3}$$

where $\{n_{ijk}\}=\partial(\{s_{ij}\})$ represents $\partial_G([p])$, and let

$$v_{ijk}(x)=\overline{\chi_{ijk}(x,w_k(x))}\,\lambda_{ij}(w_j(x)^{-1}\cdot x,s_{jk}(p(x)))\,\mu_{ijk}(p(x)). \tag{4.4}$$

Since we could multiply μ by any cocycle in $Z^2(Z,\mathscr{S})$ without affecting Eq. (4.3), the class $\delta(q)$ of $[v_{ijk}]$ depends on the choice of μ, and we shall rather define $d_p([q])$ to be the image of $[v_{ijk}]$ in the quotient of $\check{H}^3(X;\mathbb{Z})$ by the range of $p^*: \check{H}^3(Z;\mathbb{Z}) \to \check{H}^3(X;\mathbb{Z})$.

LEMMA 4.1. *Suppose that G is locally compact Abelian group, and that N is a closed subgroup such that $G \to G/N$ and $\hat{G} \to \hat{G}/N^{\perp}$ have local sections. Let $[p] \in H^1(Z, \mathscr{G}/\mathscr{N})$ and $[q] \in H^1(Z, \hat{\mathscr{N}})$ satisfy $\langle [p], [q] \rangle_G = 0$. Thus the class $d_p([q])$ in the quotient $\check{H}^3(X;Z)/p^*(\check{H}^2(Z;\mathbb{Z}))$ defined above is independent of any of the choices made, and d_p is a homomorphism of* $\ker(\langle [p], \cdot \rangle_G)$ *into* $H^3(X;\mathbb{Z})/p^*\check{H}^3(Z;\mathbb{Z})$.

Proof. Any other cochain $\{\mu'_{ijk}\}$ satisfying Eq. (4.3) will differ from $\{\mu_{ijk}\}$ by a cocycle $\{\rho_{ijk}\} \in Z^2(Z;\mathscr{S})$, and then the corresponding $\{v'_{ijk}\}$ will differ from $\{v_{ijk}\}$ by $p^*(\{\rho_{ijk}\})$, and hence define the same class in $H^3(X;\mathbb{Z})/\operatorname{Im} p^*$. Replacing the lifting s_{ij} for the transition functions t_{ij} by $s_{ij} m_{ij}$ will change n_{ijk} by $\partial(m)_{ijk}$; however, multiplying μ_{ijk} by $\gamma_{ij}(m_{jk})$ cancels the resulting change on the right-hand side of Eq. (4.3), and also cancels the change inside λ_{ij} on the right-hand side of Eq. (4.4), so v_{ijk} is unchanged. Similarly, if we change the liftings $\tilde{\gamma}_{ij}$ for γ_{ij}, we get compensating changes in λ_{ij} and μ_{ijk}, and v_{ijk} is unchanged.

Thus we only have to worry about different choices of equivariant projections w_i and transition functions γ_{ij} for q. Another choice of equivariant projections $v_i: p^{-1}(N_i) \to G/N$ must differ from the first by continuous maps $y_i: N_i \to G/N$; that is, $v_i(x) = w_i(x)\, y_i(p(x))$. We may as well suppose $y_i: N_i \to G$, and then s_{ij} is replaced by $y_i^{-1} s_{ij} y_j$, and v_{ijk} by

$$\begin{aligned} v'_{ijk}(x) &= \overline{\chi_{ijk}(x, w_k(x)\, y_k(p(x))\, N)} \\ &\quad \cdot \lambda_{ij}(y_j(p(x))^{-1}\, w_j(x)^{-1} \cdot x, (y_j^{-1} s_{jk}\, y_k)(p(x)))\, \mu_{ijk}(p(x)) \\ &= \overline{(\partial\tilde{\gamma})_{ijk}\,(p(x))(y_k(p(x)))}\, \tilde{\gamma}_{ij}(p(x))(y_j(p(x))^{-1}\, y_k(p(x)))\, v_{ijk}(x) \\ &= \overline{(\partial\pi)_{ijk}\,(p(x))}\, v_{ijk}(x), \end{aligned}$$

where $\pi_{ij}(z) = \tilde{\gamma}_{ij}(z)(y_j(z))$. Next, suppose we had started with $\xi_i \gamma_{ij} \xi_j^{-1}$ instead of γ_{ij}. As usual, we can by refining the cover assume $\xi_i = \tilde{\xi}_i|_N$, and replace $\tilde{\gamma}_{ij}$ by $\tilde{\xi}_i \tilde{\gamma}_{ij} \tilde{\xi}_j^{-1}$. This does not change χ_{ijk}, though we have to multiply μ_{ijk} by $\tilde{\xi}_i(n_{ijk}^{-1})$, and adjust λ_{ij}; v_{ijk} becomes

$$\begin{aligned} v'_{ijk}(x) &= \overline{\chi_{ijk}(x, w_k(x))}\, \lambda_{ij}(w_j(x)^{-1} \cdot x, s_{jk}(p(x)))\, \tilde{\xi}_i \tilde{\xi}_j^{-1}(p(x))(s_{jk}(p(x))) \\ &\quad \cdot \tilde{\xi}_i(p(x))(n_{ijk}(p(x))^{-1})\, \mu_{ijk}(p(x)). \end{aligned}$$

But

$$[\tilde{\xi}_i \tilde{\xi}_j^{-1}(s_{jk})]^{-1}\, \tilde{\xi}_i(n_{ijk}^{-1}) = \tilde{\xi}_j(s_{jk})^{-1}\, \tilde{\xi}_i(s_{ij})^{-1}\, \tilde{\xi}_i(s_{ik})$$

is a coboundary, and hence $\{v'_{ijk}\}$ is again equivalent to $\{v_{ijk}\}$.

Finally, to see that d_p is a homomorphism, we just have to observe that, when dealing with a product $\{\gamma_{ij}\zeta_{ij}\}$, we can just multiply the liftings $\tilde{\gamma}_{ij}$, $\tilde{\xi}_{ij}$ and the cochains satisfying Eq. (4.3). ∎

THEOREM 4.1. *Suppose that G is a second countable locally compact Abelian group, and that N is a closed subgroup such that $G \to G/N$ and $\hat{G} \to \hat{N}$ have local sections. Let $p: X \to Z$ be a principal G/N-bundle, let A be a stable continuous trace C^*-algebra with spectrum X, and let $q: Y \to Z$ be a principal $\hat{N}$-bundle. Then there is an N-principal system (A, G, α) with spectrum $p: X \to Z$ such that $(A \rtimes_\alpha G)^\wedge$ is $\hat{N}$-isomorphic to Y if and only if $d_p([q]) = \delta(A) + \operatorname{Im} p^*$ in $\check{H}^3(X; \mathbb{Z})/p^*(\check{H}^3(Z; \mathbb{Z}))$.*

Proof. Suppose first that $\delta(A) + \operatorname{Im} p^* = d_p([q])$, and that $\{\nu_{ijk}\}$ is the cocycle representing $d_p([q])$ constructed above. If we refine $\{N_i\}$ to ensure that $\delta(A) - [\nu_{ijk}]$ can be realized by a cocycle $p^*(\{\mu_{ijk}\})$ in $p^*(Z^2(Z; \mathscr{S}))$, then $\{\nu_{ijk}(\mu_{ijk} \circ \rho)\}$ is a representative for $\delta(A)$. But it is straightforward to check that the family λ_{ij} given by Eq. (4.1) still satisfies conditions a and b of Proposition 3.3 relative to the cocycle $\nu(\mu \circ \rho)$, and hence it follows from Proposition 3.6 that there is an N-principal system (A, G, α) with spectrum $p: X \to Z$ implementing $[q]$.

Next suppose that there is an N-principal system (A, G, α) with $[(A \rtimes_\alpha G)^\wedge] = [q]$, and let (ρ, δ) denote the corresponding data satisfying conditions a and b of Proposition 3.3 (i.e., ρ_{ijk} and δ_{ij} are, respectively, the ν_{ijk} and λ_{ij} of that proposition). As usual we define $\gamma_{ij}: N_{ij} \to \hat{N}$ by $\gamma_{ij}(p(x))(n) = \delta_{ij}(x, n)$, so that $[q] = [\gamma_{ij}]$. As in the proof of Proposition 3.3(1), if we define

$$\begin{aligned} \varepsilon_{ijk}(x, s) &= (\partial\delta(\cdot, s))_{ijk}(x) \\ \mu_{ijk}(p(x)) &= \varepsilon_{ijk}(x, w_k(x))\, \overline{\delta_{ij}(w_j(x)^{-1} \cdot x, s_{jk}(p(x)))}\, \rho_{ijk}(x), \end{aligned} \tag{4.5}$$

then $\partial\mu$ is a representative for $\langle [p], [q] \rangle_G$. We use this cochain μ to construct a representative $\{\nu_{ijk}\}$ for $d_p([q])$ by extending γ_{ij} to $\tilde{\gamma}_{ij}$ and defining λ_{ij}, χ_{ijk}, and ν_{ijk} by Eqs. (4.1), (4.2), and (4.4), respectively. We have to find $\pi_{ij}: p^{-1}(N_{ij}) \to \mathbb{T}$ such that $(\partial\pi)\,\rho\nu^{-1}$ is constant on orbits, and hence has the form $p^*\eta$ for some $\eta \in Z^2(\{N_i\}; \mathscr{S})$.

Comparing Eqs. (4.4) and (4.5) shows that

$$\rho_{ijk}(x)\, \overline{\nu_{ijk}(x)} = \chi_{ijk}(x, w_k(x))\, \overline{\varepsilon_{ijk}(x, w_k(x))}\, T(x),$$

where T is constant on orbits, and hence it is enough to find $\{\pi_{ij}\}$ such that

$$R(x) = (\partial\pi)_{ijk}(x)\, \chi_{ijk}(x, w_k(x))\, \overline{\varepsilon_{ijk}(x, w_k(x))}$$

is constant on orbits. We take

$$\pi_{ij}(x) = \overline{\tilde{\gamma}_{ij}(p(x))(w_j(x))}\, \lambda_{ij}(x, w_j(x)),$$

which is well-defined since two representatives in G for $w_j(x)$ differ by an element of N, and since

$$\tilde{\gamma}_{ij}(p(x))(n) = \gamma_{ij}(p(x))(n) = \lambda_{ij}(x, n) \qquad \text{for all} \quad n \in N.$$

Then

$$\begin{aligned}(\partial\pi)_{ijk}\,(s\cdot x) &= \overline{(\partial\tilde{\gamma})_{ijk}\,(p(x))(s)}\,(\partial\lambda(\cdot, s))_{ijk}\,(s\cdot x)(\partial\pi)_{ijk}\,(x)\\ &= \overline{\chi_{ijk}(s\cdot x, s)}\,\varepsilon_{ijk}(s\cdot x, s)(\partial\pi)_{ijk}\,(x),\end{aligned}$$

and hence

$$\begin{aligned}R(s\cdot x) &= (\partial\pi)_{ijk}\,(s\cdot x)\,\chi_{ijk}(s\cdot x, s\cdot w_k(x))\,\overline{\varepsilon_{ijk}(s\cdot x, s\cdot w_k(x))}\\ &= (\partial\pi)_{ijk}\,(s\cdot x)\,\chi_{ijk}(s\cdot x, s)\,\chi_{ijk}(x, w_k(x))\,\overline{\varepsilon_{ijk}(s\cdot x, s)\,\varepsilon_{ijk}(x, w_k(x))}\\ &= R(x),\end{aligned}$$

and the theorem is proved. ∎

Remark 4.2. For a more C^*-algebraic proof of the first part, suppose (A, G, α) is an N-principal system, $\delta = \delta(A) + p^*\varepsilon$ and B is a continuous-trace algebra with spectrum Z and with $\delta(B) = \varepsilon$. Then

$$C = A \otimes_{C(Z)} B = A \otimes_{C(X)} (C_0(X) \otimes_{C(Z)} B) = A \otimes_{C(X)} p^*B$$

has Dixmier–Douady class

$$\delta(C) = \delta(A) + \delta(p^*B) = \delta(A) + p^*(\delta(B)) = \delta(A) + p^*\varepsilon = \delta$$

(cf. [19]), and the action $\beta = \alpha \otimes_{C(Z)} \mathrm{id}$ of G on C satisfies

$$C \rtimes_\beta G = (A \otimes_{C(Z)} B) \rtimes_{\alpha \otimes_{C(Z)} \mathrm{id}} G = (A \rtimes_\alpha G) \otimes_{C(Z)} B,$$

which has the same spectrum as $A \rtimes_\alpha G$ since $\hat{B} = Z$.

5. Examples and Applications

(A) *The Case* $G = \mathbb{R}$, $N = \mathbb{Z}$

We now want to show how to recover most of [17, Sect. 4] from our work. The main points made there are as follows. First of all, it is shown that if A is continuous-trace algebra whose spectrum is a principal $\mathbb{T}$-bundle $p\colon X \to Z$ then there is an action α of $\mathbb{R}$ on A inducing the given action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on $X = \hat{A}$ [17, first part of Theorem 4.12].

Next, it is deduced from the vanishing of the Moore cohomology group $H^2(\mathbb{R}, C(X, \mathbb{T}))$ [17, Theorem 4.1] that this action α on A is unique up to exterior equivalence [17, Corollary 4.3]. And third, it is shown that the class $[q]$ in $H^1(Z, \mathscr{S}) \cong \check{H}^2(Z; \mathbb{Z})$ of the principal $\mathbb{T}$-bundle $q: (A \rtimes_\alpha G)^\wedge \to Z$ is the image of the Dixmier–Douady class $\delta(A)$ under the map $p_!: \check{H}^3(X; \mathbb{Z}) \to \check{H}^2(Z; \mathbb{Z})$ given by "integration along the fibres" [17; second part of Theorem 4.12]. Thus it follows from exactness of the Gysin sequence

$$\cdots \longrightarrow \check{H}^3(Z; \mathbb{Z}) \xrightarrow{p^*} \check{H}^3(X; \mathbb{Z}) \xrightarrow{p_!} \check{H}^2(Z; \mathbb{Z}) \xrightarrow{\cup [p]} \check{H}^4(Z; \mathbb{Z}) \longrightarrow \cdots$$

that a class $c \in \check{H}^2(Z; \mathbb{Z})$ can arise as $[q]$ if and only if $c \cup [p] = 0$ in $\check{H}^4(Z; \mathbb{Z})$, and since a stable algebra A then carries an action α with $[(A \rtimes_\alpha G)^\wedge] = c$ if and only if $p_!(\delta(A)) = c$, $\delta(A)$ is uniquely determined modulo the image of p^*.

Given the existence and uniqueness of the action α, and our results, it is quite easy to believe the rest of this. Suppose for a moment, then, that we have established this existence and uniqueness. They imply that the Dixmier–Douady class $\delta(A)$ uniquely determines a system (A, G, α), and hence in particular the class $[q]$ of the bundle $q: (A \rtimes_\alpha G)^\wedge \to Z$. In this case, $\hat{N} = \mathbb{T}$ and $[q]$ lies in $H^1(Z, \mathscr{S}) \cong \check{H}^2(Z; \mathbb{Z})$, so $\delta(A) \mapsto [q]$ defines a map $\psi: \check{H}^3(X; \mathbb{Z}) \to \check{H}^2(Z; \mathbb{Z})$. The class of the bundle $p: X \to Z$ also lies in $\check{H}^2(Z; \mathbb{Z})$, and our theorem says precisely that there is an exact sequence

$$\check{H}^3(Z; \mathbb{Z}) \xrightarrow{p^*} \check{H}^3(X; \mathbb{Z}) \xrightarrow{\psi} \check{H}^2(Z; \mathbb{Z}) \xrightarrow{\langle [p], \cdot \rangle_{\mathbb{R}}} \check{H}^4(Z; \mathbb{Z}). \quad (5.1)$$

It is, of course, clear that this should be the Gysin sequence, but while it is relatively easy to identify the pairing $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ with the usual cup product in Čech cohomology (cf. Lemma 1.3), it is not so obvious that our ψ is the map $p_!$ appearing in the Gysin sequence.

We plan to show elsewhere [22] how, for more general pairs (G, N), there is an exact sequence like Eq. (5.1). Showing that this sequence specializes to the usual Gysin sequence when $G = \mathbb{R}$ and $N = \mathbb{Z}$ involves an analysis which, in our setting, gives an alternative proof of all the results of [17, Sect. 4] mentioned above. Here we merely outline the argument.

Let $p: X \to Z$ be a fixed $\mathbb{T}$-bundle, and let A be a stable continuous-trace C^*-algebra with spectrum X. To see that there is an action α of $\mathbb{R}$ on A covering the action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on $X = \hat{A}$, we first show that $\delta(A)$ can be realized by a cocycle $\{n_{ijk}\}$ defined relative to a $\mathbb{T}$-invariant cover $\{M_i\}$ of X. Next, we construct a family $\lambda_{ij}: M_{ij} \times \mathbb{R} \to \mathbb{T}$ satisfying consistency conditions a, b and c of Proposition 3.3, and then use [18] as in Proposition 3.6 to produce an action of $\mathbb{R}$ on a continuous-trace C^*-algebra stably isomorphic to A. To see that the action is unique up to exterior equivalence, we prove directly that any two families $\{\lambda_{ij}\}$ are equivalent in a natural sense. It follows from the construction of $\{\lambda_{ij}\}$ that the map ψ is the usual "integration over the fibres" map $p_!$ appearing in the Gysin sequence.

In our general Gysin sequence, we replace $\check{H}^3(X; \mathbb{Z}) \cong \check{H}^2(X, \mathscr{S})$ with an equivariant cohomology group $H^2_G(X, \mathscr{S})$, in which the cocycles are pairs (v, λ) with $v \in Z^2(Z, \mathscr{S})$ and $\lambda = \{\lambda_{ij}\}$ satisfying conditions a, b and c of Proposition 3.3. Using Proposition 3.6, we can identify $H^2_G(X, \mathscr{S})$ with Morita equivalence classes of systems (A, G, α) [21], and the argument we outlined above amounts to showing that, if $p: X \to Z$ is a $\mathbb{T}$-bundle, then $(v, \lambda) \mapsto v$ induces an isomorphism of $H^2_{\mathbb{R}}(X, \mathscr{S})$ onto $H^2(X, \mathscr{S})$. We defer the details to [22] largely because, to get a *long* exact sequence, we need equivariant cohomology groups $H^n_G(X, \mathscr{S})$ of other dimensions, and we shall want $H^n_{\mathbb{R}}(X, \mathscr{S}) \cong H^n(X, \mathscr{S})$ in general in order to deduce the usual Gysin sequence from ours.

(B) *The Case* $G = \mathbb{T}$, $N = \mathbb{Z}_n$

As in the previous example, the pairing $\langle \cdot, \cdot \rangle_{\mathbb{T}}$ can be expressed in terms of the usual cup product in integral cohomology: if $\partial_1 : H^1(Z; \mathbb{Z}_n) \to \check{H}^2(Z; \mathbb{Z})$ is the coboundary map associated to the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0,$$

then we have

$$\langle [p], [q] \rangle_{\mathbb{T}} = \partial([p]) \cup \partial_1([q]) \qquad \text{for} \quad [q] \in \check{H}^1(Z; \mathbb{Z}_n).$$

The easiest way for us to see this is to reduce to the case $G = \mathbb{R}$; our claim follows from the next lemma and the equation $\partial_1 = \partial \circ i_*$.

LEMMA 5.1. *Let $i: \mathbb{Z}_n \to \mathbb{T}$ denote the inclusion of $\mathbb{Z}_n$ as the nth roots of unity, and suppose $p: X \to Z$ is a principal $\mathbb{T}$-bundle. Then if $G = \mathbb{T}$ and $N = i(\mathbb{Z}_n)$, we have*

$$\langle [p], [q] \rangle_{\mathbb{T}} = \langle [p], i_*([q]) \rangle_{\mathbb{R}} \qquad \textit{for} \quad [q] \in \check{H}^1(Z; \mathbb{Z}_n),$$

and the homomorphism $d^{\mathbb{T}}_p$ of $\ker(\langle [p], \cdot \rangle_{\mathbb{T}})$ *into $\check{H}^3(X; \mathbb{Z})$ is given in terms of the one for $G = \mathbb{R}$, $N = \mathbb{Z}$ by $d^{\mathbb{T}}_p = d^{\mathbb{R}}_p \circ i_*$.*

Proof. We identify both N and $\hat{N}$ with $i(\mathbb{Z}_n)$, so that if we view $\exp(2\pi ik/n) \in N$ and $\exp(2\pi il/n) \in \hat{N}$, then

$$\exp(2\pi il/n)(\exp(2\pi ik/n)) = \exp(2\pi ikl/n). \tag{5.2}$$

Then the bundles q and i_*q have the same transition functions $\gamma_{ij}=\exp(2\pi i m_{ij}/n)$, where $m_{ij}\colon N_{ij}\to\mathbb{Z}$ is a cocycle mod n. If $[p]$ has transition functions $t_{ij}=\exp(2\pi i r_{ij})$, then the lifting $s_{ij}\colon N_{ij}\to G=\mathbb{T}$ is given by $s_{ij}=\exp(2\pi i r_{ij}/n)$, and $\partial_{\mathbb{T}}([p])$ is represented by $n_{ijk}=\exp(2\pi i\rho_{ijk})$, where $\rho_{ijk}=r_{ij}+r_{jk}-r_{ik}\in\mathbb{Z}$. From Eq. (5.2) we have

$$\begin{aligned}\langle[p],[q]\rangle_{\mathbb{T}}&=[\gamma_{ij}(n_{jkl})]=[\exp(2\pi i m_{ij}/n)(\exp(2\pi i\rho_{jkl}/n))]\\&=[\exp(2\pi i m_{ij}\,\rho_{jkl}/n)].\end{aligned}$$

On the other hand, to compute $\langle[p],i_*[q]\rangle_{\mathbb{R}}$, we lift $t_{ij}\colon N_{ij}\to\mathbb{T}=\mathbb{R}/\mathbb{Z}$ to $s'_{ij}\colon N_{ij}\to\mathbb{R}$ (so we can take $s'_{ij}=r_{ij}$), let $\{n'_{ijk}\}=\partial(\{s'_{ij}\})$ (so $n'_{ijk}=\rho_{ijk}$), and

$$\begin{aligned}\langle[p],i_*[q]\rangle_{\mathbb{R}}&=[\gamma_{ij}(n'_{jkl})]=[\exp(2\pi i m_{ij}/n)(\rho_{jkl})]=(\exp(2\pi i m_{ij}/n))^{\rho_{jkl}}]\\&=\langle[p],[q]\rangle_{\mathbb{T}}.\end{aligned}$$

To calculate $d_p^{\mathbb{T}}$, we need to lift $\gamma_{ij}\colon N_{ij}\to\hat{N}$ to $\tilde{\gamma}_{ij}^{q}\colon N_{ij}\to\hat{G}=\hat{\mathbb{T}}=\mathbb{Z}$ along the quotient map $k\mapsto\exp(2\pi i k/n)\colon\mathbb{Z}\to\mathbb{Z}_n\subseteq\mathbb{T}$. So we take $\tilde{\gamma}_{ij}^{q}(w)=w^{m_{ij}}$. Now $\chi_{ijk}^{q}=\partial(\{\tilde{\gamma}_{ij}^{q}\})_{ijk}$ takes values in $N^{\perp}$, and hence it makes sense to apply $\chi_{ijk}^{q}(z)$ to $w\in\mathbb{T}\cong G/N$: note that $w\in\mathbb{T}\cong G/N$ is really identified with $w^{1/n}\mathbb{Z}_n\in\mathbb{T}/\mathbb{Z}_n=G/N$, and hence if $\gamma\in N^{\perp}$ corresponds to $\gamma'\in(G/N)^{\wedge}$, then $\gamma'(w)=\gamma(w^{1/n})$ (which is well-defined because $\gamma\in\mathbb{Z}_n^{\perp}$).Thus if $\partial\mu=\{\gamma_{ij}(n_{jkl})\}$, then a cocycle representing $\partial_{\mathbb{T}}([q])$ is given by

$$\begin{aligned}v_{ijk}(x)&=w_k(x)^{(m_{ij}+m_{jk}-m_{ik})(p(x))/n}\,s_{jk}(p(x))^{m_{ij}(p(x))}\,\mu_{ijk}(x)\\&=w_k(x)^{(m_{ij}+m_{jk}-m_{ik})(p(x))/n}\exp(2\pi i r_{ij}(p(x))\,m_{ij}(p(x))/n)\,\mu_{ijk}(x).\end{aligned}$$

To compute $d_p^{\mathbb{R}}([i_*[q]])$, we need a lifting $\tilde{\gamma}_{ij}^{i_*q}$ for $\gamma_{ij}\colon N_{ij}\to\hat{\mathbb{Z}}$ along the quotient map $x\mapsto e^{2\pi i x}\colon\mathbb{R}=\hat{\mathbb{R}}\to\mathbb{T}=\hat{\mathbb{Z}}$, and we can take $\tilde{\gamma}_{ij}^{i_*q}(z)=m_{ij}(z)/n$. Saying that $\partial(\{\tilde{\gamma}_{ij}^{i_*q}\})$ takes values in $\mathbb{Z}=\mathbb{T}^{\perp}$ just means that $\partial(m_{ij})\in n\mathbb{Z}$, so $w\mapsto w^{\partial(m_{ij})/n}$ is well-defined on $\mathbb{T}$, and a cocycle representing $d_p^{\mathbb{R}}([i_*q])$ is

$$\begin{aligned}\pi_{ijk}&=w_k(x)^{\partial(m_{ij})(p(x))/n}\,\tilde{\gamma}_{ij}^{i_*q}(p(x))(s'_{ij}(p(x))\,\mu_{ijk}(x)\\&=w_k(x)^{(m_{ij}+m_{jk}-m_{ik})(p(x))/n}\exp(2\pi i r_{ij}(p(x))\,m_{ij}(p(x))/n)\,\mu_{ijk}(x),\end{aligned}$$

which is exactly the same as $\{v_{ijk}\}$. ∎

It is easy to construct examples in which $\check{H}^1(Z;\mathbb{Z})$ and $\check{H}^2(Z;\mathbb{Z})$ are torsion-free groups, and $\langle[p],\cdot\rangle_{\mathbb{R}}=\cup\partial([p])=0$—for example, take $Z=S^1\times S^2$. For such a space i_* will be the zero map, and hence $d_p^{\mathbb{T}}=d_p^{\mathbb{R}}\circ i_*$ is also zero, even though $\ker(\langle[p],\cdot\rangle_{\mathbb{T}})$ is all of $\check{H}^1(Z;\mathbb{Z}_n)$. Thus for any A satisfying the consistency condition $\delta(A)+\operatorname{Im}p^*\in$ range $d_p^{\mathbb{T}}$, there will be many q's satisfying $d_p^{\mathbb{T}}([q])=\delta(A)\in\operatorname{Im}p^*$. Since for each of these there is a $\mathbb{Z}_n$-principal action of $\mathbb{T}$ on A with

$(A \rtimes_\alpha \mathbb{T})^\wedge \cong q$, there must be many non-equivalent $\mathbb{Z}_n$-principal actions of $\mathbb{T}$ on the same A. Thus this example is quite different from the previous one, in spite of the similarity in the calculations of $\langle [p], \cdot \rangle_G$ and d_p.

(C) *Actions on Quotients of Induced C*-Algebras*

If $\beta: N \to \mathrm{Aut}(B)$ is locally unitary and $(A, G, \alpha) = (\mathrm{Ind}_N^G(B, \beta), G, \tau)$, then $(A \rtimes_\alpha G)^\wedge$ is $\hat{N}$-isomorphic to $(B \rtimes_\beta N)^\wedge$, and hence $[q]$ can be any class in $\check{H}^1(Z, \hat{\mathcal{N}})$ [17, Sect. 3(a)]; for these examples, however, $p: \hat{A} \to Z$ is trivial as a G/N-bundle. In [16, Proposition 3.5], it was shown that by inducing along other principal G-bundles and taking quotients, we could construct many examples in which both $[p]$ and $[q]$ were non-trivial. This construction only worked when $p: X \to Z$ was the quotient $Y/N \to Z$ of some G-bundle, but for any such $[p]$, we could obtain any $[q]$. This is easily seen to be consistent with our theorem: the existence of such a bundle Y is equivalent to the vanishing of $\partial_G([p]) \in H^2(Z, \mathcal{N})$, and hence it follows immediately from Definition 1.1 that $\langle [p], [q] \rangle_G = 0$ for any $[q] \in H^1(Z, \hat{\mathcal{N}})$.

In [16, Proposition 3.5], we gave a formula for the Dixmier–Douady class $\delta(A)$ also, and we now check its compatible with our theorem. In this formula, $\langle \cdot, \cdot \rangle$ denotes the pairing of $H^1(Z, \hat{\mathcal{N}}) \times H^1(Z, \mathcal{N})$ into $\check{H}^3(Z; \mathbb{Z}) \cong \partial(H^2(Z, \mathcal{S}))$ given by

$$\langle [\gamma_{ij}], [n_{ij}] \rangle = \partial([\lambda_{ijk}]), \qquad \text{where} \quad \lambda_{ijk}(t) = \gamma_{ij}(z)(n_{jk}(t)),$$

which is just the cup product composed with evaluation as in Definition 1.1.

PROPOSITION 5.2. *Let $r: Y \to Y/N = X$ be the quotient map. Then for any $[q] \in H^1(Z, \hat{\mathcal{N}})$, we have*

$$d_p([q]) = \langle p^*[q], [r] \rangle + \mathrm{Im}\, p^* \qquad \textit{in} \quad \check{H}^3(X; \mathbb{Z})/\mathrm{Im}\, p^*.$$

Proof. We retain the usual notation, so $\{\gamma_{ij}\}$ are transition functions for $[q]$, the functions $\tilde{\gamma}_{ij}: N_{ij} \to \hat{G}$ extend the γ_{ij}, and $\{\chi_{ijk}\} = \{\tilde{\gamma}_{ij} \tilde{\gamma}_{jk} \tilde{\gamma}_{ik}^{-1}\}$ represents $\partial([q])$ in $H^2(Z, \mathcal{N}^\perp)$. In addition, we fix local trivializations $k_i: r^{-1}(p^{-1}(N_i)) \to N_i \times G$ such that $k_i \circ k_j^{-1}(z, s) = (t, ss_{ij}(z))$ and assume that the trivializations of $p: Y/N \to Z$ are induced by k_i, in the sense that $h_i(yN) = k_i(y) N$. We define $w_i: p^{-1}(N_i) \to G/N$ by $h_i(x) = (p(x), w_i(x))$. To define $d_p([q])$ we need a cochain $\mu_{ijk}: N_{ijk} \to \mathbb{T}$ such that

$$\partial(\{\mu_{ijk}\})_{ijkl} = \gamma_{ij}(n_{jkl}),$$

but here $\{s_{ij}\}$ is a cocycle, so $n_{ijk} = 1$, and we can take $\mu_{ijk} = 1$. We now set

$$v_{ijk}(x) = \overline{\chi_{ijk}(p(x))(w_k(x))}\, \tilde{\gamma}_{ij}(p(x))(s_{jk}(p(x))) \qquad \text{for} \quad x \in p^{-1}(N_{ijk}),$$

and then $d_p([q])$ is the image of $[v_{ijk}]$ in $H^2(X, \mathscr{S})/\operatorname{Im} p^* \cong \check{H}^3(X; \mathbb{Z})/\operatorname{Im} p^*$.

To find transition functions for $r: Y \to X$, we choose local sections $c_p: M_p \to G$ for $G \to G/N$, and consider the cover $F_{(i,p)} = h_i^{-1}(N_i \times M_p)$ of X. On $F_{(i,p)}$, we have sections

$$d_{(i,p)}(x) = k_i^{-1}(p(x), c_p(w_i(x))),$$

and r is represented by the cocycle $\{n_{(i,p)(j,q)}\}$ where

$$d_{(i,p)}(x) = n_{(i,p)(j,q)}(x)\, d_{(j,q)}(x);$$

observe that since $k_i \cdot k_j^{-1}$ is implemented by s_{ij}, we have

$$c_p(w_i(x)) = n_{(i,p)(j,q)}(x)\, c_q(w_j(x))\, s_{ij}(p(x))^{-1}. \tag{5.3}$$

Now we let $x \in F_{(i,p)(j,q)(k,r)}$, and compute $\{v_{ijk}\}$. Since $\chi_{ijk}(p(x)) \in N^{\perp}$, we can use any representative for $w_k(x)$ when calculating $\chi_{ijk}(p(x))(w_k(x))$, and we obtain from Eq. (5.3):

$$\begin{aligned} v_{ijk}(x) &= \overline{\chi_{ijk}(p(x))(c_r(w_k(x)))}\, \tilde{\gamma}_{ij}(p(x))(s_{jk}(p(x))) \\ &= \overline{\tilde{\gamma}_{ij}(p(x))(c_r(w_k(x))\, s_{jk}(p(x))^{-1})} \\ &\quad \cdot \overline{\tilde{\gamma}_{jk}(p(x))(c_r(w_k(x)))}\, \tilde{\gamma}_{ik}(p(x))(c_r(w_k(x))) \\ &= \gamma_{ij}(p(x))(n_{(j,q)(k,r)}(x))\, \overline{\tilde{\gamma}_{ij}(p(x))(c_q(w_j(x)))} \\ &\quad \cdot \overline{\tilde{\gamma}_{jk}(p(x))(c_r(w_k(x)))}\, \tilde{\gamma}_{ik}(p(x))(c_r(w_k(x))). \end{aligned} \tag{5.4}$$

If we define $\rho_{(i,p)(j,q)}: F_{(i,p)(j,q)} \to \mathbb{T}$ by

$$\rho_{(i,p)(j,q)}(x) = \overline{\tilde{\gamma}_{ij}(p(x))(c_q(w_j(x)))},$$

then Eq. (5.4) says that

$$v_{ijk}(x) = \partial(\rho)_{(i,p)(j,q)(k,r)}(x)\, \gamma_{ij}(p(x))(n_{(j,q)(k,r)}(x)).$$

Thus we have shown that $\{v_{ijk}\}$ differs by a coboundary from the cocycle defined on the cover $\{F_{(i,p)}\}$ by

$$b_{(i,p)(j,q)(k,r)}(x) = \gamma_{ij}(p(x))(n_{(j,q)(k,r)}(x)),$$

which represents $\langle [\gamma_{ij} \circ p], [n_{(i,p)(j,q)}] \rangle = \langle p^*[q], [r] \rangle$. ∎

In light of the proposition, our main theorem predicts that, given q and an algebra A over X, there will be an N-principal action of G on A with $(A \rtimes_\alpha G)^\wedge$ isomorphic to q if and only if

$$\delta(A) = p^*c + \langle p^*[q], r \rangle \qquad \text{for some} \quad c \in \check{H}^3(Z; \mathbb{Z}),$$

and [16, Proposition 3.5] provides concrete examples for each of these possibilities.

(D) *Actions on Pull-Back C^*-Algebras*

In [19], we studied diagonal actions on the pull-backs of C^*-algebras along principal bundles, and, by Theorem 1.1 of [17], these are essentially the only examples (A, G, α) in which $\hat{A}$ is a principal G-bundle. There are also natural diagonal actions of G on algebras p^*B pulled back along a G/N-bundle $p: X \to Z$, and if original action of G on B was locally unitary, these actions are N-principal. In fact, the systems we obtain this way turn out to be dual to those of the previous section: now $(A \rtimes_\alpha G)^\wedge$ has the form $Y/N^\perp$ for some $\hat{G}$-bundle Y over Z.

PROPOSITION 5.3. *Let N be a closed subgoup of a locally compact Abelian group G, let $p: X \to Z$ be a G/N-bundle, and let $\beta: G \to \operatorname{Aut}(B)$ be a locally unitary action of G on a C^*-algebra B with spectrum Z. Then the diagonal action $p^*\beta = \mathrm{id} \otimes_{C(Z)} \beta$ on $p^*B = C_0(X) \otimes_{C(Z)} B$ is locally unitary on N, and the spectrum $q: (p^*B \rtimes_{p^*\beta} G)^\wedge \to Z$ is $\hat{N}$-isomorphic to $(B \rtimes_\beta G)^\wedge / N^\perp$.*

Proof. If $u: G \to UM(B)$ implements β over M, then $1 \otimes_{C(Z)} u|_N$ implements $p^*\beta|_N$ over $p^{-1}(M)$, and $p^*\beta|_N$ is locally unitary. We also have a commutative diagram

$$\begin{array}{ccc} (\varepsilon_x, \gamma) \in M \times \hat{G} & \longrightarrow & \varepsilon_x \times \gamma u(x) \in (B \rtimes_\beta G)^\wedge \\ \downarrow & & \downarrow \scriptstyle\mathrm{Res} \\ (\varepsilon_x, \gamma N^\perp) \in M \times (\hat{G}/N^\perp) = M \times \hat{N} & \longrightarrow & \varepsilon_x \times (\gamma|_N\, u(x)|_N) \in (B \rtimes_\beta N)^\wedge, \end{array}$$

which implies in particular that $\mathrm{Res}: (B \rtimes_\beta G)^\wedge \to (B \rtimes_\beta N)^\wedge$ is continuous and open, and induces a homeomorphism of $(B \rtimes_\beta G)^\wedge / N^\perp$ onto $(B \rtimes_\beta N)^\wedge$. Since the last two maps are homeomophism, it follows that the composition h given by

$$\begin{array}{ccccccc} X \times_Z (B \rtimes_\beta G)^\wedge & \xrightarrow{\mathrm{id} \times \mathrm{Res}} & X \times_Z (B \rtimes_\beta N)^\wedge & \longrightarrow & p^*(B \rtimes_\beta N)^\wedge & \longrightarrow & ((p^*B) \rtimes_{p^*\beta} N)^\wedge \\ (x, (\varepsilon_{p(x)} \times u)) & \longmapsto & (x, (\varepsilon_{p(x)} \times u|_N)) & \longmapsto & \varepsilon_x \otimes_{C(Z)} (\varepsilon_{p(x)} \times u|_N) & \longmapsto & \varepsilon_x \times (u|_N) \end{array}$$

induces a homeomorphism of the fibre product $X \times_Z (B \rtimes_\beta G)^\wedge / N^\perp$ onto $((p^*B) \rtimes_{p^*\beta} N)^\wedge$.

We next claim that if α is the action of G on $p^*B \rtimes N$, then the map h satisfies

$$h(s \cdot x, (\varepsilon_{p(x)} \times u)) \cong h(x, (\varepsilon_{p(x)} \times u)) \circ \alpha_s^{-1}.$$

For we have from [17, bottom of p.21] that

$$(\varepsilon_x \times u|_N) \circ \alpha_s^{-1} = (\varepsilon_x \circ (p^*\beta)_s^{-1}) \times u|_N,$$

and we can compute $\varepsilon_x \circ (p^*\beta)_s^{-1}$: if $f \otimes_{C(Z)} b \in p^*B$, then

$$\begin{aligned}\varepsilon_x \circ (p^*\beta)_s^{-1} (f \otimes_{C(Z)} b) &= \varepsilon_x \otimes_{C(Z)} \varepsilon_{p(x)}(\tau_s^{-1}(f) \otimes_{C(Z)} \beta_s^{-1}(b)) \\ &= f(s \cdot x) \operatorname{Ad} u_s^{-1}(\varepsilon_{p(x)}(b)) \\ &= \operatorname{Ad} u_s^{-1}(\varepsilon_{s \cdot x}(f \otimes_{C(Z)} b)).\end{aligned}$$

Thus

$$\begin{aligned}h(x, (\varepsilon_{p(x)} \times u)) \circ \alpha_s^{-1} &= \operatorname{Ad} u_s^{-1}(\varepsilon_{s \cdot x}) \times u|_N \\ &= \operatorname{Ad} u_s^{-1}(\varepsilon_{s \cdot x} \times u|_N) \\ &\sim h(s \cdot x, \varepsilon_{p(x)} \times u),\end{aligned}$$

as claimed.

Since we know that Ind induces a homeomorphism of $(p^*B \rtimes N)^\wedge/G$ onto $(p^*B \rtimes G)^\wedge$ [17, Proposition 2.1], it follows that $\operatorname{Ind} \circ h$ induces a homeomorphism of

$$(X \times_Z ((B \rtimes_\beta G)^\wedge/N^\perp))/G = Z \times_Z ((B \rtimes_\beta G)^\wedge/N^\perp) = (B \rtimes_\beta G)^\wedge/N^\perp$$

onto $(p^*B \rtimes G)^\wedge$. But $\operatorname{Ind} \circ h$ is clearly $\hat{N}$-equivalent, and the result follows. ∎

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