Separable Injectivity for C^*-algebras

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1. Introduction. A unital C^*–algebra A is said to be separably injective if, given separable unital operator systems $E \subseteq F$ and a completely positive unital map $\psi : E \to A$, there exists a completely positive extension $\psi : F \to A$. This concept arose naturally in previous work of the authors [17] in connection with the study of completely bounded maps. It is weaker than the related idea of injectivity, but in many situations the object on which an extension is desired are separable, and so the notion of separable injectivity is appropriate. In the case of a commutative C^*–algebra separable injectivity was characterized in [17] in terms of the topology of the maximal ideal space, which must be substonean. In this paper we wish to study separable injectivity in the noncommutative case, which was left untouched in [17].

After the commutative C^*–algebras, the next most tractable class is the collection of subhomogeneous C^*–algebras consisting of algebras which have only finite dimensional representations of bounded degree. It is this class of C^*–algebras which we study here. All such algebras are type I and so we do not distinguish between the spectrum and the primitive ideal space. We also note that any type I C^*–algebras is nuclear. Since we wish to consider ideals, we do not assume that our C^*–algebra is unital. This necessitates a more general definition of separable injectivity (Definition 2.3) which is equivalent to the previous definition for unital algebras.

The second section contains some technical preliminaries concerning separable injectivity for use in the main results. The third section is concerned with the case of n–homogeneous algebras and it is shown that A is separably injective if and only if $\hat{A}$ is substonean. It is perhaps surprising that separable injectivity depends only on the topology of $\hat{A}$ since many nonisomorphic n–homogeneous C^*–algebras can have the same primitive ideal space [11]. At the end of the section a general method is given for constructing examples of nontrivial separable injective homogeneous C^*–algebras.

The last section is concerned with the subhomogeneous case, and once again the topology of $\hat{A}$ plays a role. We show that if each $\hat{A}_r$ (the subset of

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r–dimensional irreducible representations) is substonean in the relative topology, then A is separably injective. The converse is not true (Theorem 4.7) and on the way to establishing this we construct an infinite class of nontrivial separably injective homogeneous C^*–algebras (Remark 4.6). In contrast, we point out that all injective homogeneous C^*–algebras are trivial [17]. We conclude the paper by showing that, as a consequence of our results, homogeneous algebras need not have homogeneous multiplier algebras.

The authors would like to thank Professor Maurice Dupré for some helpful comments concerning the material in the last section.

2. Separable Injectivity. In this section we give some basic definitions and results concerning separable injectivity. The term "operator subspace" refers to a subspace of a C^*–algebra. If E is an operator subspace, then E^* denotes the operator subspace consisting of the adjoints of elements of E, and if $E = E^*$, then E is called an operator system. When an operator system contains the identity of the C^*–algebra in which it is embedded, it is called unital.

We begin by recalling a result of Choi [3], which will be used frequently below.

Lemma 2.1. *Let $\varphi : A \to B$ be a contractive completely positive map between C^*–algebras. If*

$$\varphi(a^*a) = \varphi(a)^*\varphi(a) \qquad \text{and} \qquad \varphi(aa^*) = \varphi(a)\varphi(a)^*,$$

then for all $b \in A$,

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad \text{and} \qquad \varphi(ba) = \varphi(b)\varphi(a).$$

Such elements are said to lie in the multiplicative domain of φ.

Lemma 2.2. *Let A be a unital C^*–algebra. The following are equivalent:*

(1) *Given separable unital operator systems $E \subseteq F$ and a completely positive unital map $\varphi : E \to A$, there exists a completely positive extension $\tilde{\varphi} : F \to A$.*
(2) *Given separable unital operator systems $E \subseteq F$ and a completely positive map $\varphi : E \to A$, there exists a completely positive extension $\tilde{\varphi} : F \to A$.*
(3) *Given separable operator subspaces $E \subseteq F$ and a completely bounded map $\varphi : E \to A$, there exists a completely positive extension $\tilde{\varphi} : F \to A$ satisfying $\|\varphi\|_{cb} = \|\tilde{\varphi}\|_{cb}$.*

Proof. (2) $\Longrightarrow$ (1). This is obvious.

(3) $\Longrightarrow$ (2). Let $E \subseteq F$ be separable unital operator systems, let $\varphi : E \to A$ be completely positive, and assume that A is faithfully represented as a subalgebra of $B(H)$. Set $a = \varphi(1) \geq 0$. Then there exists a unital completely positive map $\psi : E \to B(H)$ such that

$$\varphi(x) = a^{1/2}\psi(x)a^{1/2} \qquad (x \in E)$$

by [5]. Since $B(H)$ is injective [2], there exists a unital completely positive extension $\tilde{\psi} : F \to B(H)$ of ψ. Let B be the separable subalgebra of A generated by 1 and the range of φ, and let C be the separable subalgebra of $B(H)$ generated by B and the range of $\tilde{\psi}$. By hypothesis there exists a complete contraction $\vartheta : C \to A$ which extends the identity embedding of B into A. Since ϑ is unital, it follows that ϑ is completely positive. Observe that $a^{1/2}$ is in the multiplicative domain of ϑ.

Now define $\tilde{\varphi} : F \to A$ by

$$\tilde{\varphi}(x) = a^{1/2}\vartheta\big(\tilde{\psi}(x)\big)a^{1/2} \qquad (x \in F).$$

This map is completely positive, and if $x \in E$, then

$$\begin{aligned}\tilde{\varphi}(x) &= a^{1/2}\vartheta\big(\tilde{\psi}(x)\big)a^{1/2} = a^{1/2}\vartheta\big(\psi(x)\big)a^{1/2}\\ &= \vartheta\big(a^{1/2}\psi(x)a^{1/2}\big) = \vartheta\big(\varphi(x)\big)\\ &= \varphi(x),\end{aligned}$$

the third equality following from Lemma 2.1. Thus $\tilde{\varphi}$ is a completely positive extension of φ.

(1) $\Longrightarrow$ (3). First recall that if A satisfies (1), then so also does $A \otimes M_2$ [17].

Let $E \subseteq F$ be separable operator subspaces and let $\varphi : E \to A$ be a complete contraction. Define unital operator systems $\widetilde{E} \subseteq \widetilde{F}$ by

$$\widetilde{E} = \left\{ \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} : \lambda,\, \mu \in \mathbf{C},\, x,\, y \in E \right\}$$

$$\widetilde{F} = \left\{ \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} : \lambda,\, \mu \in \mathbf{C},\, x,\, y \in F \right\}$$

and define $\vartheta : \widetilde{E} \to A \otimes M_2$ by

$$\vartheta \begin{pmatrix} \lambda & x \\ y^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \varphi(x) \\ \varphi(y)^* & \mu \end{pmatrix}.$$

From [12] this is a completely positive unital map and so, by hypothesis, has a completely positive unital extension $\tilde{\vartheta} : \widetilde{F} \to A \otimes M_2$. For $x \in F$, define

$\widetilde{\varphi} : F \to A$ by

$$\widetilde{\varphi}(x) = (1,0)\widetilde{\vartheta}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\widetilde{\varphi}$ is completely contractive and extends φ. □

In [17] condition (1) was taken as the definition of separable injectivity for unital C^*–algebras. We wish to widen this to include the nonunital case, and from above it is consistent to make the following:

Definition 2.3. A C^*–algebra A is said to be separable injective if, given separable operator subspaces $E \subseteq F$ and a completely bounded map $\varphi : E \to A$, there exists a completely bounded extension $\widetilde{\varphi} : F \to A$ satisfying $\|\widetilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.

Definition 2.4. A C^*–algebra A is said to be countably unital if, given any countable set $\{a_n\}_{n=1}^{\infty}$ from A there exists a positive element $a \in A$ of unit norm for which $aa_n = a_n a = a_n$.

Note that this could be reformulated in two equivalent ways. The countable set could be replaced by a separable subalgebra, or by a single element. An element a acts as the identity for $\{a_n\}_{n=1}^{\infty}$ if and only if it acts as the identity for

$$\sum_{n=1}^{\infty} 2^{-n} a_n^* a_n .$$

The following result gives a necessary condition for a C^*–algebra to be separable injective.

Proposition 2.5. *If a C^*–algebra A is separable injective then it is countably unital.*

Proof. Let B be a separable subalgebra of A and let B_1 denote a B with unit adjoined. By hypothesis there exists a completely contractive map $\varphi : B_1 \to A$ which extends the identity embedding of B into A. It may be assumed that φ is self–adjoint since it may otherwise be replaced by $\frac{1}{2}(\varphi + \varphi^*)$. Let x denote $\varphi(1)$, which is thus self–adjoint.

The map $\varphi^{**} : B_1^{**} \to A^{**}$ is also completely contractive. Let p denote the identity of B^{**} in B_1^{**}. Then $\varphi^{**}(p) = p$ by w^*–continuity of φ^{**}. Since

$$\left\|e^{i\vartheta} p + (1-p)\right\| = 1,$$

it follows that

$$\left\|e^{i\vartheta} p + x - p\right\| \le 1.$$

Now multiply on both sides by p to obtain

$$\left\|e^{i\vartheta} p + pxp - p\right\| \le 1.$$

These elements lie in $pA^{**}p$ with identity p, and so for any state ω

$$\left|e^{i\vartheta} + \omega(pxp) - 1\right| \leq 1,$$

from which it follows that

$$\omega(pxp) = 1 \qquad \text{and} \qquad pxp = p.$$

Since $1 - x \geq 0$ and $p(1-x)p = 0$, it follows that

$$px = xp = p.$$

Thus, for all $b \in B$,

$$bx = bpx = bp = b$$

and

$$xb = xpb = pb = b,$$

and these equalities remain valid if x is replaced by x^2. Thus x^2 is a positive element of unit norm which acts as the identity for B. □

Proposition 2.6

(1) *If A is separably injective, then A_1 is separably injective.*

(2) *If A_1 is separably injective and A is countably unital, then A is separably injective.*

Proof. (1) Let $E \subseteq F$ be separable operator subspaces and let $\varphi : E \to A_1$ be a complete contraction. Let B be a separable subalgebra of A for which the range of φ is contained in $B + \mathbf{C}1$. By Proposition 2.5 there exist two positive elements a_1, $a_2 \in A$ of unit norm such that a_1 acts as the identity on B and a_2 acts as the identity on $C^*(B,a_1)$. Let ω be the state on A_1 which annihilates A. Define $\psi : E \to A$ by

$$\psi(x) = a_2\varphi(x)a_2 \qquad (x \in E)$$

and $\vartheta : E \to \mathbf{C}$ by

$$\vartheta(x) = \omega\big(\varphi(x)\big) \qquad (x \in E).$$

By the Hahn–Banach theorem ϑ has an extension $\widetilde{\vartheta} : F \to \mathbf{C}$ which may be regarded as a completely contractive map $x \to \widetilde{\vartheta}(x)1$ into A_1. By hypothesis ψ has a completely contractive extension $\widetilde{\psi} : F \to A$.

Define $\widetilde{\varphi} : F \to A_1$ by

$$\widetilde{\varphi}(x) = a_1\widetilde{\psi}(x)a_1 + (1 - a_1^2)\widetilde{\vartheta}(x) \qquad (x \in F).$$

Since $\widetilde{\varphi}(x)$ may be written

$$\left(a_1, (1-a_1^2)^{1/2}\right)\begin{pmatrix}\widetilde{\psi}(x) & 0 \\ 0 & \widetilde{\vartheta}(x)\end{pmatrix}\begin{pmatrix}a_1 \\ (1-a_1^2)^{1/2}\end{pmatrix},$$

it is clear that $\widetilde{\varphi}$ is completely contractive. To see that $\widetilde{\varphi}$ extends φ, choose $x \in E$ and write $\varphi(x) = b + \lambda 1$, $b \in B$. Then $\widetilde{\psi}(x) = b + \lambda a_2^2$ and $\widetilde{\vartheta}(x) = \lambda 1$. Thus

$$\begin{aligned}\widetilde{\varphi}(x) &= a_1(b+\lambda a_2^2)a_1 + \lambda(1-a_2^2)\\ &= b + \lambda a_1^2 + \lambda 1 + \lambda a_1^2\\ &= \rho + \lambda 1\\ &= \varphi(x).\end{aligned}$$

(2) Let $\varphi : E \to A$ be a complete contraction, and choose a positive element $a \in A$ of unit norm which acts as the identity on the range of φ. By hypothesis φ has a completely contractive extension $\psi : F \to A_1$, and now define $\widetilde{\varphi} : F \to A$ by

$$\widetilde{\varphi}(x) = a\psi(x)a \qquad (x \in F).$$

Then $\widetilde{\varphi}$ is completely contractive and extends φ. □

Recall from [14] and [17] respectively the following definitions:

A C^*–algebra A is said to be an SAW^*–algebra if given a, $b \in A_+$ with $ab = 0$, there exists $e \in A_+$ of unit norm satisfying $ae = a$ and $be = 0$.

A C^*–algebra A is said to have the countable Riesz separation property if, given an increasing sequence $\{x_n\}$ and decreasing sequence $\{y_m\}$ of self–adjoint elements of A satisfying $x_n \le y_m$ for all m, $n \ge 1$, there exists $z \in A$ satisfying $x_n \le z \le y_m$ for all m, $n \ge 1$.

We are indebted to Professor G. K. Pedersen for pointing out the next result.

Proposition 2.7. *Consider the following conditions on a C^*–algebra A:*

(i) *A is separably injective,*
(ii) *A has the countable Riesz separation property,*
(iii) *A is an SAW^*–algebra.*

Then (i) $\Longrightarrow$ (ii), (i) $\Longrightarrow$ (iii), *and, if A is unital (or, more generally, countably unital), then* (ii) $\Longrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii). Let A be separably injective and let A_1 be A with unit adjoined. By Proposition 2.6 (1) A_1 is separably injective. Consider two monotone sequences $\{x_n\}$ and $\{y_m\}$ from A_{sa} satisfying

$$x_1 \le x_2 \le \ldots \le y_2 \le y_1 .$$

Since A^{**} is a W^*–algebra, there exists $t \in A^{**}$ (which may be taken to be the supremum of $\{x_n\}$) satisfying

$$x_1 \le x_2 \le \ldots \le t \le \ldots \le y_2 \le y_1 .$$

Consider the two separable operator systems

$$E = \text{span}\{x_1, y_1, x_2, y_2, \ldots\} \qquad \text{and}$$
$$F = \text{span}\{E, t\}.$$

By Lemma 2.2 (2), the identity embedding of E into A has a completely positive extension $\varphi : F \to A_1$. Let $z = \varphi(t)$. Then

$$x_1 \leq x_2 \leq \ldots \leq z \leq \ldots \leq y_2 \leq y_1 \qquad (\text{in } A_1)$$

and, since A is an ideal in A_1, $z \in A$. Thus A has the countable Riesz separation property.

(ii) $\Longrightarrow$ (iii) for countably unital algebras.

Let a, $b \in A_+$ satisfy $ab = 0$, and without loss of generality suppose that $\|a\|$, $\|b\| \leq 1$. Since A is countably unital, there exists $f \in A_+$, $\|f\| = 1$ such that $fa = af = a$, $fb = bf = b$. For $n \geq 1$, $m \geq 1$ write $x_n = a^{1/n}$, $y_m = (f - b)^m$. Then

$$x_1 \leq x_2 \leq \ldots \leq y_2 \leq y_1,$$

and, by assumption, there exists $e \in A$ satisfying

$$x_1 \leq x_2 \leq \ldots \leq e \leq \ldots \leq y_2 \leq y_1.$$

It is easy to check that

$$ea = ae = a \qquad \text{and} \qquad eb = be = 0.$$

(i) $\Longrightarrow$ (iii). If A is separably injective, then it has the countable Riesz separation property, from above, and it is countably unital by Proposition 2.5. Condition (iii) now follows from (ii) $\Longrightarrow$ (iii) for countably unital algebras. □

Remark. In general (iii) does not follow from (ii). It is easy to check that c_0 has the countable Riesz separation property, but no infinite dimensional separable C^*-algebra can be an SAW^*-algebra [14, Corollary 2]. All W^*-algebras satisfy (ii) and (iii) but only satisfy (i) if they are injective. Thus, the implications (iii) $\Longrightarrow$ (i) and (ii) $\Longrightarrow$ (i) fail, but (iii) $\Longrightarrow$ (ii) is an open conjecture.

Proposition 2.8. *Let J be an ideal in a C^*-algebra A.*
(1) *If A is countably unital, then A/J is countably unital.*
(2) *If J and A/J are countably unital, then A is countably unital.*

Proof. (1) Let $\rho : A \to A/J$ denote the quotient homomorphism. If C is a separable subalgebra of A/J, choose a separable subalgebra B of A for which $\rho(B) = C$. By hypothesis there exists a positive element $a \in A$ of unit norm which acts as the identity for B, and then $\rho(a)$ acts as the identity for C.

(2) Let B be a separable subalgebra of A which in turn is considered to be an ideal in A_1. By hypothesis there exists a positive element $x \in A$ of unit norm for which

$$\rho(x)\rho(b) = \rho(b)\rho(x) = \rho(b) \qquad (b \in B),$$

or, equivalently,

$$(1-x)b,\ b(1-x) \in J \qquad (b \in B).$$

Since J is countably unital, there exists a positive element $j \in J$ of unit norm such that

$$(1-j)(1-x)b = b(1-x)(1-j) = 0 \qquad (b \in B).$$

Thus

$$(1-x)(1-j)(1-x)b = b(1-x)(1-j)(1-x) = 0 \qquad (b \in B).$$

Write $(1-x)(1-j)(1-x) = 1-a$, where $a \in A$. Then $0 \le 1-a \le 1$ and so

$$\|a\| \le 1, \qquad a \ge 0.$$

In addition, a acts as the identity on B and so A is countably–unital. □

The following is a result which will allow us to concentrate on unital algebras in future work. Let $\mathcal{C}$ be a class of C^*–algebras closed under the formation of quotients, ideals and the adjunction of units. (We have in mind the class of subhomogeneous C^*–algebras.)

Theorem 2.9. *If, for unital members A of the class $\mathcal{C}$ the implication*

$$J \text{ and } A/J \text{ separably injective} \Longrightarrow A \text{ separably injective}$$

is valid, then it is also valid for nonunital members.

Proof. Let $A \in \mathcal{C}$, and let J be an ideal in A. Assume that J and A/J are separably injective. By Proposition 2.5 J and A/J are countably unital, and so by Proposition 2.8, A is countably unital. Regard J as an ideal in A_1, and identify A_1/J with $(A/J)_1$. By Proposition 2.6 $(A/J)_1$ is separably injective, and so both J and A_1/J are separably injective in $\mathcal{C}$. By hypothesis A_1 is separably injective, and so A is separably injective, by Proposition 2.6(2). □

We close this section with a result which, while not relating directly to subsequent sections, may be of interest.

Theorem 2.10. *Let A be separably injective. If E is a separable unital operator system and $\varphi : E \to A$ is completely positive with $\varphi(1) = a \geq 0$, then there exists a unital completely positive map $\psi : E \to A_1$ such that*

$$\varphi(x) = a^{1/2}\psi(x)a^{1/2} \qquad (x \in E).$$

Proof. Let A_1 be faithfully represented as a unital subalgebra of $B(H)$. By [5] there exists a completely positive unital map $\eta : E \to B(H)$ such that

$$\varphi(x) = a^{1/2}\eta(x)a^{1/2} \qquad (x \in E).$$

Let B be the unital separable subalgebra of A_1 generated by the range of φ, and let D be the separable subalgebra of $B(H)$ generated by B and the range of η. By Proposition 2.6(1) there is a completely contractive map $\vartheta : D \to A_1$ which is the identity when restricted to B. Since $\vartheta(1) = 1$, ϑ is completely positive. Define $\psi : E \to A_1$ by

$$\psi(x) = \vartheta\big(\eta(x)\big) \qquad (x \in E).$$

Since $\vartheta(a^{1/2}) = a^{1/2}$, by definition of ϑ, it follows from Lemma 2.1 that

$$\vartheta\big(a^{1/2}\eta(x)a^{1/2}\big) = a^{1/2}\vartheta\big(\eta(x)\big)a^{1/2} \qquad (x \in E).$$

Thus

$$\begin{aligned}\varphi(x) &= \vartheta\big(\varphi(x)\big)\\ &= \vartheta\big(a^{1/2}\eta(x)a^{1/2}\big)\\ &= a^{1/2}\vartheta\big(\eta(x)\big)a^{1/2}\\ &= a^{1/2}\psi(x)a^{1/2} \qquad\qquad (x \in E),\end{aligned}$$

and the theorem is proved. □

3. Homogeneous C^*-algebras. If A is a C^*-algebra, let $\hat{A}$ denote the primitive ideal space of A. If A is n-homogeneous, then $\hat{A}$ is a locally compact Hausdorff space, and Fell [9] has classified such C^*-algebras as algebras of cross-sections of bundles with base space $\hat{A}$ and fibre M_n. The object of this section is to characterize the separably injective n-homogeneous algebras as those with substonean primitive ideal spaces. We do not wish to assume that A has in identity, and this introduces the complication of considering noncompact spectra.

The following result was proved for unital algebras in [17]. Recall from [10] that a locally compact Hausdorff space is said to be substonean if disjoint co-zero sets have disjoint compact closures.

Proposition 3.1. *A commutative C^*-algebra $C_0(X)$ is separably injective if and only if X is substonean.*

Proof. ($\Longrightarrow$) Suppose that $C_0(X)$ is separably injective. To prove that X is substonean, it suffices, by [10], to show that if $0 \le f, g \le 1$ and $fg = 0$, then there exists $h \in C_0(X)$ such that

$$kf = f \qquad \text{and} \qquad kg = 0.$$

Let $U = \{x \in X\colon\ f(x) \neq 0\}$, and let h be the characteristic function of U. Let B be the subalgebra of $C_0(X)$ generated by f and g, and let C be the algebra generated by B and h in the algebra of bounded Borel functions on X. By hypothesis, the identity embedding of B in $C_0(X)$ has an extension to a completely contractive map $\varphi : C \to C_0(X)$. Let $k = \varphi(h)$. For any integer n

$$\|h - 2f^{1/n}\| \le 1$$

and so

$$\|k - 2f^{1/n}\| \le 1.$$

If $x \in U$, then

$$\left|k(x) - 2f(x)^{1/n}\right| \le 1$$

and this leads to $k(x) = 1$ by letting $n \to \infty$. Thus $kf = f$.

Since h and g have disjoint supports,

$$\|e^{i\vartheta} h + g^{1/n}\| \le 1$$

and so

$$\|e^{i\vartheta} k + g^{1/n}\| \le 1.$$

If $x \in X$ and $g(x) > 0$, then

$$\left|e^{i\vartheta} k(x) + g(x)^{1/n}\right| \le 1,$$

leading to

$$\left|e^{i\vartheta} k(x) + 1\right| \le 1$$

in the limit. Thus, $k(x) = 0$, and $kg = 0$.

($\Longleftarrow$) Suppose that X is substonean. Let $f \in C_0(X)$, and apply the Grove–Pedersen criterion [10] to $|f|$ and 0 to obtain $h \in C_0(X)$ such that $hf = f$. If necessary, h may be replaced by $\min\{|h|,1\}$ so that $0 \le h \le 1$. Thus, $C_0(X)$ is countably unital.

Let $\widetilde{X}$ be the one point compactification of X with extra point ω. $C_0(\widetilde{X})$ may be regarded as $C_0(X)$ with unit adjoined. If $f, g \ge 0$ in $C(\widetilde{X})$ and $fg = 0$, then one, say f, vanishes at ω, so that $f \in C_0(X)$. From above there exists $h \in C_0(X)$ such that $fh = f$, and the functions fh and gh lie in $C_0(X)$ with disjoint supports. Then there exists $k \in C_0(X)$ such that

$$f(hk) = fh = f \qquad \text{and} \qquad g(hk) = 0.$$

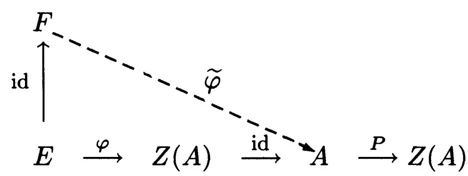
Thus $\widetilde{X}$ is substonean and compact, from which we conclude that $C(\widetilde{X})$ is separably injective [17]. By Proposition 2.6(2), $C_0(X)$ is separably injective. $\square$

Theorem 3.2. *If A is a separably injective n–homogeneous C^*–algebra, then $\hat{A}$ is substonean.*

Proof. The center $Z(A)$ may be identified with $C_0(\hat{A})$, so by Proposition 3.1 it suffices to show that $Z(A)$ is separably injective. Every n–homogeneous algebras has continuous trace [9] and moreover the ideal of continuous trace elements consists of the entire algebra. Thus there is a completely contractive projection $P : A \to Z(A)$ defined, for $\pi \in \hat{A}$, by

$$P(a)(\pi) = \tfrac{1}{n}\mathrm{Tr}\,\pi(a).$$

The diagram



shows that $Z(A)$ is separably injective. $\square$

We now turn to the converse of this theorem. The first step is to show that the general case follows from the case of a unital algebra.

Lemma 3.3. *If the implication*

$$\hat{A} \text{ substonean} \Longrightarrow A \text{ separably injective}$$

is true for unital n–homogeneous C^*–algebras, then it is also true in general.

Proof. Let A be an n–homogeneous C^*–algebra and suppose that $\hat{A}$ is substonean. Let $E \subseteq F$ be separable operator subspaces and let $\varphi : E \to A$ be completely contractive. The range of φ generates a separable subalgebra B of A. Choose a dense countable subset $\{b_i\}_{i=1}^{\infty}$ of B and set

$$U = \bigcup_{i=1}^{\infty}\{\pi \in \hat{A}\colon \mathrm{Tr}\,\pi(b_i^* b_i) > 0\}.$$

U is a co–zero set and so, since $\hat{A}$ is substonean, is substonean, $\overline{U}$ is compact. Thus there exist z_1, $z_2 \in Z(A)$ which are compactly supported and satisfy

$$0 \le z_1,\ z_2 \le 1,\ z_1 \equiv 1 \text{ on } \overline{U},\ z_1 z_2 = z_1 .$$

Notice that z_1 and z_2 act as the identity on B. Let J be the ideal defined by

$$J = \{j\colon jz_2 = 0\}$$

and observe that $(A/J)^{\wedge}$ is a compact subset of $\hat{A}$. Thus A/J is a unital n–homogeneous C^*–algebra with a substonean primitive ideal space. By hypothesis A/J is separably injective. Denote by $\rho : A \to A/J$ the quotient homomorphism.

The map $\rho\varphi : A \to A/J$ is completely contractive and so has a completely contractive extension $\psi : F \to A/J$. Let D be the separable subalgebra of A/J generated by the range of ψ and the identity. Then D is a separable type I algebra and is thus nuclear. The Choi–Effros lifting theorem [4] may be applied to obtain a completely contractive map $\vartheta : D \to A$ such that $\rho\vartheta$ is the identity on D.

Define $\widetilde{\varphi} : F \to A$ by

$$\widetilde{\varphi}(x) = z_1\vartheta\big(\psi(x)\big) \qquad (x \in F).$$

Then $\widetilde{\varphi}$ is completely contractive and it remains to be seen that $\widetilde{\varphi}$ extends φ. If $x \in E$, then $\varphi(x) \in B$ and $\rho\varphi(x) \in D$. Thus

$$\vartheta\big(\psi(x)\big) = \varphi(x) + j$$

where $j \in J$. It follows that

$$\begin{aligned}\widetilde{\varphi}(x) &= z_1\varphi(x) + z_1 j \\ &= z_1\varphi(x) + (z_1 z_2)j \\ &= z_1\varphi(x) + z_1(z_2 j) \\ &= \varphi(x)\end{aligned}$$

and so A is separably injective. □

Theorem 3.4. *Let A be an n–homogeneous C^*–algebra. Then A is separably injective if and only if $\hat{A}$ is substonean.*

Proof. In light of the previous results of this section it suffices to show that if A is a unital n–homogeneous algebra with compact substonean primitive ideal space X, then A is separably injective. By Lemma 2.2 we consider separable unital operator systems $L \subseteq K$ and a completely positive unital map $\varphi : L \to A$. It is possible to make a further simplification; we assume that L is of codimension 1 in K so that there exists a self–adjoint element $b \in K$ with $K = \operatorname{span}\{L, b\}$. If the extension can be accomplished in this case, then a countable repetition of the argument will settle the general case. All inequalities in $M_r(K)$ may be written

$$Y \le Ab \qquad (Y \in M_r(L), A \in M_r) \tag{3.1}$$

so it suffices to choose $f \in A$ such that

$$\varphi_r(Y) \le Af \qquad \text{whenever } Y \le Ab. \tag{3.2}$$

By [9] every n-homogeneous C^*-algebra with compact primitive ideal space X arises as the algebra of continuous cross sections of a locally trivial bundle B over X with fibre M_n and structure group $PU(n)$. For our purposes a more useful description may be obtained from [15]. Choose two coverings of X by compact sets $\{U_i\}_{i=1}^r$ and $\{W_i\}_{i=1}^r$ such that

(i) B is trivial over each U_i and W_i;

(ii) for each i, $U_i \subseteq$ interior (W_i);

(iii) for the bundle projection $\begin{array}{c} B \\ \downarrow \pi \\ X \end{array}$ there exist homeomorphisms

$$\varphi_i : W_i \times M_n \to \pi^{-1}(W_i);$$

(iv) there exist maps $\varphi_{ij} : W_i \cap W_j \to PU(n)$ such that for $x \in W_i \cap W_j$,

$$\varphi_i^{-1}\varphi_j(x,a) = \big(x, \varphi_{ij}(x)(a)\big);$$

(v) for $x \in U_i \cap U_j \cap U_k$,

$$\varphi_{ij}(x) = \varphi_{ik}(x)\varphi_{kj}(x);$$

(vi) the restrictions of φ_i to U_i and φ_{ij} to $U_i \cap U_j$ satisfy (i)–(v).

Here an element $\varphi_{ij}(x) \in PU(n)$ is thought of an defining an automorphism of M_n by conjugation.

Define a bundle F by

$$\begin{aligned} F &= \coprod_{i=1}^{r} W_i \times M_n \\ &= \big\{(i,x,a)\colon 1 \le i \le r,\, x \in W_i,\, a \in M_n\big\}. \end{aligned}$$

Now define an equivalence relation on F by setting $(i,x,a) \sim (j,x,b)$ if and only if $x \in W_i \cap W_j$ and

$$a = \varphi_{ij}(x)(b).$$

F is a trivial bundle over $\coprod_{i=1}^r W_i$ while $F/\sim$, the bundle obtained from the equivalence relation is isomorphic to B. Replacing W_i by U_i throughout this construction creates two other bundles E and $E/\sim$, and again $E/\sim$ is isomorphic to B. As circumstances dictate, it will be convenient to think of A as the algebra of continuous cross sections of $E/\sim$ or $F/\sim$. Let $\Gamma(D)$ denote the continuous cross sections of any bundle D.

Now A may be regarded as certain cross sections in $\Gamma(F)$ (or $\Gamma(E)$) which satisfy the relations (iv) and (v) on the overlaps of the covers. $\Gamma(F)$ is trivial of the form $C\left(\coprod_{i=1}^{r} W_i\right) \otimes M_n$, and since $\coprod_{i=1}^{r} W_i$ is substonean, $\Gamma(F)$ is separably injective. Thus there exists $f = (1,f_1),\dots,(r,f_r) \in \Gamma(F)$ satisfying the inequalities (3.2). This element need not lie in A, and the idea of the proof is to fit together the cross sections f_i on W_i to form a cross section of $E/\sim$ still satisfying the inequalities. For simplicity, we assume that $r = 3$, since this case contains the essence while keeping the details to a minimum. Define $g \in \Gamma(E/\sim)$ as follows:

$$g(1,x) = f(1,x) \qquad \text{on } \{1\} \times U_1 .$$

Choose ψ_1 and $\psi_2 \in C(X)^+$ such that

$$0 \leq \psi_i \leq 1 ,$$

$$\psi_1(x) = 1 \text{ for } x \in U_1 ,\ \text{supp}\ \psi_1 \subseteq W_1 ,$$

$$\psi_2(x) = 0 \text{ for } x \in U_1 ,\ \text{supp}\ \psi_2 \subseteq W_2 ,$$

$$\psi_1(x) + \psi_2(x) = 1 \text{ on } U_1 \cup U_2 .$$

Define

$$g(2,x) = \psi_1(x)\varphi_{21}(x)\big[f(1,x)\big] + \psi_2(x)f(2,x)$$

for $x \in U_2$. The element g has now been defined on U_1 and U_2 and satisfies (by construction) the correct relations on $U_1 \cap U_2$. It remains to define g on U_3. Choose $\psi_3 \in C(X)^+$ such that supp $\psi_3 \subseteq W_3$ and $\psi_1(x) + \psi_2(x) + \psi_3(x) = 1$ on $U_1 \cup U_2 \cup U_3$. Automatically, $\psi_3(x) = 0$ on $U_1 \cup U_2$. Set

$$g(3,x) = \psi_1(x)\varphi_{31}(x)f(1,x) + \psi_2(x)\varphi_{32}(x)f(2,x) + \psi_3(x)f(3,x)$$

for $x \in U_3$. This uses $f(1,x)$ on $U_1 \cap U_3$, $f(2,x)$ on $U_2 \cap U_3$, and $f(3,x)$ on the remainder of U_3. The relations (iv) and (v) guarantee consistency on the multiple overlap $U_1 \cap U_2 \cap U_2$, and that $g \in \Gamma(E/\sim)$.

Recall that $\varphi : L \to A \approx \Gamma(E/\sim)$ is completely positive and define $\widetilde{\varphi} : K \to \Gamma(E/\sim)$ by setting $\widetilde{\varphi}(b) = g$. It remains to be shown that g satisfies the inequalities (3.2). Let $Y \leq Ab$ be a typical inequality. Then, by construction,

$$\varphi(Y)(i,x) \leq Af(i,x)$$

for all $(i,x) \in \coprod_{i=1}^{3} U_r$. Now, for $x \in U_3$,

$$\begin{aligned} Ag(3,x) &= \psi_1(x)\varphi_{31}(x)\big(Af(1,x)\big) + \psi_2(x)\varphi_{32}(x)Af(2,x) + \psi_3(x)f(3,x) \\ &\geq \psi_1(x)\varphi_{31}(x)(\varphi Y)(1,x) + \psi_2(x)(\varphi Y)(2,x) + \psi_3(x)(\varphi Y)(3,x) \\ &= \big(\psi_1(x) + \psi_2(x) + \psi_3(x)\big)(\varphi Y)(3,x) \\ &= (\varphi Y)(3,x) \end{aligned}$$

and so the inequalities are satisfied on U_3. The verifications are similar on U_1 and U_2. Thus A is separably injective. □

This characterization may be used to construct nontrivial examples of separably injective homogeneous C^*-algebras.

Let B be any unital n-homogeneous C^*-algebra, and let $\mathbf{R}^+$ denote $[0,\infty)$. Set

$$A = C^b(\mathbf{R}^+, B)/C_0(\mathbf{R}^+, B)$$

(this is a generalization of the corona construction of [10]). As B, and hence A, is unital, it is not difficult to verify that A is n-homogeneous and that

$$Z(A) = C^b\big(\mathbf{R}^+, Z(B)\big)/C_0\big(\mathbf{R}^+, Z(B)\big).$$

By Theorem 3.4 and [17] it suffices to show that $Z(A)$ is separably injective, and so, replacing B by $Z(B)$, we may assume from the outset that a B is a commutative algebra $C(X)$. From the characterizations of substonean spaces given in [17] it is enough to verify the following condition: If

$$f_1 \le f_2 \le \ldots \le f_n \le \ldots \le g_n \le \ldots \le g_2 \le g_1 \qquad \text{in } C^b(\mathbf{R}^+, C(X)),$$

then there exists $h \in C^b\big(\mathbf{R}^+, C(X)\big)$ such that

$$\dot{f}_i \le \dot{h} \le \dot{g}_j \qquad \text{in } C^b(\mathbf{R}^+, C(X))/C_0(\mathbf{R}^+, C(X)).$$

Define $h : \mathbf{R}^+ \to C(X)$ on $[n-1, n]$ by

$$h(n-1+\lambda) = (1-\lambda)f_n(n-1+\lambda) + \lambda f_{n+1}(n-1+\lambda)$$

$(0 \le \lambda \le 1,\ n = 1,2,3,\ldots)$. Then

$$h \le g_j \text{ (on } \mathbf{R}^+\text{) and } h \ge f_i \text{ on } [i-1,\infty),$$

so that

$$\dot{f}_i \le \dot{h} \le \dot{g}_j.$$

Thus

$$C^b(\mathbf{R}^+, B)/C_0(\mathbf{R}^+, B)$$

is separably injective. □

We will denote this algebra by $\chi(B)$ (see [10]). Recall from [9] that a unital n-homogeneous C^*-algebra B is said to have a 1-dimensional projection field if there exists $p \in B$ such that $\pi(p)$ is a 1-dimensional projection for all $\pi \in \hat{B}$. Algebras of the form $C(X) \otimes M_n$ (called trivial) always admit 1-dimensional projection fields, and so the following result will guarantee the existence of nontrivial separably injective n-homogeneous C^*-algebras (see Section 4 for explicit constructions).

Proposition 3.5. *Let B be a unital n-homogeneous C^*-algebra. Then B has a 1-dimensional projection field if and only if $\chi(B)$ has a 1-dimensional projection field.*

Proof. Suppose that $p \in B$ is a 1–dimensional projection field, and define $f \in C^b(\mathbf{R}^+, B)$ to be the constant function with value p. It is easy to check that $\dot{f} \in \chi(B)$ is a 1–dimensional projection field.

Conversely suppose that $f \in C^b(\mathbf{R}^+, B)$, $0 \le f \le 1$, is such that $\dot{f} \in \chi(B)$ is a 1–dimensional projection field. Then $f^2 - f \in C_0(\mathbf{R}^+, B)$ and so

$$\lim_{t \to \infty} \left\| f(t)^2 - f(t) \right\| = 0.$$

Choose ε such that $0 < \varepsilon < \frac{1}{2}$, and choose a continuous function $g : [0,1] \to [0,1]$ which is identically 0 on $[0,\varepsilon]$ and identically 1 on $[1-\varepsilon, 1]$. Applying the functional calculus to g and f gives an element $p = g(f) \in C^b(\mathbf{R}^+, B)$ satisfying

(i) $\lim_{t\to\infty} \left\| f(t) - p(t) \right\| = 0$,

(ii) $p(t)^2 = p(t)$ for all $t \ge t_0$, where t_0 is a sufficiently large number.

Suppose that for no value of t is $p(t)$ a 1–dimensional projection field in B. Then for each integer $r \ge t_0$ there exists an irreducible representation $\pi_r \in \hat{B}$ such that $\pi_r\big(\rho(r)\big)$ is not 1–dimensional, and so

$$\operatorname{Tr} \pi_r\big(p(r)\big) \in \{0, 2, 3, \dots, n\}.$$

It follows from (i) that

$$\lim_{r \to \infty} \operatorname{Tr} \pi_r\big(f(r)\big) - \operatorname{Tr} \pi_r\big(p(r)\big) = 0,$$

and so there exists an integer $r_0 \ge t_0$ such that

$$\operatorname{Tr} \pi_r\big(f(r)\big) \in \{0, 2, 3, \dots, n\} + \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \qquad (r \ge r_0).$$

Let γ_r be the representation of $C^b(\mathbf{R}^+, B)$ defined, for $h \in C^b(\mathbf{R}^+, B)$, by

$$\gamma_r(h) = \pi_r\big(h(r)\big).$$

Then $\{\gamma_r\}_{r \ge r_0}$ contains a limit point $\gamma \in \widehat{\chi(B)}$. By continuity of the trace,

$$\operatorname{Tr} \gamma(f) \in \big\{0, 2, 3, \dots, n\big\} + \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$$

and so $\operatorname{Tr} \gamma(\dot{f}) = \operatorname{Tr} \gamma(f) \ne 1$, a contradiction. Thus, for r sufficiently large, $p(r)$ is a 1–dimensional projection field in B. □

4. Subhomogeneous C^*–algebras. A C^*–algebra is subhomogeneous if there is an integer n for which $\dim \pi \le n$ for all irreducible representations π of A. The smallest such integer n will be called the degree, and A will be said to be n–subhomogeneous. In this section we give sufficient conditions for A to be separably injective.

Recall that for each integer r, $\hat{A}_r$ denotes the subset of $\hat{A}$ consisting of r–dimensional irreducible representations. Each $\hat{A}_r$ is a Hausdorff space in the relative topology from $\hat{A}$ [13], and if A is n–subhomogeneous, then $\hat{A}_n$ is open.

Theorem 4.1. *Let A be a subhomogeneous C^*-algebra. If each $\hat{A}_r$ is substonean in its relative topology, then A is separably injective.*

Proof. The proof will be by induction on the degree n of A. Observe that the case $n = 1$ is the commutative situation, handled by Proposition 3.1. Now suppose that the result is true for all subhomogeneous algebras of degree at most n, and let A have degree $n+1$.

Let $J = \{a \in A\colon \pi(a) = 0 \text{ for } \dim\pi \le n\}$. Then J is homogeneous of degree $n+1$ while A/J is subhomogeneous of degree at most n. Since $(A/J)^\wedge = \bigcup_{r=1}^{n} \hat{A}_r$ and $\hat{J} = \hat{A}_{n+1}$ [13], the induction hypothesis and Theorem 3.4 imply that A/J and J are separably injective. We now wish to show that A is separably injective.

Consider two separable operator subspaces $E \subseteq F$ and a completely contractive map $\varphi : E \to A$. Let $\rho : A \to A/J$ denote the quotient map. Then $\rho\varphi : E \to A/J$ is completely contractive, and so has a completely contractive extension $\vartheta : F \to A/J$. Denote by B the separable subalgebra of A/J generated by the range of ϑ. Adjoin units to A and B, and consider the identity embedding $B_1 \to A_1/J$. Since B_1 is type I, it is nuclear, and so there exists a completely positive unital lifting $\eta : B_1 \to A_1$ by the Choi–Effros lifting theorem [4]. It is easy to check that η maps B into A, and so $\psi = \eta\vartheta : F \to A$ is completely contractive. Observe that if $x \in E$, then

$$\begin{aligned}\rho\varphi(x) - \rho\psi(x) &= \rho\varphi(x) - \rho\eta\vartheta(x)\\ &= \rho\varphi(x) - \rho\eta\rho\vartheta(x)\\ &= \rho\varphi(x) - \rho\varphi(x)\\ &= 0\end{aligned}$$

in A/J, and so

$$\varphi(x) - \psi(x) \in J \qquad (x \in E).$$

The C^*-algebra C generated by the elements $\{\varphi(x) - \psi(x)\colon x \in E\}$ is separable and so, as in the proof of Lemma 3.3, there exist $j_1, j_2 \in Z(J)$ such that $0 \le j_1$, $j_1 \le 1$ and j_1 acts as the identity on C and j_2 acts as the identity on $C^*(C, j_1)$. If J is an ideal in any C^*-algebra, then it is easy to prove that $Z(J) = Z(A) \cap J$, and so $j_1, j_2 \in Z(A)$.

The map $x \to j_2\varphi(x)$ is a complete contraction of E into J, and so has a completely contractive extension $\xi : F \to J$, since J is separably injective. Now define $\lambda : F \to A$ by

$$\lambda(x) = j_1\xi(x) + (1 - j_1)\psi(x) \qquad (x \in F).$$

It must now be verified that λ is completely contractive, and is an extension of φ. Since $j_1 \in Z(A)$, λ may be written

$$\lambda(x) = \left(j_1^{1/2}, (1-j_1)^{1/2}\right)\begin{pmatrix}\xi(x) & 0\\ 0 & \psi(x)\end{pmatrix}\begin{pmatrix}j_1^{1/2}\\ (1-j_1)^{1/2}\end{pmatrix}$$

and so is completely contractive. If $x \in E$, then

$$\begin{aligned}\lambda(x) - \varphi(x) &= j_1\xi(x) + (1-j_1)\psi(x) - \varphi(x)\\ &= j_1 j_2\varphi(x) + (1-j_1)\psi(x) - \varphi(x)\\ &= j_1\varphi(x) + (1-j_1)\psi(x) - \varphi(x)\\ &= (1-j_1)\big(\psi(x) - \varphi(x)\big) = 0\end{aligned}$$

since $\psi(x) - \varphi(x) \in C$ and j_1 acts as the identity on C. Thus λ extends φ and A is separably injective. This completes the induction step, and the result follows. □

As will be seen, the converse of this theorem is not true (Theorem 4.7). Let A be an n–homogeneous unital C^*–algebra with compact primitive ideal space $\hat{A}$. By compactness and the results of [9], there exists a finite collection $\{U_i\}_{i=1}^r$ of compact subsets covering $\hat{A}$ on each of which the restriction of A is isomorphic to $C(U_i) \otimes M_n$. Thus rn^2 elements $\{a_1, \dots, a_{rn^2}\}$ may be chosen from A such that, for every irreducible representation π, the set $\{\pi(a_1), \dots, \pi(a_{rn^2})\}$ spans M_n. We may now introduce an integer valued index $\ell(A)$, defined to be the minimal number of elements of A required to form a spanning set $\pi(A)$ for all irreducible representations of A. The index is invariant under isomorphism and so if $\ell(A) \neq \ell(B)$, then A and B cannot be isomorphic. However, the two nonisomorphic 2–homogeneous C^*–algebras over the sphere S^2, constructed in [11], both have index 4 by direct calculation. We omit the details since we only mention this fact in passing.

Proposition 4.2. *Let $\{A_r\}_{r=1}^\infty$ be a sequence of n–homogeneous unital C^*–algebras satisfying $\lim_{r\to\infty} \ell(A_r) = \infty$. Then A_∞, the ℓ_∞–direct sum $\bigoplus A_r$, is n–subhomogeneous but not n–homogeneous.*

Proof. If A_∞ were not n–subhomogeneous, then it would have an irreducible representation π of dimension at least $n+1$, and possibly infinite. By [16, Section 3] there exists a C^*–algebra B and a map $\lambda : B \to A_\infty$ which is n–positive but not $(n+1)$–positive. It would follow that the map fails to be $(n+1)$–positive in some factor A_r, and hence that A_r possesses an irreducible representation of dimension at least $n+1$ [16, Section 3]. This contradiction ensures that A_∞ is n–subhomogeneous.

Now if A_∞ were n–homogeneous, then it is simple to see that $\ell(A_r) \leq \ell(A_\infty)$ for each r. The sequence $\{\ell(A_r)\}_{r=1}^\infty$ would then be bounded, in contradiction of the hypothesis. □

We now describe examples of k–homogeneous unital C^*–algebras A for which $\ell(A)$ may be arbitrarily large. We need a technical result.

Lemma 4.3. *Let S_c^n be the unit sphere in $\mathbf{C}^{n+1}$, and let $g_1, \ldots, g_r : S_c^n \to \mathbf{C}$ be continuous functions satisfying*

(a) $g_i(-\xi) = -g_i(\xi)$ $(\xi \in S_c^n,\ 1 \leq i \leq r)$
(b) *for each $\xi \in S_c^n$, at least one of $g_1(\xi), \ldots, g_r(\xi)$ is nonzero.*

Then $r \geq \frac{2}{3^n} + 1$.

Proof. By compactness there exists $\delta > 0$ such that, for all $\xi \in S_c^n$,

$$\sup_{1 \leq i \leq r} |g_i(\xi)| \geq \delta .$$

Define three closed subsets of the unit circle

$$T_1 = \left\{e^{i\vartheta} : 0 \leq \vartheta \leq \tfrac{2\pi}{3}\right\},$$

$$T_2 = \left\{e^{i\vartheta} : \tfrac{2\pi}{3} \leq \vartheta \leq \tfrac{4\pi}{3}\right\},$$

$$T_3 = \left\{e^{i\vartheta} : \tfrac{4\pi}{3} \leq \vartheta \leq 2\pi\right\}.$$

Now define $3r$ closed subsets of S_c^n by

$$Y_{ij} = \left\{\xi \in S_c^n : |g_i(\xi)| \geq \delta \text{ and } \frac{g_i(\xi)}{|g_i(\xi)|} \in T_j\right\}$$

for $1 \leq i \leq r$, $1 \leq j \leq 3$. Since $g_i(-\xi) = -g_i(\xi)$, it follows that if $\xi,\ -\xi \in Y_{ij}$, then

$$\pm \frac{g_i(\xi)}{|g_i(\xi)|} \in T_j .$$

However, the T_j's have been constructed to preclude this possibility, and so no Y_{ij} contains a pair of antipodal points.

S_c^n is naturally identified with the real unit sphere S^{2n+1} in $\mathbf{R}^{2n+2}$. By the Lusternik–Schnirelmann Theorem [1, p. 205], it is necessary that $3r \geq 2n+3$, since otherwise at least one of the Y_{ij} would contain a pair of antipodal points. Thus $r \geq \frac{2}{3^n} + 1$. □

Example 4.4. Fix two integers n, k with $n \geq k+1$, $k \geq 2$, and fix an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbf{C}^n$. Let $X_{n,k}$ be the set of k–dimensional subspaces of $\mathbf{C}^n$ containing $\{e_1, \ldots, e_{k-1}\}$. We identify a subspace with its associated orthogonal projection in M_n, and then $X_{n,k}$ becomes a compact Hausdorff space in the relative norm topology. We identify S_c^{n-k} with the unit sphere of span $\{e_k, \ldots, e_n\}$, and define P_ξ, for each $\xi \in S_c^{n-k}$, to be the projection onto span $\{e_1, \ldots, e_{k-1}, \xi\} \in X_{n,k}$. The map $\xi \to P_\xi$ is clearly continuous on S_c^{n-k}.

For each $P \in X_{n,k}$ let $A(P)$ be the subalgebra PM_nP of M_n. Since P is k–dimensional, $A(P)$ is isomorphic to M_k. Now define

$$A_{n,k} = \left\{f : X_{n,k} \to M_n : f \text{ is continuous and } f(P) \in A(P)\right\}.$$

It is easy to check that $A_{n,k}$ is a unital k–homogeneous C^*–algebra whose primitive ideal space is $X_{n,k}$. Suppose that $\{f_1,\dots,f_r\} \in A_{n,k}$ spans M_k at every point in $X_{n,k}$.

For each $\xi \in S_c^{n-k}$, define $B_\xi \in M_n$ by

$$B_\xi \eta = \langle \eta, e_1 \rangle \xi \qquad (\eta \in \mathbf{C}^n).$$

Then $B_\xi \in A(P_\xi)$ and so there exists constants $\lambda_1,\dots,\lambda_r$ such that

$$\sum_{i=1}^r \lambda_i f_i(P_\xi) = B_\xi .$$

Consequently,

$$\sum_{i=1}^r \lambda_i \langle f_i(P_\xi)e_1,\xi\rangle = \langle B_\xi e_1,\xi\rangle = 1,$$

and so at least one of the numbers $\langle f_i(P_\xi)e_1,\xi\rangle$ is nonzero. Let

$$g_i(\xi) = \langle f_i(P_\xi)e_1,\xi\rangle .$$

Then $g_1,\dots,g_r$ are continuous complex valued functions on S_c^{n-k} satisfying the hypothesis of Lemma 4.3, since

$$g_i(-\xi) = \langle f_i(P_{-\xi})e_1, -\xi\rangle = -g_i(\xi).$$

Thus

$$r \geq \tfrac{2}{3}(n-k)+1,$$

and so $\ell(A_{n,k}) \geq \frac{2}{3}(n-k)+1$.

Now, for any fixed integer $k \geq 2$, we may let n increase to obtain k–homogeneous unital C^*–algebras of arbitrarily large index. □

We require the further refinement that our C^*–algebras should be separably injective. This is achieved by the next result.

Proposition 4.5. *Let A be a unital n–homogeneous C^*–algebra. Then $\ell(A) = \ell(\chi(A))$.*

Proof. For convenience, we write B for $C^b(\mathbf{R}^+,A)$, I for $C_0(\mathbf{R}^+,A)$ and $\rho : B \to B/I$ for the quotient map. We first show that $\ell(A) \geq \ell(\chi(A))$.

Let $r = \ell(A)$, and choose a spanning set $\{a_1,\dots,a_r\}$ from A. For each i let f_i be the constant function in B with value a_i. Consider an irreducible representation $\pi : B/I \to M_n$. Then $\pi\rho$ is an irreducible representation of B whose restriction to the constant functions in B induces an irreducible representation π_0 of A. Then

$$\begin{aligned}\operatorname{span}\{\pi(\rho(f_1)),\dots,\pi(\rho(f_r))\} &= \operatorname{span}\{\pi_0(a_1),\dots,\pi_0(a_r)\}\\ &= M_n\end{aligned}$$

and so $\{\rho(f_1),\dots,\rho(f_r)\}$ is a spanning set for B/I. Thus

$$\ell(\chi(A)) \le r = \ell(A).$$

Conversely, let $r = \ell(\chi(A))$, and choose a spanning set $\{\rho(f_1),\dots,\rho(f_r)\}$, $f_i \in B$. Let $E_1,\dots,E_k$ be an enumeration of the subsets of $\{f_1,\dots,f_r\}$ of length n^2, and define

$$S_i = \{\pi \in \hat{B} : \pi(E_i) \text{ fails to span } M_n\},\ 1 \le i \le k.$$

We wish to show that each S_i is closed, so fix i, and let $g_1,\dots,g_{n^2}$ be the elements of E_i. Let U be the unit sphere in $\ell_\infty(n^2)$ and let $\{\pi_\alpha\} \in S_i$ be a net converging to $\pi \in \hat{B}$.

Since $\pi_\alpha \in S_i$, there exists $\Lambda_\alpha = (\lambda_1^\alpha,\lambda_2^\alpha,\dots,\lambda_{n^2}^\alpha) \in U$ such that

$$\sum_{m=1}^{n^2} \lambda_m^\alpha \pi_\alpha(g_m) = 0.$$

Choose a convergent subnet (Λ_β) with limit $\Lambda = (\lambda_1,\lambda_2,\dots,\lambda_{n^2}) \in U$ and observe that

$$\sum_{m-1}^{n^2} \lambda_m \pi(g_m) = \lim_\beta \sum_{m=1}^{n^2} \lambda_n^\beta \pi_\beta(g_m) = 0.$$

Thus $\pi(E_i)$ fails to span M_n, and so $\pi \in S_i$.

Thus $S = \bigcap_{i=1}^k S_i$ is a compact subset of $\hat{B}$, and consists of those representations π for which $\{\pi(f_1),\dots,\pi(f_r)\}$ fails to span M_n.

If, for every integer p, the set $\{f_1(p),\dots,f_r(p)\}$ failed to be a spanning set for A, then there would be a sequence $\{\pi_p\} \in \hat{A}$ for which

$$\dim \operatorname{span}\{\pi_p(f_1(p)),\dots,\pi_p(f_r(p))\} \le n^2 - 1.$$

Let $\gamma_p : B \to M_n$ be the irreducible representation defined by

$$\gamma_p(f) = \pi_p(f(p)) \qquad (f \in B).$$

Then each $\gamma_p \in S$, and so any limit point of $\{\gamma_p\}_{p=1}^\infty$ is also in S. Clearly this set has a limit point in $(B/I)^\wedge$ and so $S \cap (B/I)^\wedge$ is nonempty. This would contradict the original choice of $\{f_1,\dots,f_r\}$, invalidating the assumption. Thus, for some sufficiently large integer p, $\{f_1(p),\dots,f_r(p)\}$ is a spanning set for A. We conclude that

$$\ell(A) \le r = \ell(\chi(A)),$$

establishing the reverse inequality. □

Remark 4.6. Recall that a k–homogeneous unital C^*–algebra A is said to be trivial if A is isomorphic to $C(\hat{A}) \otimes M_k$. The set $\{1 \otimes E_{ij}\}$ is a spanning set for $C(\hat{A}) \otimes M_k$, and so $\ell(C(\hat{A}) \otimes M_k) = k^2$. Thus, if $\ell(A) > k^2$, then A is nontrivial. Applying this to the algebras $A_{n,k}$ constructed in Example 4.4, we find that $A_{n,k}$ and $\chi(A_{n,k})$ are nontrivial provided that

$$n > \frac{3(k^2-1)}{2} + k. \qquad \square$$

It is now possible to disprove the converse of Theorem 4.1.

Theorem 4.7. *There exists a unital 2–subhomogeneous separably injective C^*–algebra A for which $\hat{A}$ is not substonean.*

Proof. From Example 4.4 there exists a sequence

$$\{B_n\}_{n=1}^{\infty}$$

of 2–homogeneous unital C^*–algebras for which $\ell(B_n) \geq n$. By Proposition 4.5, $\ell(\chi(B_n)) \geq n$.

Let A and A_0 be respectively the ℓ_∞– and c_0–directs sums of $\{\chi(B_n)\}_{n=1}^{\infty}$. By Proposition 4.2 A is 2–subhomogeneous but fails to be 2–homogeneous. Let I be $\bigcap_{\pi \in \hat{A}_1} \ker \pi$. Clearly $A_0 \subseteq I$, and $\hat{I} = \hat{A}_2$. If $\hat{A}_2$ were substonean, then I would be separably injective, by Theorem 3.4, and hence I would be countably unital, by Proposition 2.5.

Let 1_n be the identity in $\chi(B_n)$ and consider $a = (1, 2^{-1}1_2, 3^{-1}1_3, \ldots) \in A_0$. Then $a \in I$, so there exists $x \in I$ such that $xa = ax = a$. This would force x to be the identity, implying that $I = A$, which would in turn imply that A is 2–homogeneous. This contradiction means that $\hat{A}_2$ is not substonean. $\square$

The techniques of this section lead to a surprising result which runs counter to intuition. To the best of our knowledge it is new.

Corollary 4.8. *For each integer $k \geq 2$ there is a k–homogeneous C^*–algebra A whose multiplier algebra $M(A)$ is k–subhomogeneous but not k–homogeneous.*

Proof. Fix $k \geq 2$. By Example 4.4, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of k–homogeneous unital C^*–algebras for which $\ell(B_n) \geq n$. Let A be the c_0–direct sum of $\{B_n\}_{n=1}^{\infty}$. Then A is k–homogeneous, since each irreducible representation is an irreducible representation of one of the factors. $M(A)$ is the ℓ_∞–direct sum of $\{B_n\}_{n=1}^{\infty}$, which, by Proposition 4.2, is k–subhomogeneous but not k–homogeneous. $\square$

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