Dynamically Motivated Models for Multiplex Networks

Daryl DeFord

Dartmouth College
Department of Mathematics

Santa Fe Institute
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1 Joint work with Scott Pauls
Abstract

In this talk I will present a dynamically motivated model for a class of multiplex networks that provides natural extensions for many of the standard network tools to the multiplex setting, including centralities, diffusion, and clustering. I will also present some spectral results related to the Laplacian formulation of this model for diffusion and clustering applications on multiplex networks.
Outline

1. Introduction
   - Abstract
   - Outline
2. Motivation
   - Setup
   - World Trade Web
   - Social Networks
   - Goals
3. Methodology
   - Projections
   - Simplifications
4. Applications
   - Adjacency
   - Diffusion
   - Random Walks
5. Acknowledgements
In the terminology introduced in\(^2\) these are the diagonal, node–aligned multilayer networks.
Motivation

• In the terminology introduced in\textsuperscript{2} these are the diagonal, node–aligned multilayer networks.

• Disaggregated Data
  • A single set of objects of interest
  • Many different types of relations or connections
  • Intra–object interactions that are distinct from the inter–layer dynamics

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- Examples:
  - World Trade Web
  - Social Networks
  - Neural Networks
  - Many others ...

WTW Setup

- Nodes → Countries
- Edges → Trade Volume
- Disaggregation: Commodity Type

<table>
<thead>
<tr>
<th>Layer</th>
<th>Description</th>
<th>Volume</th>
<th>% Total</th>
<th>Transitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Food and live animals</td>
<td>29,155,437</td>
<td>5.1</td>
<td>.82</td>
</tr>
<tr>
<td>1</td>
<td>Beverages and tobacco</td>
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<td>Crude materials</td>
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<td>.79</td>
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<td>.62</td>
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<td>4</td>
<td>Animal and vegetable oils</td>
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<td>0.3</td>
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<tr>
<td>5</td>
<td>Chemicals</td>
<td>53,570,315</td>
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<td>.83</td>
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<tr>
<td>6</td>
<td>Manufactured Goods</td>
<td>79,058,2194</td>
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<td>.87</td>
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<tr>
<td>7</td>
<td>Machinery</td>
<td>23,878,28874</td>
<td>42.1</td>
<td>.85</td>
</tr>
<tr>
<td>8</td>
<td>Miscellaneous manufacturing</td>
<td>7,366,42890</td>
<td>13.0</td>
<td>.83</td>
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<tr>
<td>9</td>
<td>Other commodities</td>
<td>1,076,85024</td>
<td>1.9</td>
<td>.56</td>
</tr>
</tbody>
</table>

All Aggregate Trade: 56,673,805,93

Table: Commodity information for the 2000 WTW
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<td>5.1</td>
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<td>1</td>
<td>Beverages and tobacco</td>
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<td>0.9</td>
<td>.67</td>
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<td>3.3</td>
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<td>Animal and vegetable oils</td>
<td>14578671</td>
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<td>.56</td>
</tr>
<tr>
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<td>5667380593</td>
<td>100</td>
<td>.93</td>
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Table: Commodity information for the 2000 WTW
Figure: Aggregate 2000 World Trade Web

• Edge weights reflect the volume of trade flow
WTW Dynamics

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• Stability analysis\(^4\) can reveal sensitivity of the global network to various perturbations. This approach can be refined by considering:
  • Exchanges between the various industries within each country
  • Net trade surpluses and deficits
• A model that allows us to distinguish intra–country dynamics from international dynamics permits a more nuanced view of the data and hence a more complete analysis.

Social Networks

- Survey Data
  - Heterogeneous layers

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- Information dynamics$^5$
  - Diffusive
  - Transactional

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Social Layers

Figure: Krackhardt Hi-Tech Manager Relationships

Advice

Friendship

Report

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- Refined Aggregation
- Layer effects pass through copies. Copies don’t interact directly.
Notation

- \( n \) nodes
- \( k \) layers
- \( v \) a \( nk \times 1 \) vector of “quantities”
- \( v^i_j \) the quantity at node \( j \) on layer \( i \)
- \( i \) and \( \ell \) layer indices
- \( j \) node index
General Approach

Two–step iterative process

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  - Linear Case: Given a collection of dynamic operators $\{D_i\}$, one for each layer, form $D = \text{diag}(D_1, D_2, \ldots, D_n)$. 
- Step 2: Intra–node mixing: mix the effects of the $D_i$ as a scaled, convex combination of the resulting values at each copy of each node.
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\[
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$$
(v')^i_j = \alpha^i_j \sum_{\ell=1}^{k} c^{i,\ell}_j (Dv)^\ell_j
$$

- Return to Step 1
Step 2: Scaled Projections

- Orthogonal projections onto the “node subspaces”
- Gather and redistribute
- $c_{j}^{i,\ell}$ – “pass-through proportion”
- $\alpha_{j}^{i}$ – scaling coefficient
Matrix Representations

Step 2 can be expressed as a single block matrix acting on \( v \), with
\[
C^{i,\ell} = \text{diag}(\alpha_1^\ell c_1^{i,\ell}, \alpha_2^\ell c_2^{i,\ell}, \ldots, \alpha_n^\ell c_n^{i,\ell})
\]
\[
M = \begin{bmatrix}
C^{1,1} & C^{1,2} & \cdots & C^{1,k} \\
C^{2,1} & C^{2,2} & \cdots & C^{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
C^{k,1} & C^{k,2} & \cdots & C^{k,k}
\end{bmatrix}
\]

The final multiplex dynamic operator is a product of the layer dynamics matrix \( D \) and the redistribution matrix \( M \)
\[
\mathcal{D} = MD = \begin{bmatrix}
C^{1,1}D_1 & C^{1,2}D_2 & \cdots & C^{1,k}D_k \\
C^{2,1}D_1 & C^{2,2}D_2 & \cdots & C^{2,k}D_k \\
\vdots & \vdots & \ddots & \vdots \\
C^{k,1}D_1 & C^{k,2}D_2 & \cdots & C^{k,k}D_k
\end{bmatrix}
\]
The **unified node model** assumes that each node has a distinct set of weighted preferences between its copies.
Unified Node

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As an example, in the WTW network each country can evaluate each commodity’s importance as the total volume of trade at that country in the respective layer. Then the total trade flow at each country can be redistributed proportionally to these weighted degrees.
Hierarchical Layers

A further natural simplification occurs if we assume that the global network has an ordering of layers, so that the effect of layer $i$ on layer $\ell$ is fixed across all nodes. In this hierarchical layer model the blocks $C_{i,\ell}$ are just scalar multiples of $I_n$.

In the absence of application specific choices of $C_{i,\ell}$ the layer densities provide a natural hierarchy either by taking $C_{i,\ell}$ to be the density itself or the ratio of the density of layer $i$ to the density of layer $j$.

When the layer dynamics are the adjacency matrices, this simplification is the asymmetric influence matrix introduced in$^7$.

The simplest version of this operator, the **equidistribution model**, sets $c_{ij} = \frac{1}{k}$ for all $1 \leq j \leq n$ and $1 \leq i, \ell \leq k$. At every step, this operator simply averages the quantities at each node copy. This is a natural simplification for applications where the flow is equally likely to move between layers or represents the probabilities of a binary process.
Centrality Score Comparison

Figure: Monoplex Comparison
Centrality Score Comparison

Figure: Unified Node Comparison
Centrality Score Comparison

Figure: Hierarchical Layer Comparison
Centrality Score Comparison

Figure: Equidistribution Comparison
To extend the standard interpretation of the Laplacian operator to the multiplex setting, we allow the $c_{j}^{i,\ell}$ to represent the proportion of the effect on layer $\ell$ that passes to the $j$th node on layer $i$:

$$
\frac{dv_{j}^{i}}{dt} = K \sum_{\ell=1}^{k} c_{j}^{i,\ell} \sum_{n_{j}^{\ell} \sim n_{m}^{\ell}} (v_{j}^{\ell} - v_{m}^{\ell}).
$$

Here $K$ is the diffusion constant and the $c_{j}^{i,\ell}$ represent the proportion of the effect on layer $\ell$ that passes through to $n_{j}^{i}$. Linear algebraically, this is:

$$
\frac{dv_{j}^{i}}{dt} = K \left[ \sum_{\ell=1}^{k} c_{j}^{i,\ell} L^{\ell} v^{\ell} \right]_{j}
$$

(1)
The theory of Hermitian matrices, and in particular the Weyl bounds, allow us to bound the eigenvalues of this diffusion operator using the hierarchical layer or equidistribution models:

- **Fiedler Value:** \( \max_i (\lambda^i_f) \leq k \lambda_f \leq \lambda^m_f + \sum_{j \neq \ell} \lambda^j_1 \),
- **Leading Value:** \( \max_i (\lambda^i_1) \leq k \lambda_1 \leq \sum_i \lambda^i_1 \).

These bounds are special cases of the following more general but less computationally feasible bounds:

\[
\max_i (\lambda^i_{n-\ell}) \leq k \lambda_{n-\ell} \leq \min_{J \subseteq \{1, \ldots, n\} \cup \{-\ell, \ldots, n-\ell\}} \left( \min_{\sigma \in S_n} \left( \sum_{i=1}^{k} \lambda^{\sigma(i)}_{j_i} \right) \right),
\]
Preserved Properties

If we take the original layer dynamics to be the corresponding random walk matrices then many of the matrix properties are preserved in the equidistribution model:

- Stochastic
- Irreducible
- Primitive

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Additionally, we may project the random walk to the original node space to derive a $n \times n$ transition matrix. This is an example of the refined aggregation aspect of our model.

\textsuperscript{8}De Domenico et al.: *Navigability of interconnected networks under random failures*, PNAS (2014).

WTW Applications

- Commute Time Clustering
  - Distance proxy for gravity model of trade
WTW Applications

• Commute Time Clustering
  • Distance proxy for gravity model of trade

• Random Walk Betweenness Centrality
  • Aggregate: US/Canada
  • Individual Layers: Sources and sinks
  • Multiplex: Good measure of global flow
Conclusions

- Our operator represents a dynamically motivated approach to better understanding the properties of multiplex networks.
- Control of the intra-node mixing allows us to examine a continuum of results for each data set that reveals different aspects of the underlying data.
- This approach generalizes several standard methodologies from both monoplex and multiplex perspectives.
That’s all...

Thank You!
Small Example

Figure: A toy multiplex model
Eigenvector Centrality

<table>
<thead>
<tr>
<th>Node</th>
<th>Level 1</th>
<th>Level 2</th>
<th>Level 3</th>
<th>$\hat{D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>.7071</td>
<td>.6438</td>
</tr>
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<td>.4714</td>
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<td>.3922</td>
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<td>4</td>
<td>.5883</td>
<td>.5</td>
<td>.2357</td>
<td>.4636</td>
</tr>
</tbody>
</table>

Table: Eigenvector centrality scores for the toy multiplex network

In\textsuperscript{10} the authors use diffusion centrality as a proxy for their communication centrality. This approach also translates directly to this multiplex operator.

\textsuperscript{10}Banerjee et al.: The Diffusion of Microfinance, Science (2013)