COMBINATORIAL CHESSBOARD REARRANGEMENTS

DARYL DEFORD

ABSTRACT. Many problems concerning tilings of rectangular boards are of significant combinatorial interest. In this paper we introduce a similar type of counting problem based on game piece rearrangements. Many of these rearrangements satisfy recurrence relations which can be computed using various combinatorial techniques. We also present the solution to a rearrangement counterpart to the well-known knight’s tour problem.

1. Introduction

Tiling problems on rectangles form a very commonly studied set of problems in combinatorics. Tilings with dominoes [7, 10, 18] are very common, as are tilings with square tiles [3, 8]. The fantastic book by Benjamin and Quinn [2] contains a wide variety of tiling problems, mostly on $1 \times n$ rectangles, which are used to derive and prove identities concerning the Fibonacci, Lucas, and other well-known sequences. These tiling problems are often solved using the theory of recurrence relations and their equivalent generating functions [5].

In this paper we consider a generalization of these problems that contains a similar combinatorial structure. In particular, we will use homogeneous, linear recurrence relations and their corresponding generating functions and generalized power sums to provide solutions to these problems. In addition, we will describe a method based on matrix permanents that allows particular instances of these problems to be numerically computed, and demonstrate its efficacy by giving the solution to an $8 \times 8$ knight chessboard rearrangement problem.

1.1. Seating Rearrangements. Our counting problems are motivated by the seating rearrangement model introduced by Honsberger [9], and further studied by Cooper and Kennedy [11, 17]. These problems concern rectangular arrays of desks where each desk is occupied by a single student. The teacher directs the students to rearrange themselves, by each moving to an adjacent desk. In their second paper, Cooper, Kennedy, and Otake give a formula to calculate the number of these rearrangements on a general $2m \times n$ classroom [17].

In this paper we are interested in counting rearrangements of markers on rectangular chessboards. Given an $m \times n$ chessboard along with $mn$ indistinguishable markers, one on each square, a legitimate rearrangement satisfies the following rules:

- Each marker must make one permissible move.
- After all $mn$ markers have moved, each square must contain exactly 1 marker.

Date: October 3, 2012.
2010 Mathematics Subject Classification. Primary 05A19.
Key words and phrases. Tilings; Recurrence Relations; Knight Tours;
Thus, given a set of permissible moves for the markers, we want to count the number of legitimate rearrangements that exist, on a rectangular board, where the row dimension $m$, is fixed, and the column dimension $n$, varies. To solve specific cases, for fixed $m$ and $n$, we can use the theory of matrix permanents [13]. Kuperberg’s paper contains a survey of a variety of other combinatorial results that can be derived using matrix permanents [12]. Although matrix permanents are difficult to compute in general [1, 19], we can use them to provide base cases for our recurrence relations and generating functions.

1.2. Rearrangement Digraphs. For any given rearrangement problem, we can construct a digraph that represents each particular board. The digraph can be formed by adding a vertex for each square on the chessboard, and placing a directed edge between two vertices if a marker is permitted to move between the corresponding squares. It can easily be seen that by the rules given for a legitimate rearrangement, there exists a one–to–one correspondence between cycle covers on the digraph and legitimate rearrangements on the chessboard.

This correspondence is important, because the permanent of the adjacency matrix of a digraph counts the cycle covers of that digraph [6]. Thus, the number of legitimate rearrangements with a given set of movements on a fixed board can be calculated as a matrix permanent. Although this method cannot be used in general to give closed forms for our rearrangement problems, it does allow us to compute initial conditions in a systematic manner.

2. Counting 8 × 8 Knight Rearrangements

In this section we present an example of a fixed board rearrangement problem. A famous combinatorial problem is enumerating the knight’s tours of a standard chessboard, or counting the Hamiltonian cycles on a 8 × 8 knight graph. It is known that there are 26,534,728,821,064 of these cycles [14]. In our case we can consider the following related problem:

**Problem.** Given an 8 × 8 chessboard, with a knight on each square, how many ways can we rearrange the knights such that each knight makes exactly one move.
We can approach this problem by forming the digraph that corresponds to this rearrangement instance (Figure 1). Then, the solution to our problem is equal to the number of cycle covers on the digraph that we have constructed. Using matrix permanents we computed the number of cycle covers and thus knight rearrangements on an $8 \times 8$ board to be $8, 121, 130, 233, 753, 702, 400$.

3. Systems of Recurrences

As the markers are being rearranged, on a given $m \times n$ board, any particular square can be in one of four states:

1. (initial) the square contains a single marker that has not moved yet,
2. (intermediate) the square contains two markers,
3. (intermediate) the square contains no markers,
4. (completed) the square contains a single marker that has already moved.

Thus, for a particular square to be completed, the marker initially placed in that square must make a legitimate move, and a marker must move in to the original square. Note that some sets of movements will permit pieces to remain in place. Thus, the marker that completes a square may be the marker that was initially placed in that square. A legitimate rearrangement is then one in which every square is completed.

In order to construct linear recurrences for these problems we proceed by constructing systems of sub–recurrences that represent the number of rearrangements on boards where all of the squares in the first column has been completed. Generally, these sub–boards contain squares in all four states listed above. The successor operator method of DeTemple and Webb generates solutions to these systems by giving a linear recurrence that is satisfied by all of the sequences [4]. Thus, the sequence consisting of the number of rearrangements with a given set of moves, on a $m \times n$ board with $m$ fixed as $n$ varies, satisfies this recurrence. Then, given enough initial conditions, standard combinatorial methods can be used to construct a generating function or a generalized power sum that also generates the sequence.

4. Chesspiece Rearrangements

The game of chess provides a natural set of pieces to analyze. In this section we will compute rearrangements on rectangular boards using kings, queens, rooks, bishops, knights. Particularly, we will examine two cases for each board, the number of rearrangements when each piece must make exactly one legal move, and the number of rearrangements when each piece may either make a legal move or remain in place. In many of the examples the second case will lead to the more interesting results.

Some of the sequences that we have generated can be found in the OEIS in the context of array permutations contributed by R. H. Hardin [15]. For example, sequences A189145-A189150 give the number of $m \times n$ array permutations when each element makes either 0 or 1 knight moves, as $m$ ranges from 2 to 6 [16]. These particular array permutations correspond to the number of rearrangements on an appropriately sized board, where each marker can either remain in place or make a single knight move.
4.1. $1 \times n$ Rearrangements. In the $1 \times n$ case the rearrangements are very simple. When the pieces are required to make a legal move on a $1 \times n$ board there are no legitimate rearrangements of knights or bishops and exactly one possible king rearrangement if $n$ is even. In this case the number of rearrangements for queens and rooks is equal to the $n^{th}$ derangement number.

Allowing pieces to either make a single move or remain in place on a $1 \times n$ board gives our first connection to the Fibonacci numbers. The number of king rearrangements under these rules is equal to the $n^{th}$ combinatorial Fibonacci number, which can be seen by comparing these rearrangements with stays or swaps to tilings with squares and dominoes. A queen or rook under these rules can move from any square to any other square thus the number of rook or queen rearrangements in equal to $n!$.

4.2. $2 \times n$ Rearrangements. Adding an extra row to our boards gives more interesting solutions. Knight rearrangements where the pieces are not allowed to remain in place are still very limited, there is exactly 1 if $n$ is a multiple 4, otherwise there are none, while the number of bishop rearrangements is 1 for all even $n$ and 0 otherwise. For king rearrangements, computing the system of recurrences that count the number of ways to complete the first column leads to a third order recurrence:

$$a_n = 5a_{n-1} + 4a_{n-2} - 16a_{n-3}.$$  This is interesting, because if a similar method is employed to calculate the recurrence satisfied by king rearrangements when the pieces are allowed to remain in place another third order recurrence is obtained:

$$a_n = 6a_{n-1} + 12a_{n-2} - 16a_{n-3}.$$  Both of these sequences can be found in the OEIS [15].

Permitting the pieces to move or remain in place again provides more connections to the Fibonacci numbers. Both the number of bishop and knight rearrangements can be described in terms of Fibonacci numbers.

**Example 1.** The number of bishop rearrangements of a $2 \times n$ board where pieces are permitted to remain in place is equal to the square of the $n^{th}$ Fibonacci number.

**Proof.** Consider that since any bishop must remain on its original color the white and black pieces can be rearranged independently. Similar to the $1 \times n$ king rearrangements, a natural bijection arises between bishops remaining on their initial square or swapping places and tilings with squares and dominoes. Since each rearrangement can be constructed as a combination of two of these separate Fibonacci rearrangements, the total number must be $f_n^2$. □

**Example 2.** The number of knight rearrangements of a $2 \times 2n$ board where pieces are permitted to remain in place is equal to $f_n^4$ while there are $f_n^2f_{n-1}^2$ of these rearrangements on a $2 \times (2n - 1)$ board.

**Proof.** For each $n$, note that the squares may be divided into 4 disjoint classes determined by the knight movements. Since a knight initially placed on a square in one of these classes cannot move to a square in another class, we can consider the 4 sets of knights independently. Again, by considering a bijection between rearrangements within a class and Fibonacci tilings, we see that the number of knight rearrangements on a board of dimensions $2 \times 2n$ is equal to $f_n^4$, while a board of size $2 \times (2n - 1)$ has $f_n^2f_{n-1}^2$ knight rearrangements. □

Rearrangement problems concerning queens and rooks lead to more complicated expressions than the other pieces. For example, the following expression gives the
number of rook rearrangements on a $2 \times n$ board when the pieces are permitted to remain in place:

$$\sum_{i=0}^{n} \binom{n}{i}^2 ((n-i)!)^2$$

To obtain this expression, we note that a rook may either move to any available square in its original row, or remain in its column and switch rows. Conditioning on the number of rooks that switch rows gives the result. This expression, and the other similar formulas that we have derived for rook and queen rearrangement problems cannot be easily expressed as a recurrence relation. Thus, we are more inclined to study rearrangement problems about kings, bishops, and knights which we will prove in Section 5 always satisfy linear homogeneous constant coefficient recurrence relations.

4.3. $3 \times n$ Rearrangements. In order to demonstrate our method more fully, in this section we present two complete examples, counting $3 \times 2n$ knight rearrangements where each marker must make a knight move, and $3 \times n$ bishop rearrangements where the markers are allowed to remain in place.

**Example 3.** $3 \times 2n$ knight rearrangements where pieces are not permitted to remain in place.

Each knight move on a chessboard takes the marker to a square of opposite color. Thus, there can be no rearrangements on a board of dimensions $3 \times (2k+1)$ since there will not be an equal number of squares of each color and the pigeonhole principle shows that no rearrangements can exist. This further implies that we can consider only the movements of the markers initially placed on white squares because the total number of rearrangements is the square of this number.

We begin by letting $a_n$ represent the number of rearrangements of the white knights, and displaying the sub–rectangles that can be formed by completing every square in the first column. Figure 2(a) displays all of the endings that we must consider. Note that the black dots represent knight that have not moved, while the white dots represent knights that have already moved. For each ending, our goal is to express the number of rearrangements in terms of other endings with fewer columns. For example given an $a_n$ board, the initial 2 columns may be completed in the following fashion:

- The white knights in the first two columns may cover both black squares in the first column and one black square in the third column. The remaining board is equivalent to $b_{n-1}$ and may be obtained in two ways depending on which square in the third column is filled.
- The white knights in the first two columns may cover one black square in each of the first, third, and fourth columns, with the squares in the first and third columns being in separate rows. The remaining board is then equivalent to $c_{n-1}$ and again this positioning can happen in two ways.
- The white knights in the first two columns may cover one black square in the first column and both black squares in the third column. The remaining board is equivalent to $d_{n-1}$ and may be obtained in two ways depending on which square in the first column is filled.
- Finally, The white knights in the first two columns may cover one black square in each of the first, third, and fourth columns, with the squares in
the first and third columns being in the same row. The remaining board is \( e_{n-1} \) and again can occur in two ways depending on the row of black squares that is covered.

Thus, since these cases cover all legitimate options, we have that \( a_{n+1} = 2b_n + 2c_n + 2d_n + 2e_n \). For each of these endings we perform the same procedure until we have a complete system of recursively defined relations. Taken together these recurrences form the following system:

\[
\begin{align*}
a_n &= 2b_{n-1} + 2c_{n-1} + 2d_{n-1} + 2e_{n-1} \\
b_n &= 2b_{n-1} + 2c_{n-1} + 2d_{n-1} + 2e_{n-1} \\
c_n &= a_{n-1} + f_{n-1} + 2g_{n-1} + h_{n-1} \\
d_n &= 2b_{n-1} + 2i_{n-1} \\
e_n &= a_{n-1} + f_{n-1} + 2g_{n-1} + h_{n-1} \\
f_n &= b_{n-1} + d_{n-1} + 2c_{n-1} + i_{n-1} \\
g_n &= a_{n-1} + 2g_{n-1} + 2h_{n-1} \\
h_n &= b_{n-1} + c_{n-1} + d_{n-1} + e_{n-1} + i_{n-1} \\
i_n &= 2f_{n-1} + 2h_{n-1}
\end{align*}
\]

Simplifying this system, for example noticing that \( a_n = b_n, \ c_n = e_n, \) and \( f_n = h_n, \) and applying the successor operator to this system gives the following matrix whose determinant is the characteristic polynomial of the recurrence we are seeking.

\[
M = \begin{bmatrix}
E - 2 & -4 & -2 & 0 & 0 \\
-1 & E & 0 & -2 & -2 \\
-2 & 0 & E^2 & -8 & 0 \\
-E & -2 & -E & E^2 - 4 & 0 \\
-1 & 0 & 0 & -2 & E - 2
\end{bmatrix}
\]

\[det(M) = E^7 - 4E^6 - 12E^5 + 16E^4 + 32E^3 - 32E^2\]

Thus, the number of white knight rearrangements on a \( 3 \times 2n \) chessboard satisfies the recurrence relation and initial conditions:

\[
a_n = 4a_{n-1} + 12a_{n-2}16a_{n-3} - 32a_{n-4} + 32a_{n-5}
\]

Table 1. Initial conditions for \( 3 \times 2n \) knight rearrangements

<table>
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<tr>
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<th>3</th>
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As noted above, the total number of rearrangements, for each \( n \), are the squares of these values.
Figure 2. 3 × n Rearrangement Endings
Example 4. $3 \times n$ bishop rearrangements where pieces are permitted to remain in place.

We begin by letting $a_n$ represent the number of rearrangements of the white bishops, and $b_n$ represent the rearrangements of the black bishops. When $n$ is even, we have that $a_n = b_n$ since the board is symmetric. When $n$ is odd, $a_n \neq b_n$ since there are more black squares than white squares. Note that we can consider $a_n$ and $b_n$ separately, since the black and white bishops cannot interfere with each other. Thus, the total number of rearrangements for any given $n$ is $a_n b_n$.

Figure 2(b) shows the endings that are necessary to count the $3 \times n$ bishop rearrangements. Note that the dots represent bishops that have not moved, while the crosses represent bishops that have already moved. In order to compute the recurrence equations we consider all possible ways to complete the first column of each ending, and express these sequences of partial rearrangements as recurrences themselves.

For example, the first column of a board beginning $a_n$ can be completed in the following ways:

- The bishop in the first column can remain in place, leaving $b_{n-1}$.
- The bishop in the first column can swap places with either one of the two bishops in the second column, giving a $c_{n-1}$.
- The bishop in the first column may move to one of the white squares in the second column, while the empty square in the first column is filled by the bishop from the other square in the first column. Regardless of the row selected by the first bishop, the remaining pieces form a $d_{n-1}$.

Hence, $a_n = b_{n-1} + 2c_{n-1} + 2d_{n-1}$. Similarly, the first bishop of a board beginning $e_n$ has two options:

- It can remain in place, creating a $c_{n-1}$.
- It can swap places with the available bishop in the second column, leaving an $a_{n-2}$.

Thus, $e_n = c_{n-1} + a_{n-2}$.

Extending this method to all necessary endings we obtain the following system of recurrences. Note that because they are defined in terms of each other, both $a_n$ and $b_n$ satisfy the recurrence relation whose characteristic polynomial is the determinant of the successor matrix. Thus, we need to compute only one recurrence with two sets of initial conditions. The product of these two sequences, $t_n = a_n b_n$, which is the total number of rearrangements we wish to count, satisfies a separate recurrence.

With the endings shown in Figure 3 we get the following recurrences:

\[
\begin{align*}
  a_n &= b_{n-1} + 2c_{n-1} + 2d_{n-1} \\
  b_n &= a_{n-1} + 5a_{n-3} + 2b_{n-2} + 6c_{n-2} + 6d_{n-2} + 2e_{n-1} \\
  c_n &= a_{n-1} + b_{n-2} + 2c_{n-2} + 2d_{n-2} + e_{n-1} \\
  d_n &= f_n + g_n \\
  e_n &= a_{n-2} + c_{n-1} \\
  f_n &= b_{n-2} + c_{n-2} + d_{n-2} \\
  g_n &= a_{n-3} + c_{n-2} + 2d_{n-2}
\end{align*}
\]
Applying the successor operator to this system gives the following matrix whose determinant is the characteristic polynomial of the recurrence we are seeking:

\[
M = \begin{bmatrix}
E & -1 & -2 & -2 & 0 & 0 & 0 \\
-E^2 - 5 & E^3 - 2E & -6E & -6E & -2E^2 & 0 & 0 \\
-E & -1 & E^2 - 2 & -2 & -E & 0 & 0 \\
0 & 0 & 0 & E & 0 & -E & -E \\
-1 & 0 & -E & 0 & E^2 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & E^2 & 0 \\
-1 & 0 & -E & -2E & 0 & 0 & E^3
\end{bmatrix}
\]

\[
det(M) = E^{14} - 11E^{12} - 8E^{10} + 19E^{8} - E^6
\]

Thus, the number of bishop rearrangements on a $3 \times n$ chessboard can be computed with the recurrence relation given below and the separate initial conditions for $a_n$ and $b_n$:

\[
a_n = 11a_{n-2} + 8a_{n-4} - 19a_{n-6} + a_{n-8}
\]

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4.4. Larger Boards. As the fixed number of rows on our boards grows, the problems grow increasingly complex. For many of these sequences we are able to construct homogeneous, constant coefficient, linear recurrences, generating functions, and generalized power sums. However, in general, problems containing queens and rooks do not satisfy these types expressions. We will prove a theorem in Section 5 that explains this phenomena.

Another interesting question concerns the magnitude of the order of the recurrence satisfied by each problem. King rearrangement problems satisfy lower order recurrences than any of the other chess pieces we considered. Even so, for King $2 \times n$ rearrangements we have a three term recurrence, for King $3 \times n$ we have a 10 term recurrence, which grows to 27 terms for $4 \times n$ and to 53 terms for the $5 \times n$ case. A similar growth rate occurs for the knight rearrangements which satisfy an eight term recurrence for the $2 \times n$ case and a 27 term recurrence for the $3 \times n$ case. These patterns of recurrences can also be found in the OEIS.

5. General Movements

Obviously, there are many other sets of permissible marker movements that can be considered. Our final theorem gives a sufficient condition for a set of game piece movements to satisfy a linear homogeneous constant-coefficient recurrence relation. This argument is inspired by the one given by Webb, Cridge, and DeTemple, providing a similar result for tilings with arbitrary tile sets [21]. Webb also gives
a similar result on counting matrices of a given order that avoid certain forbidden sub–matrices [20].

**Theorem 1.** On any rectangular \( m \times n \) board \( B \) with \( m \) fixed, and a marker on each square, where the set of permissible marker movements has a maximum horizontal displacement, the number of rearrangements of the markers on \( B \) satisfies a linear, homogeneous, constant–coefficient recurrence relation as \( n \) varies.

**Proof.** Let \( d \) represent the maximum permissible horizontal displacement. Consider any set of marker movements that completes every square in the first column. After all of the markers in the first column have been moved, and other markers have been moved into the first column to fill the remaining empty squares, any square in the initial \( m \times d \) sub–rectangle may be in one of four states defined in Section 3. Let \( S \) be the collection of all \( 4^{md} \) possible states of the initial \( m \times d \) sub–rectangle, and let \( S^* \) represent the corresponding sequences counting the number of rearrangements of a board of length \( n \) beginning with each state as \( n \) varies. Finally, let \( a_n \) denote the sequence that describes the number of rearrangements on the original \( B \) as \( n \) varies.

For any board beginning with an element of \( S \), consider all of possible sets of movements that “complete” the initial column. The resulting board must be in a state also in \( S \), and has length \( n - k \) for some \( k \) in \([1, d]\). Hence, the corresponding sequence can be expressed as a sum of elements in \( S^* \) with subscripts bounded below by \( n - d \). This system of recurrences can be expressed as a linear, homogeneous, constant–coefficient recurrence relation in \( a_n \) either through the Cayley–Hamilton Theorem or by a successor operator matrix.

Note that any board that begins with a state in \( S \) satisfies the same recurrence. Thus, this result holds for any board whose ending is in \( S \).

This result gives an intuitive explanation for why queen and rook rearrangements do not satisfy simple recurrence relations. However, in general the converse of our theorem is not true. Consider rearrangements on a \( 1 \times n \) board where a marker in column \( c \) can either move to column \( n - c + 1 \) or remain in place. The number of legitimate rearrangements then satisfies the recurrence \( a_n = 2a_{n-2} \), but there is no upper bound on the distance that a marker can move.

Similarly, this condition cannot be a necessary condition, since it is possible to construct sets of movements that admit no legitimate rearrangements and thus trivially satisfy all such recurrence relations. For example, a set of movements where each marker in column \( c \) must move to a square in column \( c + 1 \).

**Acknowledgments**

I am very thankful to Dr. William Webb for his support and guidance. This work was supported by a grant from the Washington State University College of Sciences.

**References**


DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY
E-mail address: daryl.deford@email.wsu.edu