PULSATED FIBONACCI RECURRENCES

KRASSIMIR T. ATANASSOV, DARYL R. DEFORD, AND ANTHONY G. SHANNON

ABSTRACT. In this note we define a new type of pulsated Fibonacci sequence. Properties are developed with a successor operator. Some examples are given.

1. Introduction

The motivation for this work goes back to some research of Hall [9], Neumann [14], and Stein [19] on finite models of identities. In order to answer the question of whether every member of a variety is a quasi–group given that every finite member is, Stein [18] found it necessary to examine the intersection of Fibonacci sequences.

Subba Rao [20, 21], Horadam [10], and Shannon [17] investigated the intersection of Fibonacci and Lucas sequences and their generalizations with asymptotic proofs, while Péter Kiss adopted a different approach and supplied many relevant historical references [11]. Atanassov developed coupled recursive sequence which had some obvious intersections [1, 5]. Not considered her are various sequences, such as diatomic sequences, which by their very definitions intersect with many other sequences [14].

In this paper, following previous research (see [2, 3, 4]), a new type of pulsated Fibonacci sequence is developed: ‘pulsated’ because, in a sense, these sequences expand and contract with regular movements.

2. Definitions

Let \( a, b, \) and \( c \) be three fixed real numbers. Let us construct the following two recurrent sequences, \( \{\alpha_n\} \) and \( \{\beta_n\} \) with initial conditions:

\[
\alpha_0 = \beta_0 = a,
\]
\[
\alpha_1 = 2b,
\]
\[
\beta_1 = 2c,
\]

satisfying the combined recurrence relations:

\[
\alpha_{2k} = \beta_{2k} = \alpha_{2k-2} + \frac{\alpha_{2k-1} + \beta_{2k-1}}{2},
\]
\[
\alpha_{2k+1} = \alpha_{2k} + \beta_{2k-1},
\]
\[
\beta_{2k+1} = \beta_{2k} + \alpha_{2k-1},
\]

for every natural number \( k \geq 1 \). This pair of sequences we call a \( (a; 2b; 2c) \)-Pulsated Fibonacci sequence. The first values of the sequence are given in the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \alpha_k )</th>
<th>( \beta_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a )</td>
<td>( a )</td>
</tr>
<tr>
<td>1</td>
<td>( 2b )</td>
<td>( 2c )</td>
</tr>
<tr>
<td>2</td>
<td>( a + 2b )</td>
<td>( a + 2c )</td>
</tr>
<tr>
<td>3</td>
<td>( a + 4b + c )</td>
<td>( a + 4c + b )</td>
</tr>
<tr>
<td>4</td>
<td>( a + 6b + 3c )</td>
<td>( a + 6c + 3b )</td>
</tr>
</tbody>
</table>
Table 1. Initial values for the \((a; 2b; 2c)\)-Pulsated Fibonacci sequence.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\alpha_{2k+1})</th>
<th>(\alpha_{2k} = \beta_{2k})</th>
<th>(\beta_{2k+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-)</td>
<td>(a)</td>
<td>(-)</td>
</tr>
<tr>
<td>1</td>
<td>(2b)</td>
<td>(-)</td>
<td>(2c)</td>
</tr>
<tr>
<td>2</td>
<td>(-)</td>
<td>(a + b + c)</td>
<td>(-)</td>
</tr>
<tr>
<td>3</td>
<td>(a + b + 3c)</td>
<td>(-)</td>
<td>(a + 3b + c)</td>
</tr>
<tr>
<td>4</td>
<td>(-)</td>
<td>(2a + 3b + 3c)</td>
<td>(-)</td>
</tr>
<tr>
<td>5</td>
<td>(3a + 6b + 4c)</td>
<td>(-)</td>
<td>(3a + 4b + 6c)</td>
</tr>
<tr>
<td>6</td>
<td>(-)</td>
<td>(5a + 8b + 8c)</td>
<td>(-)</td>
</tr>
<tr>
<td>7</td>
<td>(8a + 12b + 14c)</td>
<td>(-)</td>
<td>(8a + 14b + 12c)</td>
</tr>
<tr>
<td>8</td>
<td>(-)</td>
<td>(13a + 21b + 21c)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

Theorem 2.1. For every natural number \(k \geq 1\), with the elements of the Fibonacci sequence denoted \(\{F_n\}\),

\[
\alpha_{2k} = \beta_{2k} = F_{2k-1}a + F_{2k}b + F_{2k}c, \tag{2.7}
\]

\[
\alpha_{4k-1} = F_{4k-2}a + (F_{4k-1} - 1)b + (F_{4k-1} + 1)c, \tag{2.8}
\]

\[
\beta_{4k-1} = F_{4k-2}a + (F_{4k-1} + 1)b + (F_{4k-1} - 1)c, \tag{2.9}
\]

\[
\alpha_{4k+1} = F_{4k}a + (F_{4k+1} + 1)b + (F_{4k+1} - 1)c, \tag{2.10}
\]

\[
\beta_{4k+1} = F_{4k}a + (F_{4k+1} - 1)b + (F_{4k+1} + 1)c. \tag{2.11}
\]

Proof. We proceed by mathematical induction. Obviously, for \(k = 1\) the assertion is valid. Let us assume that for some natural number \(k \geq 1\), (2.7)–(2.11) hold. For the natural number \(k + 1\), first, we check that

\[
\alpha_{4k+2} = \beta_{4k+2} \tag{2.12}
\]

\[
= \frac{\alpha_{4k} + \alpha_{4k+1} + \beta_{4k+1}}{2} \tag{2.13}
\]

\[
= F_{4k-1}a + F_{4k}b + F_{4k}c + \frac{F_{4k}a + F_{4k}b + F_{4k+1}c + F_{4k+1}b + F_{4k+1}c}{2} \tag{2.14}
\]

\[
= F_{4k-1}a + F_{4k}b + F_{4k}c + F_{4k+1}a + F_{4k+1}b + F_{4k+1}c. \tag{2.15}
\]

Secondly, we check that

\[
\alpha_{4k+1} = \alpha_{4k+2} + \beta_{4k+1} \tag{2.16}
\]

\[
= F_{4k+1}a + F_{4k+2}b + F_{4k+2}c + F_{4k+1}b + (F_{4k+1} + 1)c \tag{2.17}
\]

\[
= F_{4k+3}a + (F_{4k+3} - 1)b + (F_{4k+3} + 1)c. \tag{2.18}
\]

All of the other equalities are checked analogously. \(\square\)

For example, when \(c = -b\), the Pulsated Fibonacci sequence has the form shown in Table 2, while when \(c = b\) we obtain Table 3.
Table 2. Initial values for the \((a; 2b; -2b)\)-Pulsated Fibonacci sequence.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\alpha_{2k+1})</th>
<th>(\alpha_{2k} = \beta_{2k})</th>
<th>(\beta_{2k+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a)</td>
<td>(a)</td>
<td>(-2b)</td>
</tr>
<tr>
<td>1</td>
<td>(2b)</td>
<td>(-)</td>
<td>(-2b)</td>
</tr>
<tr>
<td>2</td>
<td>(-)</td>
<td>(a)</td>
<td>(-)</td>
</tr>
<tr>
<td>3</td>
<td>(a - 2b)</td>
<td>(-)</td>
<td>(a + 2b)</td>
</tr>
<tr>
<td>4</td>
<td>(-)</td>
<td>(2a)</td>
<td>(-)</td>
</tr>
<tr>
<td>5</td>
<td>(3a + 2b)</td>
<td>(-)</td>
<td>(3a - 2b)</td>
</tr>
<tr>
<td>6</td>
<td>(-)</td>
<td>(5a)</td>
<td>(-)</td>
</tr>
<tr>
<td>7</td>
<td>(8a - 2b)</td>
<td>(-)</td>
<td>(8a + 2b)</td>
</tr>
<tr>
<td>8</td>
<td>(-)</td>
<td>(13a)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

Table 3. Initial values for the \((a; 2b; 2b)\)-Pulsated Fibonacci sequence.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\alpha_{2k+1})</th>
<th>(\alpha_{2k} = \beta_{2k})</th>
<th>(\beta_{2k+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a)</td>
<td>(a)</td>
<td>(-)</td>
</tr>
<tr>
<td>1</td>
<td>(2b)</td>
<td>(-)</td>
<td>(2b)</td>
</tr>
<tr>
<td>2</td>
<td>(-)</td>
<td>(a + 2b)</td>
<td>(-)</td>
</tr>
<tr>
<td>3</td>
<td>(a + 4b)</td>
<td>(-)</td>
<td>(a + 4b)</td>
</tr>
<tr>
<td>4</td>
<td>(-)</td>
<td>(2a + 6b)</td>
<td>(-)</td>
</tr>
<tr>
<td>5</td>
<td>(3a + 10b)</td>
<td>(-)</td>
<td>(3a + 10b)</td>
</tr>
<tr>
<td>6</td>
<td>(-)</td>
<td>(5a + 16b)</td>
<td>(-)</td>
</tr>
<tr>
<td>7</td>
<td>(8a + 26b)</td>
<td>(-)</td>
<td>(8a + 26b)</td>
</tr>
<tr>
<td>8</td>
<td>(-)</td>
<td>(13a + 42b)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

Where the coefficients can be easily derived from the result of Theorem 1 by substitution.

3. Discussion

We note that the recursive definitions of \(\alpha\) and \(\beta\) may be rewritten in the following form:

\[
\alpha_k = \begin{cases} 
\alpha_{k-2} + \frac{\alpha_{k-1} + \beta_{k-1}}{2} & k \equiv 0 \pmod{2} \\
\alpha_{k-1} + \beta_{k-2} & k \equiv 1 \pmod{2}
\end{cases} \quad (3.1)
\]

and

\[
\beta_k = \begin{cases} 
\alpha_{k-2} + \frac{\alpha_{k-1} + \beta_{k-1}}{2} & k \equiv 0 \pmod{2} \\
\beta_{k-1} + \alpha_{k-2} & k \equiv 1 \pmod{2}
\end{cases} \quad (3.2)
\]

This interpretation permits the statement of this problem in terms of the successor operator method introduced by DeTemple and Webb in [7]. Thus, we may define helper sequences

\[
w_n = \alpha_{2n}, \quad (3.3)
\]
\[
x_n = \alpha_{2n+1}, \quad (3.4)
\]
\[
y_n = \beta_{2n}, \quad (3.5)
\]
\[
z_n = \beta_{2n+1}. \quad (3.6)
\]

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This allows us to rewrite (3.1) and (3.2) as
\[ w_n = y_n = w_{n-1} + \frac{1}{2}x_{n-1} + \frac{1}{2}z_{n-1}, \] (3.7)
\[ x_n = w_n + z_{n-1}, \] (3.8)
\[ z_n = y_n + x_{n-1}. \] (3.9)

Which in terms of the successor operator \( E \) gives the following linear system of sequences:
\[
\begin{bmatrix}
E - 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\
-E & E & 0 & -1 \\
-1 & -\frac{1}{2} & E & -\frac{1}{2} \\
0 & -1 & -E & E
\end{bmatrix}
\begin{bmatrix}
w_n \\
x_n \\
y_n \\
z_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\] (3.10)

Thus, the determinant of this system gives the characteristic polynomial of a recurrence relation that annihilates all of the sequences. The determinant is equal to \( E(E^3 - 2E^2 - 2E + 1) \) and hence the sequences \( \{w_n\}, \{x_n\}, \{y_n\} \) and \( \{z_n\} \) all satisfy the third order homogeneous, linear recurrence relation
\[ t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}. \] (3.11)

This recurrence (3.11) has eigenvalues \( \{-1, \frac{3+\sqrt{5}}{2}, \} \), and, with initial values of unity yields the `coupled’ sequence \{1, 1, 1, 3, 7, 19, 49, 129, 337, \ldots\} [6]. This sequence appears in the OEIS as A061646, with a variety of combinatorial interpretations [16]. Additionally, the polynomial factors further as \( E(E+1)(E^2 - 3E + 1) \). From this factorization the sequence \( \{w_n\} \) and \( \{y_n\} \) (the even \( \alpha \) and \( \beta \) terms) satisfy the second order relation
\[ t_n = 3t_{n-1} - t_{n-2}, \] (3.12)
which is also satisfied by alternate terms of the Fibonacci sequence (A001519 and A001906 [16]).

Finally, putting the sequences back together we would expect to need a sixth order recurrence. Instead, we find that both of the original \( \alpha_n \) and \( \beta_n \) sequences satisfy the fourth order recurrence
\[ t_n = t_{n-1} + t_{n-3} + t_{n-4}. \] (3.13)

This recurrence (3.13) has roots \( \{\pm i, \frac{1+\sqrt{5}}{2}, \} \) and with unit initial values yields the sequence \{1, 1, 1, 1, 3, 5, 7, 11, 19, 31, 49, 79, 129, \ldots\}, contained in the OEIS as A126116 [16], of which the couple sequence above is a subsequence. The connections among all these sequence are not surprising since, as is well known, \( i^2 = -1 \) and \( \left( \frac{1+\sqrt{5}}{2} \right)^2 = \frac{3+\sqrt{5}}{2} \), and so on.

4. CONCLUDING COMMENTS

In summary then, we have that the given recursive sequences satisfy the following recurrences:

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Recurrence Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_n ) and ( \beta_n )</td>
<td>( t_n = t_{n-1} + t_{n-3} + t_{n-4} )</td>
</tr>
<tr>
<td>( w_n = \alpha_{2n} = \beta_{2n} = y_n )</td>
<td>( t_n = 3t_{n-1} - t_{n-2} )</td>
</tr>
<tr>
<td>( x_n = \alpha_{2n+1} ) and ( z_n = \beta_{2n+1} )</td>
<td>( t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3} )</td>
</tr>
</tbody>
</table>

The two sequences discussed in [2, 3] we called 2–Pulsated Fibonacci sequences (from \( (a;b) \) and \( (a;b;c) \)–types). In [4] they were extended to what were called \( s \)–Pulsated Fibonacci sequences, where \( s \geq 3 \). In future research, it is planned to extend the present
2–Pulsated Fibonacci sequences from \((a; 2b; 2c)\)–type, to \(s\)–Pulsated Fibonacci sequences from \((a; 2b_1; \ldots, 2b_s)\)–type. Other related possibilities for research concern

- conjectures on the number of distinct prime divisors of these sequences [13, 22],
- connections with geometry [6, 8, 12].

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**References**

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