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7 April 1976

On transitive graphs

Introduction.

In this paper I present a number of results about transitive graphs, culminating in the classification and enumeration of transitive graphs with a prime number of vertices. Let me describe briefly where these results come from.

2.1 - 2.4 are due to Dauber and can be found in [3], p. 172.

3.5 is stated by Chao ([9]). 3.6 - 3.10 are original, though by their elementary nature they can hardly be called new.

4.1 and 4.2 are due to Chao ([9]). The terminology is that of Higman ([18]) and Sims ([15]). 4.7 and 4.8 are due to Sabidussi ([13]). 4.11 is due to Chao ([8]) and McAndrew ([12]).

5.2 is due to Tutte ([19], p. 59). The counter-example ~~on pp~~ described on pp. 21-27 is (as far as I know) new, and settles a question raised by Tutte ([19], p. 60)

~~Her approach~~

6.3, 6.4, 6.6, ~~and~~ 6.7, and the enumeration of vertex-transitive graphs with p points are due to Turner([16]). 6.5 and the enumeration of vertex-transitive digraphs come from Alspach ([6]). The approach taken is that of Berggren ([7]).

7.3 is due to Chao ([10]). Again, the approach is that of Berggren.

1. Notation

The purpose of this section is to establish the notation that will be used in the sequel. No attempt is made to define all the terms that will be used, e.g. connected graph, complete graph, bipartite graph, etc.

A <u>graph</u> (or <u>undirected graph</u>) is an ordered pair $\Gamma = (V, E)$ consisting of a set V of <u>vertices</u> and a set E of two-element subsets of V, the <u>edges</u> (All sets, graphs, etc. will be assumed to be finite.) A <u>digraph</u> (or <u>directed graph</u>) is an ordered pair $\vec{\Gamma} = (V, D)$ consisting of a set V of <u>vertices</u> and a set $D \subset V \times V - \{(a,a) : a \in V\}$ of <u>edges</u>. Let $D^r = \{(a,b) : (b,a) \in D\}$. The graph $\vec{\Gamma}^r = (V, D^r)$ is called the <u>reversal</u> of $\vec{\Gamma}$.

To any graph $\Gamma = (V, E)$ we associate the digraph $\hat{\Gamma} = (V, D)$, where $D = \{(a,b) : \{a,b\} \in E\}$. Conversely, to any digraph $\vec{\Gamma} = (V, D)$ we associate an undirected graph $\check{\vec{\Gamma}} = (V, E)$, where $E = \{\{a,b\} : (a,b) \in D \text{ or } (b,a) \in D\}$. It is clear that these associations give rise to a 1-1 correspondence between undirected graphs and digraphs $\vec{\Gamma}$ such that $\vec{\Gamma}^r = \vec{\Gamma}$.

An <u>oriented graph</u> is a digraph $\vec{\Gamma} = (V, D)$ such that $D \cap D^r = \phi$. Clearly $\vec{\Gamma}$ is an oriented graph if and only if $\vec{\Gamma}^r$ is an oriented graph. We call $\vec{\Gamma}$ an <u>orientation</u> of the associated undirected graph $\check{\vec{\Gamma}}$.

2. Transitive graphs

If $\Gamma = (V, E)$ is a graph, let S_V be the group of permutations of V. A permutation $\pi \in S_V$ is an <u>automorphism</u> of Γ if

$$\{a, b\} \in E \iff \{\pi(a), \pi(b)\} \in E$$

for all $a, b \in V$. (Of course there is a corresponding notion of an <u>isomorphism</u> of two graphs: If Γ and Γ' are isomorphic we will write $\Gamma \cong \Gamma'$ as usual.) The automorphisms of Γ form a group $A(\Gamma)$, the <u>automorphism group</u> of Γ. The automorphism group ~~of a~~ $A(\vec{\Gamma})$ of a digraph $\vec{\Gamma}$ is defined analagously. Clearly $A(\vec{\Gamma}) = A(\vec{\Gamma}^r)$ and $A(\vec{\Gamma}) \subset A(\check{\vec{\Gamma}})$.

Two vertices $a, b \in V$ are <u>similar</u> if $\exists \phi \in A(\Gamma)$ such that $\phi(a) = b$. A graph or digraph is <u>vertex-transitive</u> (abbreviated <u>v.-t.</u>) if all the vertices are similar. This is equivalent to saying that the automorphism group acts transitively on V. Clearly Γ is v.-t. $\Leftrightarrow$ $\hat{\Gamma}$ is v.-t., and $\bar{\Gamma}$ is v.-t. $\Rightarrow$ $\check{\bar{\Gamma}}$ is v.-t. A vertex-transitive graph is obviously regular.

To any automorphism $\phi \in A(\Gamma)$ we associate a permutation $\bar{\phi} \in S_E$ of the edges, where $\bar{\phi}(\{a,b\}) = \{\phi(a), \phi(b)\}$. The set $\Lambda(\Gamma) = \{\bar{\phi} : \phi \in A(\Gamma)\}$ is a subgroup of S_E, called the ~~the g~~ edge-group of Γ. The edge-group of a digraph is defined analogously. A graph (digraph) is ~~symmetric~~ edge-transitive (abbreviated e.-t.) if the edge-group acts transitively on the set of edges. A graph (digraph) is symmetric (abbreviated sym) if it is both v.-t. and e.-t.. A graph Γ is 1-transitive (abbreviated 1-t.) if the associated digraph $\hat{\Gamma}$ is sym. Thus a graph is 1-t. if and only if it is v.-t. and any edge can be mapped "either way" onto any other. A sym. graph is not necessarily 1-t., as we will see later on. Thus in particular the fact that a graph is e.-t. does not imply that $\hat{\Gamma}$ is e.-t., although the converse clearly holds. A simpler example is the complete bipartite graph $K_{m,n}$ where $m,n \geq 1$, $m \neq n$. This is also an example of a graph that is edge-transitive but not vertex-transitive.

<u>Prop. 2.1</u>: If $\Gamma = (V, E)$ is an e.-t. graph with no isolated points then either Γ is v.-t. (and hence sym.) or it is bipartite.

<u>Pf</u>: ~~Choose this~~ Choose an edge $\{a_1, a_2\} \in E$, and let ~~$V_1 = A(\Gamma)(a_1)$~~ $V_i = A(\Gamma)(a_i)$.
Since Γ is e.-t. and there are no isolated points $V_1 \cup V_2 = V$. If $V_1 \cap V_2 = \phi$ then Γ is bipartite, since any edge must have one vertex in V_1 and the other in V_2. If $V_1 \cap V_2 \neq \phi$ then $V_1 = V_2 = V$, because two orbits of a permutation group are either disjoint or equal, and we conclude that Γ is v.-t. □

There are a number of interesting special cases to which this result can be applied. We continue to assume that Γ is e.-t. and has no isolated pts.

<u>Corollary 2.2</u>: If Γ is not regular then it is bipartite.

<u>Pf</u>: A vertex-transitive graph is regular □

<u>Corollary 2.3</u>: If Γ is regular (of degree ρ) and $p = |V|$ is odd then Γ is v.-t.

<u>Pf</u>: If Γ isn't v.-t. then $V_1 \cap V_2 = \phi$, and

$$|E| = \rho|V_1| = \rho|V_2| \Rightarrow |V_1| = |V_2|$$
$$\Rightarrow |V| = |V_1| + |V_2| \text{ is even,}$$

contradiction □

<u>Corollary 2.4</u>: If Γ is regular of degree $\rho \geq p/2$, where $p = |V|$, the Γ is v.-t.

<u>Pf</u>: If Γ isn't v.-t. then $V_1 \cap V_2 = \phi$, so $|V_1| = |V_2|$ as we just saw. Since a vertex in V_1 is connected to at most $|V_2| = p/2$ other vertices we must have $\rho = p/2$. But then Γ must be the complete bipartite graph $K_{p/2, p/2}$, which is v.-t., contradiction □.

From the last two corollaries it appears that most regular e.-t. graphs are v.-t. But Folkman ([11]) has shown there are regular e.-t. graphs that are not v.-t. The smallest of these graphs has 20 vertices.

3. Cartesian products

We have seen that an e.-t. graph is not necessarily v.-t. On the other hand, it is easy to find examples of graphs that are v.-t. but not e.-t., for instance the following graph:

This example belongs to a class of graphs described by Corollary 3.7 below. We will need the following definition: The cartesian product of two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ is the graph $\Gamma \times \Gamma' = (V \times V', F)$, where

$$F = \{ \{ (v_0, w_0), (v_1, w_1) \} : v_0, v_1 \in V,\ w_0, w_1 \in V', \text{ and either } v_0 = v_1 \text{ and } \{w_0, w_1\} \in E' \text{ or } w_0 = w_1 \text{ and } \{v_0, v_1\} \in E \}.$$

For example, the graph above is the cartesian product of the complete graphs K_3 and K_2. The cartesian product is clearly commutative and associative, that is, $\Gamma \times \Gamma' \cong \Gamma' \times \Gamma$ and $(\Gamma_1 \times \Gamma_2) \times \Gamma_3 \cong \Gamma_1 \times (\Gamma_2 \times \Gamma_3)$.

A graph $\Gamma = (V, E)$ is <u>irreducible</u> if $|V| \geq 2$ and if $\Gamma \cong \Gamma_1 \times \Gamma_2$ & $\Gamma_i = (V_i, E_i) \Rightarrow |V_1| = 1$ or $|V_2| = 1$. Sabidussi ([14]) proved the following:

<u>Theorem 3.1</u>: Any connected graph $\Gamma = (V, E)$, $|V| \geq 2$ can be expressed uniquely as a cartesian product of irreducible graphs. $\square$

Uniqueness means not only that if $\Gamma \cong \Gamma_1 \times \dots \times \Gamma_n$ and $\Gamma \cong \Gamma_1' \times \dots \times \Gamma_m'$ then $n = m$ and the Γ_i''s can be ordered so that $\Gamma_i \cong \Gamma_i'$, $i = 1, \dots, n$, but also that the order of the Γ_i''s and the isomorphisms $\phi_i : \Gamma_i \to \Gamma_i'$ can be chosen so that the following diagram commutes:

$$\begin{tikzcd} \Gamma_1 \times \dots \times \Gamma_n \arrow[rr, "\phi_1 \times \dots \times \phi_n"] \arrow[dr] & & \Gamma_1' \times \dots \times \Gamma_n' \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

<u>Corollary 3.2</u>: If $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$, where Γ_i is irreducible, $i = 1, \dots, n$, and if $\Gamma_i \cong \Gamma_j \Rightarrow \Gamma_i = \Gamma_j$, $1 \leq i, j \leq n$, then $A(\Gamma)$ is the group G generated by permutations of the form

$$\phi_1 \times \dots \times \phi_n : (u_1, \dots, u_n) \mapsto (\phi_1(u_1), \dots, \phi_n(u_n))$$

where $\phi_i \in A(\Gamma_i)$, $i = 1, \dots, n$, and permutations of the form

$$\phi_\pi : (u_1, \dots, u_n) \mapsto (u_{\pi(1)}, \dots, u_{\pi(n)}),$$

where $\pi \in S_{\{1, \dots, n\}}$ and $\Gamma_i = \Gamma_{\pi(i)}$, $i = 1, \dots, n$.

<u>Pf</u>: ~~Suppose that~~ Clearly $G \subset A(\Gamma)$. Suppose that $\phi \in A(\Gamma)$. Then by 3.1 there is a permutation $\pi \in S_{\{1,\dots,n\}}$ and a collection of isomorphisms $\phi_i : \Gamma_i \to \Gamma_{\pi(i)}$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma_1 \times \dots \times \Gamma_n \arrow[rr, "\phi_1 \times \dots \times \phi_n"] \arrow[dr, "\phi"'] & & \Gamma_{\pi(1)} \times \dots \times \Gamma_{\pi(n)} \arrow[dl, "\phi_{\pi^{-1}}"] \\ & \Gamma_1 \times \dots \times \Gamma_n, & \end{tikzcd}$$

i.e. $\phi = \phi_{\pi^{-1}} \circ (\phi_1 \times \dots \times \phi_n)$. But $\Gamma_i \cong \Gamma_{\pi(i)}$ $\Rightarrow \Gamma_i = \Gamma_{\pi(i)} \Rightarrow \phi_i \in A(\Gamma_i)$, so $\phi \in G$. □

These results can be interpreted as follows:

<u>Corollary 3.3</u>: Let ~~Γ be a c~~ $\Gamma = (V, E)$ be a connected graph with $|V| \geq 2$, and let $a \in V$. If $a = (v_1, \dots, v_n)$, where Γ has been identified with $\Gamma_1 \times \dots \times \Gamma_n$ ~~and~~ with Γ_i irreducible, $i = 1, \dots, n$, then a determines the family $\{(v_i, \Gamma_i)\}_{i=1}^{n}$ of pointed graphs up to permutation of the indices and pointed graph automorphisms of the indexed pointed graphs. ~~Furthermore, if~~ Two points $a, b \in V$ are similar iff and only if they define equivalent families of pointed graphs. □

Furthermore,

<u>Corollary 3.4</u>: Let Γ be as in corollary 3.3. If $\{a,b\} \in E$ then $\{a,b\} = \{(u_1, \ldots, u_i, \ldots, u_n), (u_1, \ldots, v_i, \ldots, u_n)\}$ where $\{u_i, v_i\} \in E_i$. If we let $\Gamma_{\{a,b\}} = \Gamma_i$ then $\Gamma_{\{a,b\}}$ is determined up to isomorphism. In other words, any edge of ~~Γ determines~~ Γ determines the irreducible factor to which it "belongs" up to isomorphism. □

We will continue to assume that Γ satisfies the assumptions of corollary 3.3.

<u>Proposition 3.5</u>: Γ is v.-t. $\Leftrightarrow$ Γ_i is v.-t., $i = 1, \ldots, n$

<u>Pf</u>: $\Leftarrow$: By 3.2

$\Rightarrow$: Suppose that $u_i, v_i \in V_i$ are dissimilar, and let u_j be an arbitrary element of V_j, $j \neq i$. Then by 3.3 the points $(u_1, \ldots, u_i, \ldots, u_n)$ and $(u_1, \ldots, v_i, \ldots, u_n)$ are dissimilar, ~~where~~ contradiction. □

Proposition 3.6: Γ is e.-t. $\Leftrightarrow$ i) $n=1$ and Γ_1 is e.-t.
or ii) $n>1$, $\Gamma_1 \cong \Gamma_2 \cong \cdots \cong \Gamma_n$
and the Γ_i's are sym.

Pf: $\Leftarrow$: By 3.2

$\Rightarrow$: It was remarked in 3.4 that any edge determines the irreducible factor to which it belongs up to isomorphism. Since all the edges are similar, all the irreducible factors must be isomorphic, i.e., $\Gamma_1 \cong \Gamma_2 \cong \cdots \cong \Gamma_n$. If Γ_1 were not e.-t. then, by 3.2, Γ would not be e.-t. Suppose that $n>1$ and Γ_1 is not v.-t.: Choose dissimilar vertices $u, v \in V_1$, and let $\phi: \Gamma_1 \to \Gamma_2$ be an isomorphism. It follows from 3.3 that the vertices

$$(u, \phi(u), u_3, \ldots, u_n)$$
$$(u, \phi(v), u_3, \ldots, u_n)$$
$$\text{and} \quad (v, \phi(v), u_3, \ldots, u_n)$$

are mutually dissimilar, where u_j is an arbitrary element of V_j, $j \geq 3$. But from the proof of 2.1 above we know that the vertices of an e.-t. ~~graph~~ connected graph fall into at most two similarity classes, contradiction. So Γ_1 is symmetric $\square$

Corollary 3.7: If $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i is an irreducible connected v.-t. graph, $i = 1, 2$, and $\Gamma_1 \not\cong \Gamma_2$, then Γ is v.-t. but not e.-t. □

Corollary 3.8: If a connected e.-t. graph is not v.-t. then it is irreducible. □

Corollary 3.9: Γ is sym. $\Leftrightarrow \Gamma_1 \cong \ldots \cong \Gamma_n$ and the Γ_i's are sym. □

Proposition 3.10: Γ is 1-t. $\Leftrightarrow \Gamma_1 \cong \ldots \cong \Gamma_n$ and the Γ_i's are 1-t.

Pf: By 3.2 □

4. Graphs with a given group of automorphisms

This section is devoted to determining those graphs whose automorphism group contains a given transitive permutation group G. Actually, we will deal mainly with digraphs: Results about graphs will be obtained from the correspondence between Γ and $\bar{\Gamma}$.

Let $G^2 = \{\phi \times \phi : \phi \in G\}$ be the permutation group on $V \times V$ corresponding to G. Let $\Delta_1 = \{(a,a) : a \in V\}, \Delta_2, \dots, \Delta_n$ be the orbits of G^2. We will call them <u>orbitals</u> of G.

<u>Proposition 4.1</u>: If $\Gamma = (V, D)$ then $A(\Gamma) \supset G \iff D = \bigcup_{i \in I} \Delta_i$, where $I \subset \{2, \dots, n\}$

<u>Pf</u>: By definition $A(\Gamma) = \{\phi \in S_V : (\phi \times \phi)(D) = D\}$
Thus $G \subset A(\Gamma) \iff D$ is a fixed block of G^2 $\iff D$ is a union of orbits of G^2. $\square$

If Δ is an orbital of G then so is $\Delta^r = \{(a,b) : (b,a) \in \Delta\}$. Δ and Δ^r are called <u>paired orbitals</u>. If $\Delta = \Delta^r$, Δ is said to be <u>self-paired</u>. If $\Delta = \Delta_i$, define i^r so that $\Delta^r = \Delta_{i^r}$.

<u>Corollary 4.2</u>: If $\Gamma = (V, E)$ and $\hat{\Gamma} = (V, D)$ then $A(\Gamma) \supset G \iff D = \bigcup_{i \in I} \Delta_i$, where $I \subset \{2, \dots, n\}$ and $i \in I \iff i^r \in I$ $\square$

Of course the automorphism groups of the digraphs and graphs in 4.1 and 4.2 may be larger than G. Also these graphs may not all be non-isomorphic, i.e., ~~[illegible]~~

$\Gamma = (V, \bigcup_{i \in I} \Delta_i) \cong \Gamma' = (V, \bigcup_{i \in I'} \Delta_i)$ does not necessarily imply that $I = I'$.

Let a be an arbitrary point of V, and let $G_a = \{\phi \in G : \phi(a) = a\}$

<u>Proposition 4.3</u>: There is a 1-1 correspondence between orbits of G_a and orbitals of G. Orbits corresponding to paired orbitals have the same length.

<u>Pf</u>: Let Δ be an orbital of G, and let $\Delta(a) = \{b : (a,b) \in \Delta\}$. If c is another element of V then by the transitivity of G $|\Delta(a)| = |\Delta(c)|$. Summing over the elements of V, this gives $|\Delta| = |V||\Delta(a)|$, so $\Delta(a)$ is not empty. But $\Delta(a)$ is clearly an orbit of G_a.

$$|V||\Delta(a)| = |\Delta| = |\Delta^r| = |V||\Delta^r(a)| \Rightarrow |\Delta(a)| = |\Delta^r(a)| \square$$

The pairing of the orbitals induces a pairing of the orbits of G_a: If $\delta = \Delta(a)$ is an orbit of G_a, we define $\delta^r = \Delta^r(a)$. By 4.3 $|\delta| = |\delta^r|$. The following is easily verified.

<u>Proposition 4.4</u>: $\delta^r = \{\phi(a) : \phi^{-1}(a) \in \delta, \phi \in G \text{ for some}\} \square$

We will call the orbit $\{a\} = \Delta_1(a)$ of G_a the <u>trivial orbit</u>. The other orbits are <u>non-trivial</u>. In terms of the orbits of G_a, 4.1 and 4.2 become

<u>Proposition 4.5</u>: Let $\Gamma = (V, D)$ be a directed graph. Then $A(\Gamma) \supset G \iff D(a) = \{b : (a,b) \in D\}$ consists of non-trivial orbits of G_a. $\square$

<u>Corollary 4.6</u>: Let $\Gamma = (V, E)$ be an undirected graph. Then $A(\Gamma) \supset G \iff D(a) = \{b : \{a,b\} \in E\}$ consists of non-trivial orbits of G_a and $\delta \subset D(a) \iff \delta^r \subset D(a)$ $\square$

These results are particularly important in the case where the group G is regular, that is, $G_a = \{1\}$. In this case $|G| = |V|$, and we can identify G with V under the correspondence $\phi \mapsto \phi(a)$, where a is an arbitrary element of V. Under this identification G becomes a group of permutations of its own elements: For $\phi, \psi \in G$,

$$\psi(\phi) = \psi(\phi(a)) = \psi \circ \phi(a) = \psi \circ \phi.$$

Thus G acts on itself by left multiplication. In effect, what we have shown here is that a regular permutation group G is isomorphic (as a permutation group) to its own left-regular representation.

Let $\Gamma = (G, D)$ be a digraph whose automorphism group contains G. Let ~~$\Delta_1, \dots, \Delta_n$ be~~: $\Delta_1 = \{(\phi, \phi) : \phi \in G\}, \Delta_2, \dots, \Delta_n$ be the orbitals of G. We know from 4.1 that $D = \bigcup_{i \in I} \Delta_i$, where $I \subset \{2, \dots, n\}$. Since G is regular, $\Delta_i(a)$ ~~consists of a sing~~ contains a single element $\phi_i(a)$ for every i, and for any $b = \phi(a) \in V$, $\Delta_i(b) = \{\phi \circ \phi_i(a)\}$. Thus $D = \{(\phi, \phi \circ \phi_i) : \phi \in G, i \in I\}$ $= \{(\phi, \phi \circ \phi') : \phi \in G, \phi' \in H\}$, where $H = \bigcup_{i \in I} \{\phi_i\} = D(a)$.

Now a digraph Γ is a <u>group-digraph</u> if Γ is isomorphic to a graph $\Gamma_{G,H} = (G, D_H)$, where G is a group, $H \subset G - \{1\}$, and $D_H = \{(g, gh) : g \in G, h \in H\}$. The left-regular representation of G clearly forms a regular subgroup of $A(\Gamma_{G,H})$. Thus we have shown:

<u>Proposition 4.7</u>: The automorphism group of a digraph Γ contains a regular subgroup G if and only if Γ is a ~~set~~ ~~group-graph~~ group-digraph. □

Now a digraph $\Gamma = (V, D)$ whose automorphism group contains a regular subgroup G corresponds to an undirected graph Γ if and only if ($i \in I \Leftrightarrow i^r \in I$), where $\Delta_1, \dots, \Delta_n$ are the orbitals of G and $D = \bigcup_{i \in I} \Delta_i$. On the other hand, a group-digraph $\Gamma_{G,H}$ corresponds to an undirected graph ~~if and only if $H = H^{-1}$~~ $\Gamma_{G,H}$, (called a <u>group-graph</u>) if and only if $H = H^{-1}$! From 4.7 we expect these two requirements to be ~~equivalent~~ one and the same: Their equivalence is due to the fact that, if ϕ_i, $i = 1, \dots, n$ are chosen so that $\Delta_i(a) = \{\phi_i(a)\}$, then $\phi_{i^r} = \phi_i^{-1}$ (cf. 4.4). Proposition 4.7, ~~restated for~~ applied to undirected graphs, becomes:

<u>Corollary 4.8</u>: The automorphism group of a graph Γ contains a regular subgroup $G \Leftrightarrow$ Γ is a group-graph. □

<u>Proposition 4.9</u>: If $\Gamma_{G,H}$ is a group-digraph, and if ϕ is a group-automorphism of G such that $\phi(H) = H$, then ϕ is a graph-automorphism of $\Gamma_{G,H}$.

~~Pf: $(g, gh) \in D_H \Leftrightarrow h \in H \Leftrightarrow \phi(h) \in H \Leftrightarrow (\phi g, \phi(gh)) \in D$~~

Pf: $(g, gh) \in D_H \Leftrightarrow h \in H \Leftrightarrow \phi(h) \in H$
$\Leftrightarrow (\phi(g), \phi(g)\phi(h)) \in D_H \Leftrightarrow \bar{\Phi}((g,gh)) \in D_H$
$\Rightarrow \phi \in A(\Gamma)$ □

Cor 4.10: If $\Gamma_{G,H}$ is a group-graph, where $H = H^{-1}$, and if ϕ is a group-automorphism of G such that $\phi(H) = H$, then ϕ is a graph-automorphism of $\Gamma_{G,H}$. □

Of course not all graph-automorphisms fixing 1 need to be group-automorphisms: The general question of the relationship between graph-automorphisms fixing 1 and group-automorphisms has not been solved. For instance, if G is a ~~regular permutation~~ group having a graphical regular representation, which is to say that there exists $H \subset G - \{1\}$, $H = H^{-1}$ such that $A(\Gamma_{G,H})$ is the left-regular representation of G, then by 4.10 there is no non-trivial group-automorphism ϕ of G such that $\phi(H) = H$. Conversely, Watkins ([17], p. 97) conjectures that if for some set $H \subset G - \{1\}$, ~~generating~~ $H = H^{-1}$ generating G there ~~are no non-trivial automorphisms of G such that~~ is no non-identity group-automorphism ϕ of G such that $\phi(H) = H$, then G has a graphical regular representation. (It is not clear whether Watkins expects that under these conditions $\Gamma_{G,H}$ will itself be a graphical regular representation of G — the conjecture only says that G has <u>some</u> g.r.r.) As far as I know, [redacted] the question of the truth of this conjecture has not yet been resolved.

<u>Proposition 4.11</u>: Let $\Gamma = (V, E)$ be a v.-t. graph. If $A(\Gamma)$ is abelian then $A(\Gamma)$ is an elementary 2-group.

<u>Pf</u>: A transitive abelian group is regular ([5], p. 9), so by 4.8 we may assume that $\Gamma = \Gamma_{G,H}$, where [redacted] $G = A(\Gamma)$ and $H = H^{-1}$. Since G is abelian, the map $\phi: g \mapsto g^{-1}$ is a group-automorphism, and hence, by 4.10, a graph-automorphism. But $\phi(1) = 1$, so $\phi = 1$, since G is regular, and this means that $g = g^{-1}$ for all $g \in G$. Hence G is an elementary 2-group. $\square$

[redacted]

<u>Corollary 4.12</u>: No v.-t. graph has as its group of automorphisms the cyclic group $\mathbb{Z}/n\mathbb{Z}$, where $n \geq 3$. $\square$

<u>5. The relation between symmetry and 1-transitivity</u>

Let Γ be an e.-t. graph.

<u>Proposition 5.1</u>: Γ is not 1-t. $\Leftrightarrow$ $\exists$ an orientation $\vec{\Gamma}$ of Γ such that $A(\vec{\Gamma}) = A(\Gamma)$. Such an orientation, if it exists, is e.-t., sym. if Γ is sym., and unique up to reversal.

<u>Pf</u>: Let $\hat{\Gamma} = (V, D)$. We may ignore the trivial case where $D = \emptyset$.

$\Leftarrow$: Let $\vec{\Gamma} = (V, D')$ be an orientation of Γ such that $A(\vec{\Gamma}) = A(\Gamma)$. Then D' is a fixed block (in fact, an orbit) of $A(\vec{\Gamma})^2 = A(\Gamma)^2 = A(\hat{\Gamma})^2$, so D is not an orbit of $A(\hat{\Gamma})^2$, and hence Γ is not 1-t.

$\Rightarrow$: Choose $(a,b) \in D$, and let $D_1 = A(\Gamma)^2((a,b))$, $D_2 = A(\Gamma)^2((b,a))$. Clearly $D_1 = D_2^r$. Since Γ is e.-t., $D_1 \cup D_2 = D$. Now $D_1 \cap D_2 \neq \emptyset \Rightarrow D_1 = D_2 = D \Rightarrow \Gamma$ is 1-t., contradiction, so

$D_1 \cap D_2 = \phi$. Hence $\vec{\Gamma}_i = (V, D_i)$ is an oriented graph, $i = 1, 2$. Since D_i is an orbit of $A(\Gamma)^2$, $A(\Gamma) \subset A(\vec{\Gamma}_i)$. On the other hand, $A(\vec{\Gamma}_i) \subset A(\check{\vec{\Gamma}}_i) = A(\Gamma)$, so $A(\vec{\Gamma}_i) = A(\Gamma)$, $i = 1, 2$.

Since $A(\vec{\Gamma}_i) = A(\Gamma)$, $\vec{\Gamma}_i$ is e-t. and sym. iff Γ is sym., $i = 1, 2$. Let $\vec{\Gamma} = (V, D')$ be any orientation of Γ such that $A(\vec{\Gamma}) = A(\Gamma)$. Then either $(a,b) \in D'$ or $(b,a) \in D'$. If $(a,b) \in D'$, then $A(\vec{\Gamma})^2((a,b)) = A(\Gamma)^2((a,b)) = D_1 \subset D' \Rightarrow D_1 = D'$ $\Rightarrow \vec{\Gamma} = \vec{\Gamma}_1$. Similarly, $(b,a) \in D' \Rightarrow \vec{\Gamma} = \vec{\Gamma}_2$. But $\vec{\Gamma}_1^{\,r} = \vec{\Gamma}_2$, so the orientation is unique up to reversal. $\square$

If Γ is sym. we can render this proposition into the language of the previous section as follows: If $\hat{\Gamma} = (V, D)$ then $D = \Delta \cup \Delta^r$, where Δ is an orbital of $A(\Gamma)$, and Γ is 1-t. $\Leftrightarrow \Delta = \Delta^r$.

<u>Corollary 5.2</u>: Any sym. graph Γ regular of odd degree is 1-t.

<u>Pf</u>: Let $\Gamma = (V, E)$, $\hat{\Gamma} = (V, D)$, and let $a \in V$. Let

$$D(a) = \{b : (a,b) \in D\}$$
$$= \{b : \{a,b\} \in E\}.$$

Then Γ is regular of degree $|D(a)|$. If Γ is not 1-t. then $D = \Delta \cup \Delta^r$, $\Delta \neq \Delta^r$ as above. But

$$
\begin{aligned}
|D(a)| &\cancel{= |\Delta(a)| + |\Delta^r(a)|} \\
&= |\Delta(a) \cup \Delta^r(a)| \\
&= |\Delta(a)| + |\Delta^r(a)| \quad \text{since } \Delta \cap \Delta^r = \phi \\
&= 2|\Delta(a)|. \square
\end{aligned}
$$

Corollary 5.2 shows that any symmetric graph that is not 1-t. must have even degree. Later we will see that a symmetric graph with a prime number of points is necessarily 1-t. The question arises whether all symmetric graphs are 1-transitive. The answer is no, as the following counter-example shows.

The counter-example will be a group-graph. The idea is to find a group G and a set of generators $H \subset G - \{1\}$ such that

i) $H \cap H^{-1} = \phi$

ii) for any $h_1, h_2 \in H$ there is an automorphism ϕ of G such that $\phi(h_1) = h_2$

iii) if ϕ is an automorphism of G such that $\phi(H \cup H^{-1}) = H \cup H^{-1}$, then $\phi(H) = H$.

The group-graph $\Gamma_{G, H \cup H^{-1}}$ will then be vertex-transitive and edge-transitive, and we may hope that conditions i)-iii) will preclude its being 1-transitive. Actually, this need not always be true: There do exist ~~graphs~~ groups G with generators H satisfying i)-iii) such that $\Gamma_{G, H \cup H^{-1}}$ is 1-transitive. But I will provide an example where 1-transitivity fails.

Let p be an odd prime and let G be the non-abelian group of order p^3 with generators a, b and relations $a^{p^2} = 1$, $b^p = 1$, $b^{-1}ab = a^{p+1}$ (Hall, Theory of Groups, p. 52) The first thing to do is to replace b by ~~a gener~~ another generator c so that the relations become symmetric in a and c. $b^{-1}ab = a^{p+1} \Rightarrow ab = ba^{p+1}$, and by induction, we can show that

$$a^i b^j = b^j a^{i(p+1)^j}$$

Using this identity, we find that

$$(ba^k)^p = b^p a^{k[1+(p+1)+\cdots+(p+1)^{p-1}]}$$
$$= a^{k[1+(p+1)+\cdots+(p+1)^{p-1}]}$$

But
$$1+(p+1)+\cdots+(p+1)^{p-1}$$
$$\equiv p + p(1+2+\cdots+p-1)$$
$$\equiv p \qquad (\text{mod } p^2)$$

since $1+2+\cdots+p-1 = p(p-1)/2$ is divisible by p (cf. Hall, p. 51), so

$$(ba^k)^p = a^{kp}.$$

Furthermore

$$(ba^k)^{-1}a(ba^k) = a^{-k}(b^{-1}ab)a^k = a^{p+1}.$$

Setting $c = ba^{-1}$, we get

$$c^p = a^{-p}, \quad c^{-1}ac = a^{p+1}.$$

But $\quad c^p = a^{-p} \Rightarrow a^p = c^{-p},$

and $c^{-1}ac = a^{p+1} \Rightarrow ac = ca^{p+1}$

$$\Rightarrow c = (a^{-1}ca)a^{p} = (a^{-1}ca)c^{-p}$$

$$\Rightarrow a^{-1}ca = c^{p+1}.$$

Thus the group G can be described as the group generated by a and c with relations

$$a^{p^2} = 1, \quad c^{p^2} = 1$$

$$c^{p} = a^{-p}, \quad a^{p} = c^{-p}$$

$$c^{-1}ac = a^{p+1}, \quad a^{-1}ca = c^{p+1}$$

Of course these relations are not independent: Their redundancy allows us to see at a glance that there is an automorphism ϕ of G ~~ex~~ such that $\phi(a) = c$, $\phi(c) = a$. Setting $H = \{a, c\}$, we see that H satisfies conditions i) and ii).

To show that iii) also holds, suppose ϕ is an automorphism of G such that $\phi(H \cup H^{-1}) = H \cup H^{-1}$, $\phi(H) \neq H$. We need only consider two cases

<u>Case I</u>: $\phi(a) = a^{-1}$, $\phi(c) = c^{-1}$

Then $c^{-1}ac = a^{p+1} \Rightarrow (c^{-1})^{-1}a^{-1}c^{-1} = ca^{-1}c^{-1} = a^{-(p+1)}$,

But $a^{-1}ca = c^{p+1} \Rightarrow a^{-1}c^{-1}a = c^{-(p+1)}$

$$\Rightarrow ca^{-1}c^{-1} = c^{-p}a^{-1} = a^{p-1},$$

so

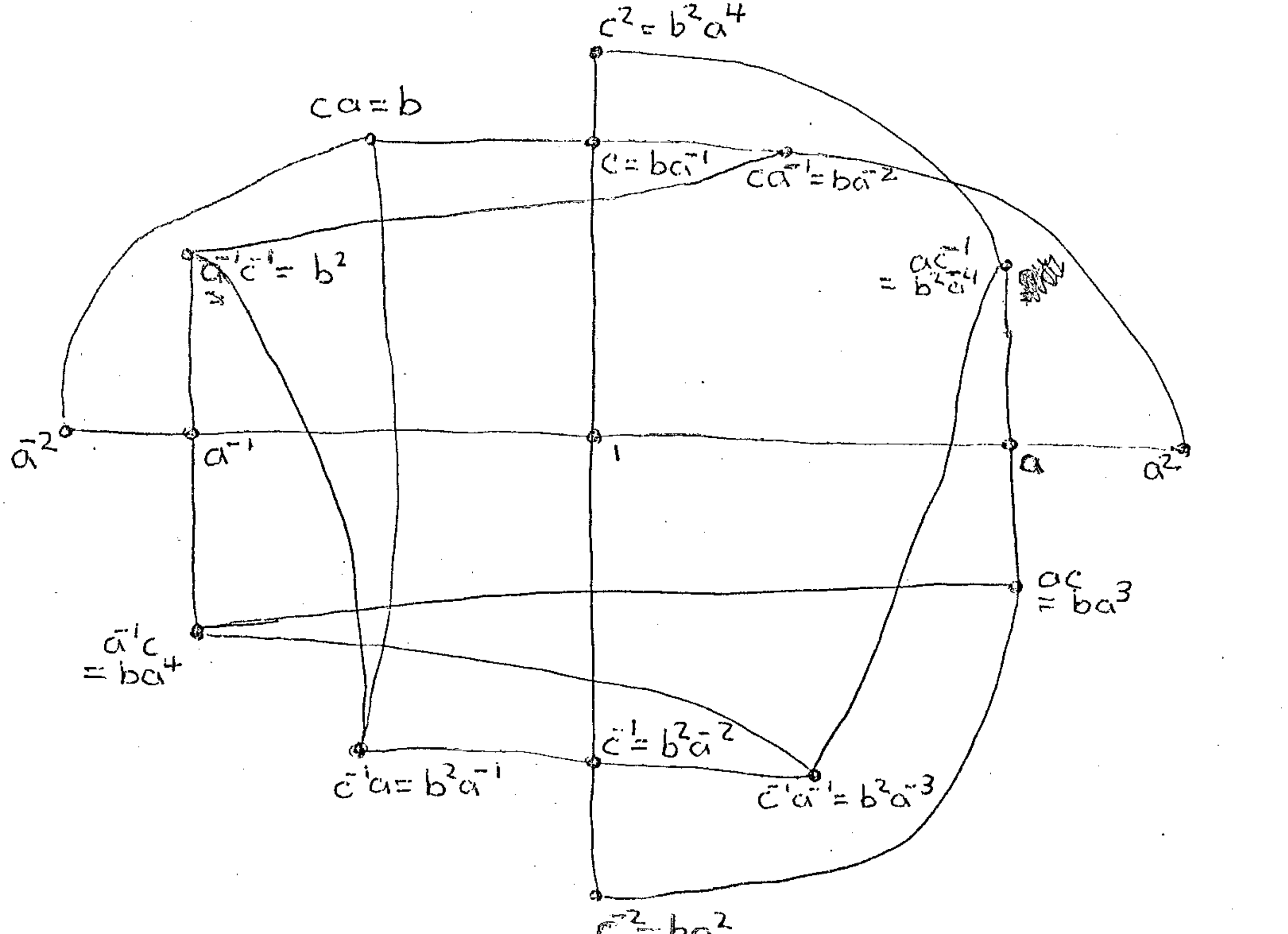
$$a^{-(p+1)} = a^{p-1} \Rightarrow a^{2p} = 1, \text{ no dice.}$$

<u>Case II</u>: $\phi(a) = a^{-1}$, $\phi(c) = c$.

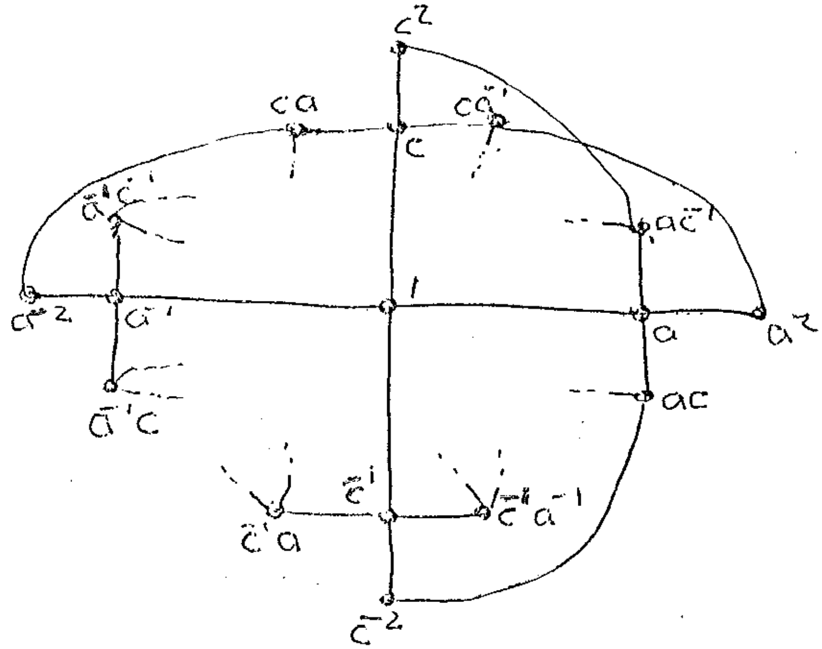
Then $a^{p} = c^{-p} \Rightarrow (a^{-1})^{p} = a^{-p} = c^{-p}$

$$\Rightarrow a^{p} = a^{-p} \Rightarrow a^{2p} = 1, \text{ no dice.}$$

Now we have an acceptable H: It remains to show that $\Gamma_{G, H \cup H^{-1}}$ is not 1-transitive ~~While I conjecture this to be the case for any odd primes all odd primes, I have only been able to establish it for $p = 3$ In this case~~ While I conjecture this to be true for all p, I have only been able to establish it for $p = 3$. The problem is to show that there is no graph-automorphism ϕ such that $\phi(1) = 1$, $\phi(a) = a^{-1}$ In the case $p=3$ we consider the subgraph Γ' of $\Gamma_{G,H\cup H^{-1}}$ generated by the ~~points~~ vertices whose distance from 1 is ≤ 2. Since any automorphism of $\Gamma_{G,H\cup H^{-1}}$ ~~restricts to an automorphism~~ fixing 1 restricts to an automorphism of Γ' fixing ~~one~~ 1, it suffices to show that there is no automorphism ϕ' of Γ' such that $\phi'(1)=1$ and $\phi'(a)=a^{-1}$. In determining Γ', one makes use of the identity $a^ib^j = b^ja^{i(p+1)^j}$. Γ' turns out to look like this:



The picture is simplified by leaving out the edges that connected two trivalent vertices:



Now if $\phi'(1)=1$, $\phi'(a)=a^{-1}$ then $\phi'(a^2)=a^{-2}$. But a^2 is ~~[illegible]~~ on a 6-cycle (in Γ') passing successively through vertices of degrees 2, 3, 4, 2, 3, 4, while a^{-2} is on no such cycle, contradiction.

Let me ~~[illegible]~~ end this section by proving an assertion made earlier, that there exist G and H satisfying i) - iii) ~~[illegible]~~ such that $\Gamma_{G,H\cup H^{-1}}$ is 1-transitive. Let G, a, b, c be as above, and let $a_1 = a$, $c_1 = c$,

$$a_m = a^{mp+1} = a_1^{mp+1}$$

$$c_n = c_1 a^{-np} = b a^{-(np+1)} = c_1^{np+1}.$$

Let $H' = \{a_1, a_2, \ldots, a_{p-1}, c_1, c_2, \ldots, c_{p-1}\}$
Condition i) clearly holds. If ϕ is the automorphism of G switching a and c, then $\phi(a_m) = c_m$ and $\phi(c_n) = a_n$. We have already seen that

$$c_n^{-1} a_1 c_n = a_1^{p+1} = a_2,$$

so $$c_n^{-1} a_m c_n = a_1^{(p+1)(mp+1)} = a^{(m+1)p+1} = a_{m+1},$$

where subscripts are taken modulo p. Applying ϕ to this identity yields

$$a_m^{-1} c_n a_m = c_{n+1}.$$

Thus conjugation by c_n fixes the c's and rotates the a's, that is, $(c_i)^{c_n} = c_i$, $(a_m)^{c_n} = a_{m+1}$, and conjugation by a_m fixes the a's and rotates the c's. We conclude that condition ii) also holds. Condition iii) holds as before, since once again we need only consider the cases $\phi(a) = a^{-1}$, $\phi(c) = c^{-1}$ and $\phi(a) = a^{-1}$, $\phi(c) = c$.

~~To see that $\Gamma(G,H)$ is 1-transitive, we~~

To see that $\Gamma_{G, H' \cup H'^{-1}}$ is 1-transitive, we notice that whether or not two elements g_1, g_2 of G are joined by an edge in $\Gamma_{G, H' \cup H'^{-1}}$ depends only on their residues modulo the commutator subgroup

$$G_c = \{1, a^p, \dots, a^{p^2-p}\} = \{1, c^p, \dots, c^{p^2-p}\}$$

of G. Thus $\Gamma_{G, H' \cup H'^{-1}}$ corresponds in an obvious way to the group-graph

$$\bar{\Gamma} = \Gamma_{G/G_c,\, H'/G_c \cup H'/G_c^{-1}}$$

Since G/G_c is abelian, $\bar{\Gamma}$ is 1-transitive. (In fact, G/G_c is of type (p, p), and $\bar{\Gamma}$ is the cartesian product of two p-cycles.) But this clearly implies that $\Gamma_{G, H' \cup H'^{-1}}$ is also 1-transitive.

6. Vertex-transitive graphs with a prime number of points

It is only when we restrict our attention to transitive graphs with a prime number of points that we get reasonably complete classification and enumeration theorems. The results depend heavily on a theorem of Burnside, which I will state here as two separate theorems:

Theorem 6.1: Let G be a transitive permutation group on a set V, where $p = |V|$ is prime. Then we can identify V with the finite field $F_p = \mathbb{Z}/p\mathbb{Z}$ so that

$$C_p = \{x \mapsto x+b : b \in F_p\} \subset G$$

Pf: Pick $a \in V$ and let $G_a = \{g \in G : g(a) = a\}$. Then for any $g' \in G$, $gG_a = \{g \in G : g(a) = g'(a)\}$. Thus there is an injective map from the set of left cosets of G_a into V. Since G is transitive the map is also surjective, so $[G : G_a] = p$. In particular, p divides $|G|$, so there is an element $g \in G$ of order p, which ~~is therefore~~ must be a p-cycle $(a_0\ a_1 \ldots a_{p-1})$.

The correspondence $a_i \longmapsto i + p\mathbb{Z}$ gives the necessary identification. □

<u>Theorem 6.2</u>: If G is a transitive subgroup of S_{F_p} (p a prime) that is not doubly transitive, and if $C_p \subset G$, then

$$G = \{T_{a,b} : a \in H, b \in F_p\}$$

where $T_{a,b}(x) = ax + b$ and H is a subgroup of $F_p^* = F_p - \{1\}$, the group of units of F_p. □

A proof of Theorem 6.2 may be found in [4], pp. 51-56.

Let $\Gamma = (V, D)$ be a v.-t. digraph, where $p = |V|$ is prime.

<u>Proposition 6.3</u>: Γ is a group-digraph.

~~Pf: If A(Γ) is doubly transitive then Γ is either the empty digraph or the complete digraph, both of which are group digraphs. If A(Γ) is not doubly transitive~~

Pf: Identify V with F_p according to 6.1. Then $A(\bar{\Gamma})$ contains the regular subgroup C_p, so by 4.7 $\bar{\Gamma}$ is a group-digraph. □

Corollary 6.4: A v.-t. graph with a prime number of points is a group-graph. □

Now let $\bar{\Gamma}$ be a v.-t. digraph with a prime number p of points. By 6.3 we may assume that $\bar{\Gamma} = \bar{\Gamma}_{F_p, A}$, where $A \subset F_p^*$. Assume that $\bar{\Gamma}$ is neither the empty digraph nor the complete digraph (i.e. $A \neq \phi, F_p^*$), so that $A(\bar{\Gamma})$ is not doubly transitive, and let H be the subgroup of F_p^* described by 6.2. If $h \in H$, then

$$\begin{aligned} a \in A &\Leftrightarrow (0, a) \in D_A \\ &\Leftrightarrow (0, ah) \in D_A \\ &\Leftrightarrow ah \in A, \end{aligned}$$

so A consists of cosets of H. Conversely, if H' is a subgroup of F_p^* such that A consists of cosets of H', then for $h \in H'$,

$$\begin{aligned} (a,b) \in D &\Leftrightarrow b - a \in A \\ &\Leftrightarrow h(b-a) = hb - ha \in A \\ &\Leftrightarrow (ha, hb) \in D, \end{aligned}$$

so $T_{h,0} \in A(\bar{\Gamma})$, hence $h \in H$. We have thus shown

<u>Proposition 6.5</u>: If H' is a subgroup of F_p^*, then $H' \subset H \iff A$ consists of cosets of H'.

If $\overline{\Gamma}$ corresponds to an undirected graph, i.e. if $A = -A$, then A consists of cosets of $\{1,-1\}$, so by 6.5 $\{1,-1\} \subset H$, and

$$D_p = \{ x \mapsto x+b,\ x \mapsto -x+b : b \in F_p \} \subset A(\overline{\Gamma}).$$

(This is obviously also true of the complete & empty graphs.) This shows that the automorphism group of a ~~graph~~ v.-t. graph with a prime number $p \geq 3$ of points contains a subgroup isomorphic to the dihedral group of order $2p$. In particular, the automorphism group of such a graph cannot be regular.

Let $\overline{\Gamma}$, p, A, H, etc. be as ~~before~~ before. The subset $A \subset F_p^*$ is not determined uniquely by $\overline{\Gamma}$: Indeed, if $a \in F_p^*$ then $\overline{\Gamma} = \overline{\Gamma}_{F_p, A} \cong \overline{\Gamma}_{F_p, aA}$ because $(b,c) \in D_A \iff c-b \in A \iff ac-ab \in aA \iff$ ~~$\iff$~~ $(ab, ac) \in D_{aA}$, so the map $b \mapsto ab$ is an isomorphism of $\overline{\Gamma}_{F_p,A}$ onto $\overline{\Gamma}_{F_p, aA}$. The group H, however, <u>is</u> determined uniquely by $\overline{\Gamma}$. One way to see this is to notice that H is the unique subgroup of F_p^* of order $|A(\overline{\Gamma})|/p$. The uniqueness of H also follows from the following theorem:

Theorem 6.6: $\Gamma_{F_p,A} \cong \Gamma_{F_p,B} \Leftrightarrow \exists c \in F_p^*$ such that $B = cA$

Pf: This is obviously true if either $\Gamma_{F_p,A}$ or $\Gamma_{F_p,B}$ is the empty digraph or the complete digraph, so we may assume that the automorphism groups of both graphs are not doubly transitive.

$\Leftarrow$: This has just been shown.

$\Rightarrow$: If $\Gamma = (V, D)$ is any digraph let $n = |V|$ and label the vertices $a_0, \dots, a_{n-1}$. Let M be the matrix $(m_{ij})_{i,j=0,\dots,n-1}$,

$$m_{ij} = \begin{cases} 0 & \text{if } (a_i, a_j) \notin D \\ 1 & \text{if } (a_i, a_j) \in D. \end{cases}$$

M will be regarded as a matrix over the complex numbers, ~~M is~~ called the adjacency matrix of Γ: It is determined up to simultaneous permutation of the rows and columns, corresponding to relabeling of the vertices. The characteristic polynomial of M, which is invariant under this kind of transformation, is thus uniquely determined by Γ, and is called the characteristic polynomial of Γ.

In the case at hand, let the ordering of the points of F_p be such that an equivalence class modulo p corresponds to its reduced representative, and let M_A (M_B) be the adjacency matrix of $\Gamma_{F_p,A}$ ($\Gamma_{F_p,B}$) relative to this ordering. Let f_A and f_B be the corresponding characteristic polynomials. $\Gamma_{F_p,A} \cong \Gamma_{F_p,B}$ $\Rightarrow f_A = f_B$.

Now $M_A = \sum_{a \in A} S^a$,
where

$$S = M_{\{1\}} = \begin{pmatrix} 0100 & \cdots & 0 \\ 0010 & \cdots & 0 \\ 0001 & \cdots & 0 \\ \vdots & & \\ 0000 & \cdots & 1 \\ 1000 & \cdots & 0 \end{pmatrix}$$

~~If we regard~~ (S^a is well-defined because $S^p = Id_p$). ~~Regarding S as a linear transformation of $\mathbb{C}^p$ into itself, the~~ ~~Let us regard S as a linear transformation~~ The vector $v_k = (1, \omega^k, \ldots, \omega^{k(p-1)})$ is easily seen to be an eigenvector of S, corresponding to the eigenvalue ω^k, where ω is a primitive p-th root of unity. Since $1, \omega, \ldots, \omega^{p-1}$ are distinct, the vectors $v_0, v_1, \ldots, v_{p-1}$ form a ~~[illegible]~~ basis of $\mathbb{C}^p$. But v_k is also an eigenvector of $M_A = \sum_{a \in A} S^a$, with ~~corresponding to the~~ eigenvalue $\alpha_k = \sum_{a \in A} \omega^{ak}$, and since $v_0, \ldots, v_{p-1}$ is a basis of $\mathbb{C}_p$,

$$f_A(X) = \prod_{k=0}^{p-1} (X - \alpha_k)$$

Similarly,

$$f_B(X) = \prod_{k=0}^{p-1} (X - \beta_k),$$

where $\beta_k = \sum_{b \in B} \omega^{bk}$.

Now $\alpha_0 = \frac{|D_A|}{p} = \frac{|D_B|}{p} = \beta_0$, since $\overline{\Gamma}_{F_p, A} \cong \overline{\Gamma}_{F_p, B}$. Since $f_A = f_B$, this implies that

$$\frac{f_A}{X - \alpha_0} = \frac{f_B}{X - \beta_0},$$

so in particular $\beta_k = \alpha_1$ for some k, $1 \le k \le p-1$. But the set $\{\omega, \omega^2, \ldots, \omega^{p-1}\}$ of primitive pth roots of unity is linearly independent over the field $\mathbb{Q}$ of rational numbers and $A, kB \subset F_p^*$, so

$$\alpha_1 = \sum_{a \in A} \omega^a = \sum_{b \in B} \omega^{ak}$$

$$\Rightarrow A = kB.$$

Setting $c = (k + p\mathbb{Z})^{-1}$, we have $B = cA$. $\square$

An examination of the foregoing proof shows that in proving that $\overline{\Gamma}_{F_p, A} \cong \overline{\Gamma}_{F_p, B} \Rightarrow B = cA$, $c \in F_p^*$, we used only the fact that $f_A = f_B$, except in the proof that $\alpha_0 = \beta_0$: But this also follows from the assumption that $f_A = f_B$, since α_0 (β_0) is the root of f_A (f_B) — i.e. the eigenvalue of M_A (M_B) — having the largest absolute value. Thus we have shown:

<u>Corollary 6.7</u>: Two v.-t. digraphs Γ, Γ' having p points are isomorphic iff and only if they have the same characteristic polynomial. □

We will say that two subsets $A, B \subset F_p^*$ are equivalent if $\exists c \in F_p^*$ such that $B = cA$. Theorem 6.6 tells us that to count the isomorphism classes of vertex-transitive digraphs with p points, it suffices to count inequivalent subsets of F_p^*. F_p^* is cyclic of order $p-1$, so this amounts to counting inequivalent subsets of $\{0, \dots, p-2\}$, where two subsets $A, B \subset \{0, \dots, p-2\}$ are equivalent if $B = \phi(A)$ for some $\phi \in C_{p-1} = \{(0\ 1\ \dots\ p-1)^i : i = 0, \dots, p-2\}$. This is done with the help of Polya's enumeration theorem (see [3], pp. 180-184). The cycle index of the cyclic group C_{p-1} is given by the formula

$$Z_{C_{p-1}}(x_1, \dots, x_{p-1}) = \frac{1}{p-1} \sum_{d \mid p-1} \phi(d)\, x_d^{\frac{p-1}{d}},$$

(where ϕ denotes the Euler ϕ-function), so the counting series for these subsets is

$Z_{C_{p-1}}(1+x, 1+x^2, \ldots, 1+x^{p-1})$. The coefficient of x^i in this polynomial is the number of subsets with i elements, i.e. the number of v.-t. digraphs with pi edges. The total number of v.-t. digraphs is $Z_{C_{p-1}}(2, 2, \ldots, 2)$.

To count the isomorphism classes of undirected v.-t. graphs with p points, we need to count the inequivalent subsets of F_p^* ~~with the property~~ ~~that~~ that consist of cosets of $\{1,-1\}$. This is the same as counting inequivalent subsets of $F_p^*/\{1,-1\}$, where two subsets $A, B \subset F_p^*/\{1,-1\}$ are equivalent if $B = cA$ for some $c \in F_p^*/\{1,-1\}$. But $F_p^*/\{1,-1\}$ is cyclic of order $\frac{p-1}{2}$, and proceeding as before we find that the counting sequence for these subsets is

$$Z_{C_{\frac{p-1}{2}}}(1+x, 1+x^2, \ldots, 1+x^{\frac{p-1}{2}})$$

where

$$Z_{C_{\frac{p-1}{2}}}(x_1, \ldots, x_{\frac{p-1}{2}}) = \frac{1}{(p-1/2)} \sum_{d \mid \frac{p-1}{2}} \phi(d)\, x_d^{\frac{p-1}{2d}}$$

The coefficient of x^i gives the number of subsets of $F_p^*/\{1,-1\}$ with i elements, which is the number of v.-t. graphs with pi edges. The total number of v.-t. graphs is $Z_{C_{\frac{p-1}{2}}}(2, 2, \ldots, 2)$.

~~These results can easily be extended to give the number of digraphs (graphs) whose automorphism group is isomorphic to~~

~~[illegible]~~

7. Symmetric graphs with a prime number of points.

The results of the last section allow us to describe all symmetric graphs ~~with a prime~~ and digraphs with a prime number of points.

Proposition 7.1: A symmetric graph with a prime number of vertices is 1-transitive

Pf: A sym. graph $\Gamma = (V, E)$ is 1-t. $\iff \forall \{a,b\} \in E$, $\exists\, \phi \in A(\Gamma)$, $\phi(a) = b$, $\phi(b) = a$. But if $|V|$ is prime then by the remarks following 6.5 $\forall a, b \in V$ $\exists \phi \in A(\Gamma)$, $\phi(a) = b$, $\phi(b) = a$ regardless of whether or not $\{a, b\} \in V$. $\square$

Let p be a prime.
Proposition 7.2: If $n \mid p-1$, let H be the subgroup of F_p^* of order n. Then $\Gamma_{F_p, H}$ is a symmetric digraph of degree n. Any symmetric digraph with

p points is isomorphic to one of these graphs.

<u>Pf</u>: The empty digraph and the complete digraph are clearly symmetric: This takes care of the cases $n=0, p-1$. If $n \neq 0, p-1$ the by 6.5

$$
\begin{aligned}
A(\Gamma_{F_p,H}) &= \{T_{a,b} : a \in H,\ b \in F_p\} \\
&\Rightarrow A(\Gamma_{F_p,H})_0 = H \\
&\Rightarrow D_H(0) = H \text{ is an orbit of } A(\Gamma_{F_p,H})_0 \\
&\Rightarrow D_H \text{ is an orbital of } A(\Gamma_{F_p,H}) \\
&\Rightarrow \Gamma_{F_p,H} \text{ is symmetric.}
\end{aligned}
$$

If, on the other hand, Γ is symmetric, and is neither the complete digraph nor the empty digraph, then $\Gamma \cong \Gamma_{F_p,A}$, ~~Let H be defined as~~ $A \subset F_p^*$. Let H be defined as usual, so that

$$A(\Gamma_{F_p,A}) = \{T_{a,b} : a \in H,\ b \in F_p\}$$

Since Γ is symmetric A consists of a single ~~can contain no more than one~~ orbit of $A(\Gamma_{F_p,A})_0$. But the non-trivial orbits of $A(\Gamma_{F_p,A})$ are the cosets of H, so A is a coset of H, so $\Gamma \cong \Gamma_{F_p,A} \cong \Gamma_{F_p,H}$ by 6.6 □

41.

Corollary 7.3: ~~There exists~~ Let p be prime. There exists a sym. graph with p points regular of degree $n \Leftrightarrow n$ is even and $n | p-1$. ~~[illegible]~~ Any two such graphs are isomorphic.

Pf: By 7.1 every symmetric graph is 1-transitive, that is, it corresponds to a symmetric digraph. But the digraph $\Gamma_{F_p, H}$ corresponds to an undirected graph ~~if and only if $H = -H$, which is the case if and only if $-1 \in H$~~ $\Leftrightarrow H = -H \Leftrightarrow -1 \in H \Leftrightarrow n = |H|$ is even. $\square$

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