I

Compound statements

1. PURPOSE OF THE THEORY

A *statement* is a verbal or written assertion. In the English language such assertions are made by means of declarative sentences. For example, "It is snowing" and "I made a mistake in signing up for this course" are statements.

The two statements quoted above are *simple statements*. A combination of two or more simple statements is a *compound statement*. For example, "It is snowing, and I wish that I were out of doors, but I made the mistake of signing up for this course," is a compound statement.

It might seem natural that one should make a study of simple statements first, and then proceed to the study of compound ones. However, the reverse order has proved to be more useful. Because of the tremendous variety of simple statements, the theory of such statements is very complex. It has been found in mathematics that it is often fruitful to assume for the moment that a difficult problem has been solved and then to go on to the next problem. Therefore we shall proceed as if we knew all about simple statements and study only the way they are compounded. The latter is a relatively easy problem.

While the first systematic treatment of such problems is found in the writings of Aristotle, mathematical methods were first employed by George Boole about 100 years ago. The more polished techniques now available are the product of twentieth century mathematical logicians.

The fundamental property of any statement is that it is either true or false (and that it cannot be both true and false). Naturally, we are interested in finding out which is the case. For a compound statement it is sufficient to know which of its components are true, since the truth values (i.e., the truth or falsity) of the components determine in a way to be described later the truth value of the compound.

Our problem then is twofold: (1) In how many different ways can statements be compounded? (2) How do we determine the truth value of a compound statement given the truth values of its components?

Let us prepare our mathematical tools. In any mathematical formula we find three kinds of symbols: *constants*, *variables*, and *auxiliary symbols*. For example, in the formula $(x + y)^2$ the plus sign and the exponent are constants, the letters x and y are variables, and the parentheses are auxiliary symbols. Constants are symbols whose meanings in a given context are fixed. Thus in the formula given above, the plus sign indicates that we are to form the sum of the two numbers x and y, while the exponent 2 indicates that we are to multiply $(x + y)$ by itself. Variables always stand for entities of a given kind, but they allow us to leave open just which particular entity we have in mind. In our example above the letters x and y stand for unspecified numbers. Auxiliary symbols function somewhat like punctuation marks. Thus if we omit the parentheses in the expression above we obtain the formula $x + y^2$, which has quite a different meaning than the formula $(x + y)^2$.

In this chapter we shall use variables of only one kind. We indicate these variables by the letters p, q, r, etc., which will stand for unspecified statements. These statements frequently will be simple statements but may also be compound. In any case we know that, since each variable stands for a statement, it has an (unknown) truth value.

The constants that we shall use will stand for certain connectives used in the compounding of statements. We will have one symbol for forming the negation of a statement and several symbols for combining two statements. It will not be necessary to introduce symbols for the compounding of three or more statements, since we can show that the same combination can also be formed by compounding them two at a time. In practice only a small number of basic constants are used and the others are defined in terms of these. It is even possible to use only a single connective! (See Section 4, Exercises 10 and 11.)

The auxiliary symbols that we shall use are, for the most part, the same ones used in elementary algebra. Any case where the usage is different will be explained.

Examples. As examples of simple statements, let us take "The weather is nice" and "It is very hot." We will let p stand for the former and q for the latter.

Suppose we wish to make the compound statement that both are true, "The weather is nice *and* it is very hot." We shall symbolize this statement by $p \wedge q$. The symbol $\wedge$, which can be read "and," is our first connective.

In place of the strong assertion above we might want to make the weak (cautious) assertion that one or the other of the statements is true. "The weather is nice *or* it is very hot." We symbolize this assertion by $p \vee q$. The symbol $\vee$, which can be read "or," is the second connective which we shall use.

Suppose we believed that one of the statements above was false, for example, "It is *not* very hot." Symbolically we would write $\sim q$. Our third connective is then $\sim$, which can be read "not."

More complex compound statements can now be made. For example, $p \wedge \sim q$ stands for "The weather is nice *and* it is *not* very hot."

EXERCISES

1. The following are compound statements or may be so interpreted. Find their simple components.

(a) It is hot and it is raining.

(b) It is hot but it is not very humid.

[*Ans.* "It is hot"; "it is very humid."]

(c) It is raining or it is very humid.

(d) Jack and Jill went up the hill.

(e) The murderer is Jones or Smith.

(f) It is neither necessary nor desirable.

(g) Either Jones wrote this book or Smith did not know who the author was.

2. In Exercise 1 assign letters to the various components, and write the statements in symbolic form. [*Ans.* (b) $p \wedge \sim q$.]

3. Write the following statements in symbolic form, letting p be "Fred is smart" and q be "George is smart."

(a) Fred is smart and George is stupid.
(b) George is smart and Fred is stupid.
(c) Fred and George are both stupid.
(d) Either Fred is smart or George is stupid.
(e) Neither Fred nor George is smart.
(f) Fred is not smart, but George is stupid.
(g) It is not true that Fred and George are both stupid.

4. Assume that Fred and George are both smart. Which of the seven compound statements in Exercise 3 are true?

5. Write the following statements in symbolic form.

(a) Fred likes George. (Statement p.)
(b) George likes Fred. (Statement q.)
(c) Fred and George like each other.
(d) Fred and George dislike each other.
(e) Fred likes George, but George does not reciprocate.
(f) George is liked by Fred, but Fred is disliked by George.
(g) Neither Fred nor George dislikes the other.
(h) It is not true that Fred and George dislike each other.

6. Suppose that Fred likes George and George dislikes Fred. Which of the eight statements in Exercise 5 are true?

7. For each statement in Exercise 5 give a condition under which it is false. [*Ans.* (c) Fred does not like George.]

8. Let p be "Stock prices are high" and q be "Stocks are rising." Give a verbal translation for each of the following.

(a) $p \wedge q$.
(b) $p \wedge \sim q$.
(c) $\sim p \wedge \sim q$.
(d) $p \vee \sim q$.
(e) $\sim(p \wedge q)$.
(f) $\sim(p \vee q)$.
(g) $\sim(\sim p \vee \sim q)$.

9. Using your answers to Exercise 8, parts (e), (f), (g), find simpler symbolic statements expressing the same idea.

10. Let p be "I have a dog" and q be "I have a cat." Translate into English and simplify: $\sim[\sim p \vee \sim\sim q] \wedge \sim\sim p$.

2. *THE MOST COMMON CONNECTIVES*

The truth value of a compound statement is determined by the truth values of its components. When discussing a connective we will want to know just how the truth of a compound statement made from this connective depends upon the truth of its components. A very convenient way of tabulating this dependency is by means of a *truth table.*

Let us consider the compound $p \wedge q$. Statement p could be either true or false and so could statement q. Thus there are four possible pairs of truth values for these statements and we want to know in each case whether or not the statement $p \wedge q$ is true. The answer is straightforward: If p and q are both true, then $p \wedge q$ is true, and otherwise $p \wedge q$ is false. This seems reasonable since the assertion $p \wedge q$ says no more and no less than that p and q are both true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1

Figure 1 gives the truth table for $p \wedge q$, the *conjunction* of p and q. The truth table contains all the information that we need to know about the connective $\wedge$, namely it tells us the truth value of the conjunction of two statements given the truth values of each of the statements.

We next look at the compound statement $p \vee q$, the *disjunction* of p and q. Here the assertion is that one or the other of these statements is true. Clearly, if one statement is true and the other false, then the disjunction is true, while if both statements are false, then the disjunction is certainly false. Thus we can fill in the last three rows of the truth table for disjunction (see Figure 2).

p	q	$p \vee q$
T	T	?
T	F	T
F	T	T
F	F	F

Figure 2

Observe that one possibility is left unsettled, namely, what happens if both components are true? Here we observe that the everyday usage of "or" is ambiguous. Does "or" mean "one or the other or both" or does it mean "one or the other but not both"?

Let us seek the answer in examples. The sentence "This summer I will date Jean or Pat" allows for the possibility that the speaker may date both girls. However the sentence "I will go to Dartmouth or to Princeton" indicates that only one of these schools will be chosen. "I will buy a TV set or a phonograph next year" could be used in either sense; the speaker may mean that he is trying to make up his mind which one of the two to buy, but it could also mean that he will buy *at least one* of these—possibly both. We see that sometimes the context makes the meaning clear, but not always.

A mathematician would never waste his time on a dispute as to which usage "should" be called the disjunction of two statements. Rather he recognizes two perfectly good usages, and calls one the *inclusive disjunction* (p or q or both) and the other the *exclusive disjunction* (p or q but not both). The symbol $\vee$ will be used for inclusive disjunction, and the symbol $\underline{\vee}$ will be used for exclusive disjunction. The truth tables for each of these are found in Figures 3 and 4. Unless we state otherwise, our disjunctions will be inclusive disjunctions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 3

p	q	$p \underline{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Figure 4

The last connective which we shall discuss in this section is *negation.* If p is a statement, the symbol $\sim p$, called the negation of p, asserts that p is false. Hence $\sim p$ is true when p is false, and false when p is true. The truth table for negation is shown in Figure 5.

p	$\sim p$
T	F
F	T

Figure 5

Besides using these basic connectives singly to form compound statements, several can be used to form a more complicated compound statement, in much the same way that complicated algebraic expressions can be formed by means of the basic arithmetic operations. For example, $\sim(p \wedge q)$, $p \wedge \sim p$, and $(p \vee q) \vee \sim p$ are all compound statements. They are to be read "from the inside out" in the same way that algebraic expressions are, namely, quantities inside the innermost parentheses are first grouped together, then these parentheses are grouped together, etc. Each compound statement has a truth table which can be constructed in a routine way. The following examples show how to construct truth tables.

Example 1. Consider the compound statement $p \vee \sim q$. We begin the construction of its truth table by writing in the first two columns the four possible pairs of truth values for the statements p and q.

Then we write the proposition in question, leaving plenty of space between symbols so that we can fill in columns below. Next we copy the truth values of p and q in the columns below their occurrences in the proposition. This completes step 1; see Figure 6.

p	q	p	$\vee$	$\sim q$
T	T	T		T
T	F	T		F
F	T	F		T
F	F	F		F
Step No.		1		1

Figure 6

Next we treat the innermost compound, the negation of the variable q, completing step 2, see Figure 7.

p	q	p	$\vee$	$\sim$	q
T	T	T		F	T
T	F	T		T	F
F	T	F		F	T
F	F	F		T	F
Step No.		1		2	1

Figure 7

Finally we fill in the column under the disjunction symbol, which gives us the truth value of the compound statement for various truth values of its variables. To indicate this we place two parallel lines on each side of the final column, completing step 3 as in Figure 8.

p	q	p	$\vee$	$\sim$	q
T	T	T	T	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	T	T	F
Step No.		1	3	2	1

Figure 8

The next two examples show truth tables of more complicated compounds worked out in the same manner. There are only two basic rules which the student must remember when working these: first, work from the "inside out"; second, the truth values of the compound statement are found in the last column filled in during this procedure.

Example 2. The truth table for the statement $(p \vee \sim q) \wedge \sim p$ together with the numbers indicating the order in which the columns are filled in appears in Figure 9.

p	q	$(p$	$\vee$	$\sim$	$q)$	$\wedge$	$\sim$	p
T	T	T	T	F	T	F	F	T
T	F	T	T	T	F	F	F	T
F	T	F	F	F	T	F	T	F
F	F	F	T	T	F	T	T	F
Step No.		1	3	2	1	4	2	1

Figure 9

Example 3. The truth table for the statement $\sim[(p \wedge q) \vee (\sim p \wedge \sim q)]$ together with the numbers indicating the order in which the columns are filled appears in Figure 10. We note that the compound statement has the same truth table as $p \veebar q$. These two statements are *equivalent* (see Section 7).

p	q	$\sim$	$[(p$	$\wedge$	$q)$	$\vee$	$(\sim$	p	$\wedge$	$\sim$	$q)]$
T	T	F	T	T	T	T	F	T	F	F	T
T	F	T	T	F	F	F	F	T	F	T	F
F	T	T	F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	T	T	F	T	T	F
Step No.		5	1	2	1	4	2	1	3	2	1

Figure 10

EXERCISES

1. Give a compound statement which symbolically states "p or q but not both," using only $\sim$, $\vee$, and $\wedge$.

2. Construct the truth table for your answer to Exercise 1, and compare this with Figure 4.

3. Construct the truth table for the symbolic form of each statement in Exercise 3 of Section 1. How does Exercise 4 of Section 1 relate to these truth tables?

4. Construct a truth table for each of the following.

(a) $\sim(p \wedge q)$. [*Ans.* FTTT.]
(b) $p \wedge \sim p$. [*Ans.* FF.]
(c) $(p \vee q) \vee \sim p$. [*Ans.* TTTT.]
(d) $\sim[(p \vee q) \wedge (\sim p \vee \sim q)]$. [*Ans.* TFFT.]

5. Let p stand for "Jones passed the course" and q stand for "Smith passed the course" and translate into symbolic form the statement "It is not the case that Jones and Smith both failed the course." Construct a truth table for this compound statement. State *in words* the circumstances under which the statement is true.

6. Construct a simpler statement about Jones and Smith that has the same truth table as the one in Exercise 5.

7. Let $p \mid q$ express that "p and q are not both true." Write a symbolic expression for $p \mid q$ using $\sim$ and $\wedge$.

8. Write a truth table for $p \mid q$.

9. Write a truth table for $p \mid p$. [*Ans.* Same as Figure 5.]

10. Write a truth table for $(p \mid q) \mid (p \mid q)$. [*Ans.* Same as Figure 1.]

11. Construct a truth table for each of the following.

(a) $\sim(p \vee q) \vee \sim(q \vee p)$. [*Ans.* FFFT.]
(b) $\sim(p \vee q) \wedge p$. [*Ans.* FFFF.]
(c) $\sim(p \veebar q)$. [*Ans.* TFFT.]
(d) $\sim(p \mid q)$. [*Ans.* TFFF.]

12. Construct two symbolic statements, using only $\sim$, $\vee$, and $\wedge$, which have the following truth tables (a) and (b), respectively.

p	q	(a)	(b)
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

13. Using only $\sim$ and $\veebar$, construct a compound statement having the same truth table as:

(a) $p \vee q$.

(b) $p \wedge q$.

3. *OTHER CONNECTIVES*

Suppose we did not wish to make an outright assertion but rather an assertion containing a condition. As examples, consider the following sentences. "If the weather is nice, I will take a walk." "If the following statement is true, then I can prove the theorem." "If the cost of living continues to rise, then the government will impose rigid curbs." Each of these statements is of the form "*if p then q*." The *conditional* is then a new connective which is symbolized by the arrow $\rightarrow$.

Of course the precise definition of this new connective must be made by means of a truth table. If both p and q are true, then $p \rightarrow q$ is certainly true, and if p is true and q false, then $p \rightarrow q$ is certainly false. Thus the first two lines of the truth table can easily be filled in, see Figure 11a. Suppose now that p is false; how shall we fill in the last two lines of the truth table in Figure 11a? At first thought one might suppose that it would be best to leave it completely undefined. However, to do so would violate our basic principle that a statement is either true or false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	?
F	F	?

Figure 11a

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 11b

Therefore we make the completely arbitrary decision that the conditional, $p \rightarrow q$, is *true* whenever p is false, regardless of the truth value of q. This decision enables us to complete the truth table for the conditional and it is given in Figure 11b. A glance at this truth table shows that the conditional $p \rightarrow q$ is considered false only if p is true and q is false. If we wished, we might rationalize the arbitrary decision made above by saying that if statement p happens to be false, then we give the conditional $p \rightarrow q$ the "benefit of the doubt" and consider it true. (For another reason, see Exercise 1.)

In everyday conversation it is customary to combine simple statements only if they are somehow related. Thus we might say "It is raining today and I will take an umbrella," but we would not say "I read a good book and I will take an umbrella." However, the rather ill-defined concept of relatedness is difficult to enforce. Concepts related to each other in one person's mind need not be related in another's. In our study of compound statements no requirement of relatedness is imposed on two statements in order that they be compounded by any of the connectives. This freedom sometimes produces strange results in the use of the conditional. For example, according to the truth table in Figure 11b, the statement "If $2 \times 2 = 5$, then black is white" is true, while the statement "If $2 \times 2 = 4$, then cows are monkeys" is false. Since we use the "if . . . then . . ." form usually only when there is a causal connection between the two statements, we might be tempted to label both of the above statements as nonsense. At this point it is important to remember that no such causal connection is intended in the usage of $\rightarrow$; the meaning of the conditional is contained in Figure 11b and nothing more is intended. This point will be discussed again in Section 7 in connection with implication.

Closely connected to the conditional connective is the *biconditional* statement, $p \leftrightarrow q$, which may be read "*p if and only if q*." The biconditional statement asserts that if p is true, then q is true, and if p is false, then q is false. Hence the biconditional is true in these cases and false in the others, so that its truth table can be filled in as in Figure 12.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 12

The biconditional is the last of the five connectives which we shall use in this chapter. The table below gives a summary of them together with the numbers of the figures giving their truth tables. Remember that the complete definition of each of these connectives is given by its truth table. The examples below show the use of the two new connectives.

Name	Symbol	Translated as	Truth Table
Conjunction	$\wedge$	"and"	Figure 1
Disjunction (inclusive)	$\vee$	"or"	Figure 3
Negation	$\sim$	"not"	Figure 5
Conditional	$\rightarrow$	"if . . . then . . ."	Figure 11b
Biconditional	$\leftrightarrow$	". . . if and only if . . ."	Figure 12

Examples. In Figures 13 and 14 the truth tables of two statements are worked out following the procedure of Section 2.

p	q	p	$\rightarrow$	$(p$	$\vee$	$q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F
Step No.		1	3	1	2	1

Figure 13

p	q	$\sim$	p	$\leftrightarrow$	$(p$	$\rightarrow$	$\sim$	$q)$
T	T	F	T	T	T	F	F	T
T	F	F	T	F	T	T	T	F
F	T	T	F	T	F	T	F	T
F	F	T	F	T	F	T	T	F
Step No.		2	1	4	1	3	2	1

Figure 14

It is also possible to form compound statements from three or more simple statements. The next example is a compound formed from three simple statements p, q, and r. Notice that there will be a total of eight possible triples of truth values for these three statements so that the truth table for our compound will have eight rows as shown in Figure 15.

p	q	r	$[p$	$\rightarrow$	$(q$	$\vee$	$r)]$	$\wedge$	$\sim$	$[p$	$\leftrightarrow$	$\sim$	$r]$
T	T	T	T	T	T	T	T	T	T	T	F	F	T
T	T	F	T	T	T	T	F	F	F	T	T	T	F
T	F	T	T	T	F	T	T	T	T	T	F	F	T
T	F	F	T	F	F	F	F	F	F	T	T	T	F
F	T	T	F	T	T	T	T	F	F	F	T	F	T
F	T	F	F	T	T	T	F	T	T	F	F	T	F
F	F	T	F	T	F	T	T	F	F	F	T	F	T
F	F	F	F	T	F	F	F	T	T	F	F	T	F
Step No.			1	3	1	2	1	5	4	1	3	2	1

Figure 15

EXERCISES

1. One way of filling in the question-marked squares in Figure 11a is given in Figure 11b. There are three other possible ways.

(a) Write the three other truth tables.

(b) Show that each one of these truth tables has an interpretation in terms of the connectives now available to us.

2. Write truth tables for $q \vee p, q \wedge p, q \rightarrow p, q \leftrightarrow p$. Compare these with the truth tables in Figures 3, 1, 11b, and 12, respectively.

3. Construct truth tables for

(a) $p \rightarrow (q \vee r)$. [*Ans.* TTTFTTTT.]

(b) $(p \vee r) \wedge (p \rightarrow q)$. [*Ans.* TTFFTFTF.]

(c) $(p \vee q) \leftrightarrow (q \vee p)$. [*Ans.* TTTT.]

(d) $p \wedge \sim p$. [*Ans.* FF.]

(e) $(p \rightarrow p) \vee (p \rightarrow \sim p)$. [*Ans.* TT.]

(f) $(p \vee \sim q) \wedge r$. [*Ans.* TFTFFFTF.]

(g) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$. [*Ans.* TTTTTTTT.]

4. For each of the following statements (i) find a symbolic form, and (ii) construct the truth table. Use the notation: p for "Joe is smart," q for "Jim is stupid," r for "Joe will get the prize."

(a) If Joe is smart and Jim is stupid, then Joe will get the prize.
[*Ans.* TFTTTTTT.]

(b) Joe will get the prize if and only if either he is smart or Jim is stupid.
[*Ans.* TFTFTFFT.]

(c) If Jim is stupid but Joe fails to get the prize, then Joe is not smart.
[*Ans.* Same as (a).]

5. Construct truth tables for each of the following, and give an interpretation.

(a) $(p \rightarrow q) \wedge (q \rightarrow p)$. (Compare with Figure 12.)
(b) $(p \wedge q) \rightarrow p$.
(c) $q \rightarrow (p \vee q)$.
(d) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$.

6. The truth table for a statement compounded from two simple statements has four rows, and the truth table for a statement compounded from three simple statements has eight rows. How many rows would the truth table for a statement compounded from four simple statements have? How many for five? For n? Devise a systematic way of writing down these latter truth tables.

7. Let p be "It is raining" and q be "The wind is blowing." Translate each of the following into symbolic form.

(a) If it rains, then the wind blows.
(b) If the wind blows, then it rains.
(c) The wind blows if and only if it rains.
(d) If the wind blows, then it does not rain.
(e) It is not the case that the wind blows if and only if it does not rain.

8. Construct truth tables for the statements in Exercise 7.
[*Ans.* TFTT; TTFT; TFFT; FTTT; TFFT.]

9. Construct truth tables for

(a) $(p \vee q) \leftrightarrow (\sim r \wedge \sim s)$.
(b) $(p \wedge q) \rightarrow \sim[\sim p \wedge (r \vee s)]$.

10. Construct a truth table for $\sim[(\sim p \wedge \sim q) \wedge (p \vee r)]$.
[*Ans.* TTTTTTFT.]

11. Find a simpler statement having the same truth table as the one found in Exercise 10.

SUPPLEMENTARY EXERCISES

12. A compound statement in p and q must have one of 16 possible truth tables. Find all of these tables.

13. For the 16 truth tables found in Exercise 12, show that eight represent negations of eight others.

14. Find a simple compound statement for each of the 16 truth tables found in Exercise 12. [*Hint:* Use the result of Exercise 13.]

15. Construct the truth table of

$$p \rightarrow ((r \vee q) \leftrightarrow \sim(r \wedge s))$$

16. Show that the truth table in Exercise 15 can be constructed much more quickly by identifying the cases in which the statement is false.

[*Ans.* False in cases TTTT, TFTT, TFFT, TFFF.]

*4. STATEMENTS HAVING GIVEN TRUTH TABLES

In the preceding two sections we showed how to construct the truth table for any compound statement. It is also interesting to consider the converse problem, namely, given a truth table to find one or more statements having this truth table. The converse problem always has a solution and, in fact, a solution using only the connectives $\wedge$, $\vee$, and $\sim$. The discussion which we give here is valid only for a truth table in three variables but can easily be extended to cover the case of n variables.

As observed in the last section, a truth table with three variables has eight rows, one for each of the eight possible triples of truth values. Suppose that our given truth table has its last column consisting entirely of F's. Then it is easy to check that the truth table of the statement $p \wedge \sim p$ also has only F's in its last column, so that this statement serves as an answer to our problem. We now need consider only truth tables having one or more T's. The method that we shall use is to construct statements that are true in one case only, and then to construct the desired statement as a disjunction of these.

It is not hard to construct statements that are true in only one case. In Figure 16 are listed eight such statements, each true in exactly

p	q	r	Basic Conjunctions
T	T	T	$p \wedge q \wedge r$
T	T	F	$p \wedge q \wedge \sim r$
T	F	T	$p \wedge \sim q \wedge r$
T	F	F	$p \wedge \sim q \wedge \sim r$
F	T	T	$\sim p \wedge q \wedge r$
F	T	F	$\sim p \wedge q \wedge \sim r$
F	F	T	$\sim p \wedge \sim q \wedge r$
F	F	F	$\sim p \wedge \sim q \wedge \sim r$

Figure 16

one case. We shall call such statements *basic conjunctions.* Such a basic conjunction contains each variable or its negation, depending on whether the line on which it appears in Figure 16 has a T or an F under the variable. Observe that the disjunction of two such basic conjunctions will be true in exactly two cases, the disjunction of three in three cases, etc. Therefore, to find a statement having a given truth table simply form the disjunction of those basic conjunctions which occur in Figure 16 on the rows where the given truth table has T's.

Example 1. Find a statement whose truth table has T's in the first, second, and last rows, and F's in the other rows. The required statement is the disjunction of the first, second, and eighth basic conjunctions, that is,

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r).$$

Exercise 2 asks the student to show that this statement has the required truth table.

Example 2. A logician is captured by a tribe of savages and placed in a jail having two exits. The savage chief offers the captive the following chance to escape: "One of the doors leads to certain death and the other to freedom. You can leave by either door. To help you in making a decision, two of my warriors will stay with you and answer any one question which you wish to ask of them. I must warn you, however, that one of my warriors is completely truthful while the other always lies." The chief then leaves, believing that he has given his captive only a sporting chance to escape.

After thinking a moment, our quick-witted logician asks one question and then chooses the door leading to freedom. What question did he ask?

Let p be the statement "The first door leads to freedom" and q be the statement "You are truthful." It is clear that p and q are useless questions in themselves, so let us try compound statements. We want to ask a single question for which a "yes" answer means that p is true and a "no" answer means that p is false, regardless of which warrior is asked the question. The answers desired to these questions are listed in Figure 17.

p	q	Desired Answer	Truth Table of Question
T	T	yes	T
T	F	yes	F
F	T	no	F
F	F	no	T

Figure 17

The next thing to consider is, what would be the truth table of a question having the desired answers. If the warrior answers "yes" and if he is truthful, that is, if q is true, then the truth value is T. But if he answers "yes" and he is a liar, that is, if q is false, then the truth value is F. A similar analysis holds if the answer is "no." The truth values of the desired question are shown in Figure 17.

Therefore we have reduced the problem to that of finding a statement having the truth table of Figure 17. Following the general method outlined above, we see that the statement

$$(p \land q) \lor (\sim p \land \sim q)$$

will do. Hence the logician asks the question: "Does the first door lead to freedom and are you truthful, or does the second door lead to freedom and are you lying?" The reader can show (Exercise 3) that the statement $p \leftrightarrow q$ also has the truth table given in Figure 17, hence a shorter equivalent question would be: "Does the first door lead to freedom if and only if you are truthful?"

As can be seen in Example 2, the method does not necessarily yield the simplest possible compound statement. However it has two advantages: (1) It gives us a mechanical method of finding a statement that solves the problem. (2) The statement appears in a standard form. The latter will be made use of in designing switching circuits (see Section 12).

EXERCISES

1. Show that each of the basic conjunctions in Figure 16 has a truth table consisting of one T appearing in the row in which the statement appears in Figure 16, and all the rest F's.

2. Find the truth table of the compound statement constructed in Example 1.

3. Show in Example 2 that the statement $p \leftrightarrow q$ has the truth table of Figure 17.

4. Construct one or more compound statements having each of the following truth tables, (a), (b), and (c).

p	q	r	(a)	(b)	(c)
T	T	T	T	F	T
T	T	F	F	F	T
T	F	T	T	F	T
T	F	F	F	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	F	F	T

5. Using only $\vee$, $\wedge$, and $\sim$, write a statement equivalent to each of the following.
   (a) $p \leftrightarrow q$.
   (b) $p \rightarrow q$.
   (c) $\sim(p \rightarrow q)$.

6. Using only $\vee$ and $\sim$, write down a statement equivalent to $p \wedge q$. Use this result to prove that any truth table can be represented by means of the two connectives $\vee$ and $\sim$.

In Exercises 7–10 we will study the new connective $\downarrow$, where $p \downarrow q$ expresses "neither p nor q."

7. Construct the truth table of $p \downarrow q$.

8. Construct the truth table for $p \downarrow p$. What other compound has this truth table? [*Ans.* Same as Figure 5.]

9. Construct the truth table for $(p \downarrow q) \downarrow (p \downarrow q)$. What other compound has this truth table? [*Ans.* Same as Figure 3.]

10. Use the results of Exercises 6, 8, and 9 to show that any truth table can be represented by means of the single connective $\downarrow$.

11. Use the results of Exercises 9 and 10 following Section 2 to show that any truth table can be represented by means of the single connective $|$.

12. Write down a compound of p, q, r which is true if and only if exactly one of the three components is true.

13. The "basic conjunctions" for statements having only one variable are p and $\sim p$. Discuss the various compound statements that can be formed by disjunctions of these. How do these relate to the possible truth tables for statements of one variable? What can be asserted about an arbitrary compound, no matter how long, that contains only the variable p?

[*Partial Ans.* There are four possible truth tables.]

14. In Example 2 there is a second question, having a different truth table than that in Figure 17, which the logician can ask. What is it?

15. A student is confronted with a true-false exam, consisting of five questions. He knows that his instructor always has more true than false questions, and that he never has three questions in a row with the same answer. From the nature of the first and last questions he knows that these must have the opposite answer. The only question to which he knows the answer is number two. And this *assures* him of having all answers correct. What did he know about question two? What is the answer to the five questions?

[*Ans.* TFTTF.]

5. *LOGICAL POSSIBILITIES*

One of the most important contributions that mathematics can make to the solution of a scientific problem is to provide an exhaustive analysis of the logical possibilities for the problem. The role of science is then to discover facts which will eliminate all but one possibility. Or, if this cannot be achieved, at least science tries to estimate the probabilities of the various possibilities.

So far we have considered only a very special case of the analysis of logical possibilities, namely truth tables. We started with a small number of given statements, say p, q, and r, and we assumed that all the truth table cases were possible. This amounts to assuming that the three statements are logically unrelated (see Section 8). Then we could determine the truth or falsity of every compound statement formed from p, q, and r for every truth table case (every logical possibility).

But there are many more statements whose truth cannot be analyzed in terms of the eight truth table cases discussed above. For example, $\sim p \vee (q \wedge r \wedge \sim s)$ requires a finer analysis, a truth table with 16 cases.

Many of these ideas are applicable in a more general setting. Let us suppose that we have an analysis of logical possibilities. That is, we have a list of eventualities, such that one and only one of them can possibly be true. We know this partly from the framework in which the problem is considered, and partly as a matter of pure logic. We then consider *statements relative to this set of possibilities*. These are statements whose truth or falsity can be determined for each logical possibility. For example, the set of possibilities may be the eight truth table cases, and the statements relative to these possibilities are the compound statements formed from p, q, and r. But we should consider a more typical example.

Example 1. Let us consider the following problem, which is of a type often studied in probability theory. "There are two urns; the first contains two black balls and one white ball, while the second contains one black ball and two white balls. Select an urn at random and draw two balls in succession from it. What is the probability

Case	Urn	First Ball	Second Ball
1	1	black	black
2	1	black	white
3	1	white	black
4	2	black	white
5	2	white	black
6	2	white	white

Figure 18

that . . . ?" Without raising questions of probability, let us ask what the possibilities are. Figures 18 and 19 give us two ways of analyzing the logical possibilities.

Case	Urn	First Ball	Second Ball
1	1	black no. 1	black no. 2
2	1	black no. 2	black no. 1
3	1	black no. 1	white
4	1	black no. 2	white
5	1	white	black no. 1
6	1	white	black no. 2
7	2	black	white no. 1
8	2	black	white no. 2
9	2	white no. 1	black
10	2	white no. 2	black
11	2	white no. 1	white no. 2
12	2	white no. 2	white no. 1

Figure 19

In Figure 18 we have analyzed the possibilities as far as colors of balls drawn was concerned. Such an analysis may be sufficient for many purposes. In Figure 19 we have carried out a finer analysis, in which we distinguished between balls of the same color in an urn. For some purposes the finer analysis may be necessary.

It is important to realize that the possibilities in a given problem may be analyzed in many different ways, from a very rough grouping to a highly refined one. The only requirements on an analysis of logical possibilities are:

(1) That under any conceivable circumstances one and only one of these possibilities must be the case, and
(2) that the analysis is fine enough so that the truth value of each statement under consideration in the problem is determined in each case.

It is easy to verify that both analyses (Figures 18 and 19) satisfy the first condition. Whether the rougher analysis will satisfy the second condition depends on the nature of the problem. If we can limit ourselves to statements like "Two black balls are drawn from the first urn," then it suffices. But if we wish to consider "The first black ball is drawn after the second black ball from the first urn," then the finer analysis is needed.

Given the analysis of logical possibilities, we can ask for each assertion about the problem, and for each logical possibility, whether the assertion is true in this case. Normally, for a given statement there will be many cases in which it is true and many in which it is false. Logic will be able to do no more than to point out the cases in which the statement is true. In Example 1, the statement "One white ball and one black ball is drawn" is true (in Figure 18) in cases 2, 3, 4, and 5, and false in cases 1 and 6. However, there are two notable exceptions, namely, a statement that is true in every logically possible case, and one that is false in every case. Here logic alone suffices to determine the truth value.

A statement that is true in every logically possible case is said to be *logically true*. The truth of such a statement follows from the meaning of the words and the form of the statement, together with the context of the problem about which the statement is made. We will see several examples of logically true statements below. A statement that is false in every logically possible case is said to be *logically false*, or to be a *self-contradiction*. For example, the conjunction of any statement with its own negation will always be a self-contradiction, since it cannot be true under any circumstances.

In Example 1, the statement "At most two black balls are drawn" is true in every case, in either analysis. Hence this statement is logically true. It follows from the very definition of the problem that we cannot draw more than two balls. Hence, also, the statement "Draw three white balls" is logically false.

What the logical possibilities are for a given set of statements will depend on the context, i.e., on the problem that is being considered. Unless we know what the possibilities are, we have not understood the task before us. This does not preclude that there may be several ways of analyzing the logical possibilities. In Example 1 above, for example, we gave two different analyses, and others could be found. In general, the question "How many cases are there in which p is true" will depend on the analysis given. (This will be of importance in our study of probability theory.) However, note that a statement that is logically true (false) according to one analysis will be logically true (false) according to every other analysis of the given problem.

The truth table analysis is often the roughest possible analysis. There may be hundreds of logical possibilities, but if all we are interested in are compounds formed from p and q, we need only know when p and q are true or false. For example, a statement of the form $p \rightarrow (p \vee q)$ will have to be true in every conceivable case. We may have a hundred cases, giving varying truth values for p and q, but every such case must correspond to one of the four truth table cases, as far as the compound is concerned. In each of these four cases the compound is true, and therefore such a statement is logically true. An example of it is "If Jones is smart, then he is smart or lucky."

However, if the components are logically related, then a truth table analysis may not be adequate. Let p be the statement "Jim is taller than Bill," while q is "Bill is taller than Jim." And consider the statement, "Either Jim is not taller than Bill or Bill is not taller than Jim," i.e., $\sim p \vee \sim q$. If we work the truth table of this compound, we find that it is false in the first case. But this case is not logically possible, since under no circumstances can p and q both be true! Our compound is logically true, but a truth table will not show this. Had we made a careful analysis of the possibilities as to the heights of the two men, we would have found that the compound statement is true in every case. (Such relations will be considered in Section 8. This particular pair of statements will be considered in Exercise 11 in that section.)

Example 2. The Miracle Filter Company conducts an annual survey of the smoking habits of adult Americans. The results of the survey are organized into 25 files, corresponding to the 25 cases in Figure 20.

Case	Sex	Educational Level	Occupation
1	male	0	prof.
2	male	0	non-prof.
3	male	1	prof.
4	male	1	non-prof.
5	male	2	prof.
6	male	2	non-prof.
7	male	3	prof.
8	male	3	non-prof.
9	male	4	prof.
10	male	4	non-prof.
11	female	0	housewife
12	female	0	prof.
13	female	0	non-prof.
14	female	1	housewife
15	female	1	prof.
16	female	1	non-prof.
17	female	2	housewife
18	female	2	prof.
19	female	2	non-prof.
20	female	3	housewife
21	female	3	prof.
22	female	3	non-prof.
23	female	4	housewife
24	female	4	prof.
25	female	4	non-prof.

Figure 20

First, figures are kept separately for men and women. Secondly, the educational level is noted according to the following code:

0 did not finish high school
1 finished high school, no college
2 some college, but no degree
3 college graduate, but no graduate work
4 did some graduate work

Finally, there is a rough occupational classification: housewife, salaried professional, or salaried non-professional.

They have found that this classification is adequate for their purposes. For instance, to get figures on all adults in their survey who did not go beyond high school, they pull out the files numbered 1, 2, 3, 4, 11, 12, 13, 14, 15, and 16. Or they can locate data on male professional workers by looking at files 1, 3, 5, 7, and 9.

According to their analysis, the statement "The person is a housewife, professional, or non-professional" is logically true, while the statement "The person has educational level greater than 3, is neither professional nor non-professional, but not a female with graduate education" is a self-contradiction. The former statement is true about all 25 files, the latter about none.

Of course, they may at some time be forced to consider a finer analysis of logical possibilities. For instance, "The person is a male with annual income over $10,000" is *not* a statement relative to the given possibilities. We could choose a case—say case 6—and the given statement may be either true or false in this case. Thus the analysis is not fine enough.

Of all the logical possibilities, one and only one represents the facts as they are. That is, for a given person, one and only one of the 25 cases is a correct description. To know which one, we need factual information. When we say that a certain statement is "true," without qualifying it, we mean that it is true in this one case. But, as we have said before, what the case actually is lies outside the domain of logic. Logic can tell us only what the circumstances (logical possibilities) are under which a statement is true.

EXERCISES

1. Prove that the negation of a logically true statement is logically false, and the negation of a logically false statement is logically true.

2. Classify the following as (i) logically true, (ii) a self-contradiction, (iii) neither.

(a) $p \leftrightarrow p$. [*Ans.* Logically true.]
(b) $p \rightarrow \sim p$.
(c) $(p \vee q) \leftrightarrow (p \wedge q)$. [*Ans.* Neither.]
(d) $(p \rightarrow \sim q) \rightarrow (q \rightarrow \sim p)$.
(e) $(p \rightarrow q) \wedge (q \rightarrow r) \wedge \sim(p \rightarrow r)$. [*Ans.* Self-contradiction.]
(f) $(p \rightarrow q) \rightarrow p$.
(g) $[(p \rightarrow q) \rightarrow p] \rightarrow p$.

3. Figure 20 gives the possible classifications of one person in the survey. How many cases do we get if we classify two people jointly? [*Ans.* 625.]

4. For each of the 25 cases in Figure 20 state whether the following statement is true: "The person has had some college education, and if the person is female then she is a housewife."

5. In Example 1, with the logical possibilities given by Figure 18, state the cases in which the following statements are true.

(a) Urn one is selected.
(b) At least one white ball is drawn.
(c) At most one white ball is drawn.
(d) If the first ball drawn is white, then the second is black.
(e) Two balls of different color are drawn if and only if urn one is selected.

6. In Example 1 give two logically true and two logically false statements (other than those in the text).

7. In a college using grades A, B, C, D, and F, how many logically possible report cards are there for a student taking four courses? [*Ans.* 625.]

8. A man has nine coins totaling 78 cents. What are the logical possibilities for the distribution of the coins? [*Hint:* There are three possibilities.]

9. In Exercise 8, which of the following statements are logically true and which are logically false?

(a) He has at least one penny. [*Ans.* Logically true.]
(b) He has at least one nickel. [*Ans.* Neither.]
(c) He has exactly two nickels. [*Ans.* Logically false.]
(d) He has exactly three nickels if and only if he has exactly one dime. [*Ans.* Logically true.]

10. In Exercise 8 we are told that the man has no nickel in his possession. What can we infer from this?

11. Two dice are rolled. Which of the following analyses satisfy the first condition for logical possibilities? What is wrong with the others?

The sum of the numbers shown is:

(a): (1) 6, (2) not 6.
(b): (1) an even number, (2) less than 6, (3) greater than 6.
(c): (1) 2, (2) 3, (3) 4, (4) more than 4.
(d): (1) 7 or 11, (2) 2, 3, or 12, (3) 4, 5, 6, 8, 9, or 10.
(e): (1) 2, 4, or 6, (2) an odd number, (3) 10 or 12.
(f): (1) less than 5 or more than 8, (2) 5 or 6, (3) 7, (4) 8.
(g): (1) more than 5 and less than 10, (2) at most 4, (3) 7, (4) 11 or 12.

[*Ans.* (a), (c), (d), (f) satisfy the condition.]

SUPPLEMENTARY EXERCISES

Note: These exercises refer to the following example: There are three urns. The first one contains two black balls. The second one contains one black and two white balls, while the third contains two black and two white balls. We select an urn, and draw two balls.

12. Construct a table of the logical possibilities, similar to Figure 18.
[*Partial Ans.* There are eight cases.]

13. In which cases is the statement "One black and one white ball is drawn" true?

14. What is the status of the statement "Urn 1 is selected, and two different color balls are drawn"? [*Ans.* Logically false.]

15. Find the cases in which the statement "Urn 1 is selected if and only if two black balls are drawn" is true.

16. How does the list of possibilities change if we don't care about the order in which the balls are drawn?

6. TREE DIAGRAMS

A very useful tool for the analysis of logical possibilities is the drawing of a "tree." This device will be illustrated by several examples.

Example 1. Consider again the survey of the Miracle Filter Company. They keep two large filing cabinets, one for men and one for women. Each cabinet has five drawers, corresponding to the five educational levels. Each drawer is subdivided according to occupations; drawers in the filing cabinet for men have two large folders, while in the other cabinet each drawer has three folders.

When a clerk files a new piece of information, he first has to find the right cabinet, then the correct drawer, and then the appropriate folder. This three-step process of filing is shown in Figure 21. For obvious reasons we shall call a figure like this, which starts at a point and branches out, a *tree.*

Observe that the tree contains all the information relevant to classifying a person interviewed. There are 25 ways of starting at the bottom and following a path to the top. The 25 paths represent the 25 cases in Figure 20. The order in which we performed the classification is

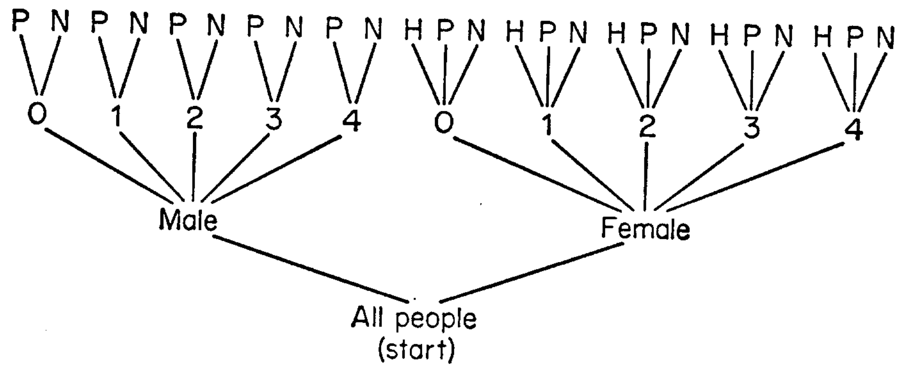


Figure 21

arbitrary. We might as well have classified first according to educational level, then according to occupation, and then according to sex. We would still obtain a tree representing the 25 logical possibilities, but the tree would look quite different. (See Exercise 1.)

Example 2. Next let us consider the example of Figure 18. This is a three-stage process; first we select an urn, then draw a ball and then draw a second ball. The tree of logical possibilities is shown in Figure 22. We note that six is the correct number of logical possibilities. The reason for this is: If we choose the first urn (which contains two black balls and one white ball) and draw from it a black ball, then the second draw may be of either color; however, if we draw a white ball first, then the second ball drawn is necessarily black. Similar remarks apply if the second urn is chosen.

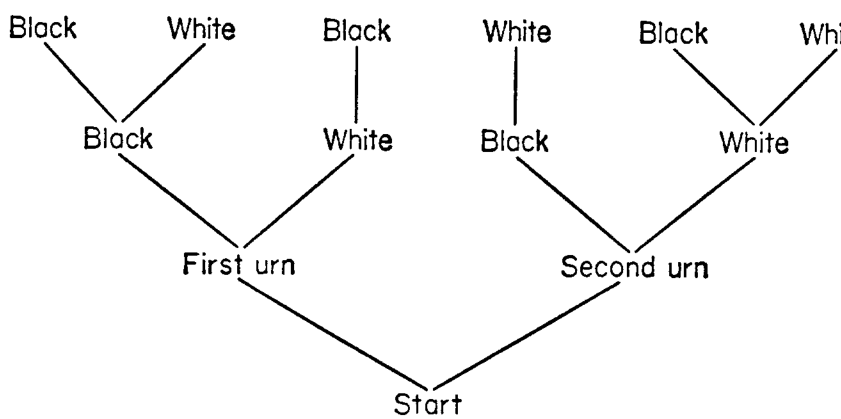


Figure 22

Example 3. As a final example, let us construct the tree of logical possibilities for the outcomes of a World Series played between the Dodgers and the Yankees. In Figure 23 is shown half of the tree,

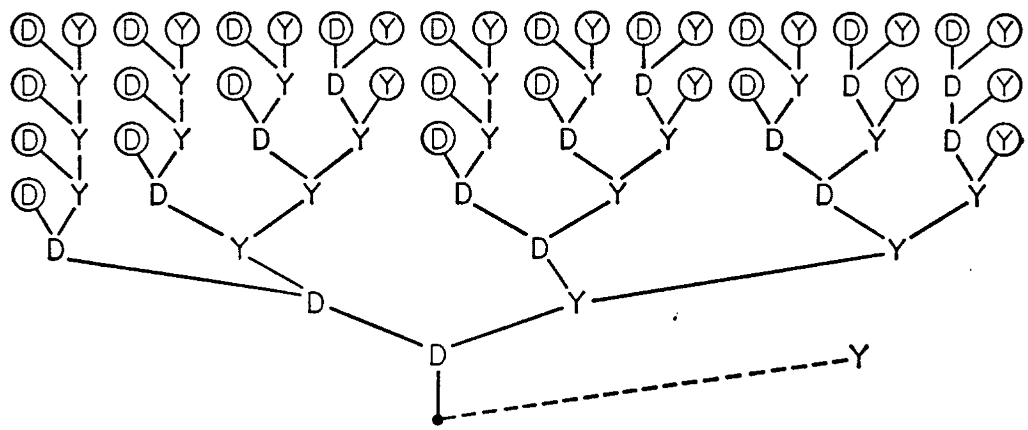


Figure 23

corresponding to the case when the Dodgers win the first game (the dotted line at the bottom leads to the other half of the tree). In the figure a "D" stands for a Dodger win and "Y" for a Yankee win. There are 35 possible outcomes (corresponding to the circled letters) in the half-tree shown, so that the World Series can end in 70 ways.

This example is different from the previous two in that the paths of the tree end at different levels, corresponding to the fact that the World Series ends whenever one of the teams has won four games.

Not always do we wish as detailed an analysis as that provided in the examples above. If, in Example 2, we wanted to know only the color and order in which the balls were drawn and not which urn they came from, then there would be only four logical possibilities instead of six. Then in Figure 22 the second and fourth paths (counting from the left) represent the same outcome, namely, a black ball followed by a white ball. Similarly, the third and fifth paths represent the same outcome. Finally, if we cared only about the color of the balls drawn, not the order, then there are only three logical possibilities: two black balls, two white balls, or one black and one white ball.

A less detailed analysis of the possibilities for the World Series is also possible. For example, we can analyze the possibilities as follows: Dodgers in four, five, six, or seven games, and Yankees in four, five, six, or seven games. The new classification reduced the number of possibilities from 70 to eight. The other possibilities have not been eliminated but merely grouped together. Thus the statement "Dodgers in four games" can happen in only one way, while "Dodgers in seven games" can happen in 20 ways (see Figure 23). A still less detailed analysis would be a classification according to the number of games in the series. Here there are only four logical possibilities.

The student will find that it often requires several trials before the "best" way of listing logical possibilities is found for a given problem.

EXERCISES

1. Construct a tree for Example 1, if people are first classified according to educational level, then according to profession, and finally according to sex. Is the shape of the tree the same as in Figure 21? Does it represent the same possibilities?

2. In 1965 the Dodgers lost the first two games of the World Series, but won the series in the end. In how many ways can the series go so that the losing team wins the first two games? [*Ans.* 10.]

3. The following is a typical process in genetics: Each parent has two genes for a given trait, AA or Aa or aa. The child will inherit one gene from each parent. What are the possibilities for a child if both parents are AA? What if one is AA and the other aa? What if one is AA and the other Aa? What if both are Aa? Construct a tree for each process. [Let stage one be the choice of a gene from the first parent, stage two from the second parent. Then see how many different types the resulting branches represent.]

4. It is often the case that types AA and Aa (see Exercise 3) are indistinguishable from the outside, but easily distinguishable from type aa. What are the logical possibilities if the two parents are of noticeably different types?

5. In nominating candidates for President and Vice President, a major party takes into account the sex of the candidate, and the part of the country from which he comes. For the latter purpose they identify four regions: East, Midwest, South, and West. Draw a tree for the various possibilities in selecting a pair of candidates.

(a) How many cases are there?

(b) How many are there if the two candidates must not come from the same part of the country?

(c) How many are there if, in addition, the party refuses to nominate two women? [*Ans.* (c) 36.]

6. We set up an experiment similar to that of Figure 18, but urn 1 has two black balls and two white balls, while urn 2 has one white ball and four black balls. We select an urn, and draw three balls from it. Construct the tree of logical possibilities. How many cases are there? [*Ans.* 10.]

7. From the tree constructed in Exercise 6 answer the following questions.

(a) In how many cases do we draw three black balls?
(b) In how many cases do we draw two black balls and one white ball?
(c) In how many cases do we draw three white balls?
(d) How many cases does this leave? What cases are these?

[*Partial Ans.* 3.]

8. In how many ways can the World Series be played (see Figure 23) if the Dodgers win the first game and

(a) No team wins two games in a row. [*Ans.* 1.]
(b) The Dodgers win at least the odd-numbered games. [*Ans.* 5.]
(c) The winning team wins four games in a row. [*Ans.* 4.]
(d) The losing team wins four games. [*Ans.* 0.]

9. A man is considering the purchase of one of four types of stocks. Each stock may go up, go down, or stay the same after his purchase. Draw the tree of logical possibilities.

10. For the tree constructed in Exercise 9 give a statement which

(a) Is true in half the cases.
(b) Is false in all but one case.
(c) Is true in all but one case.
(d) Is logically true.
(e) Is logically false.

11. In Exercise 6 we wish to make a rougher classification of logical possibilities. What branches (in the tree there constructed) become identical if

(a) We do not care about the order in which the balls are drawn.
(b) We care neither about the order of balls, nor about the number of the urn selected.
(c) We care only about what urn is selected, and whether the balls drawn are all the same color.

12. Work Exercise 7 of the last section by sketching a tree diagram.

13. A menu lists a choice of soup or orange juice for an appetizer, a choice of steak, chicken, or fish for the entree, and a choice of pie or cake for dessert. A complete dinner consists of one choice in each case. Draw the tree for the possible complete dinners.

(a) How many different complete dinners are possible? [*Ans.* 12.]
(b) How many complete dinners are there which have chicken for the entree? [*Ans.* 4.]
(c) How many complete dinners are there available for a man who will eat pie only if he had steak for the entree? [*Ans.* 8.]

SUPPLEMENTARY EXERCISES

14. In how many different ways can 55 cents change be given, using quarters, dimes, and nickels? Draw a tree. [*Hint:* To eliminate duplication, require that larger coins be handed out before smaller ones. Let the branches of the tree be labelled with the number of coins of each type handed out.]

[*Ans.* 11.]

15. Redraw the tree of Exercise 14, requiring that smaller coins be handed out before larger ones.

16. What is the answer to Exercise 14 if only one nickel is available?

17. Draw a tree for Exercise 12 in Section 5.

18. In electing the chairman of a small committee, candidate A receives two votes, and candidate B receives one. Draw a tree to represent the possible orders in which the three ballots are counted. In what fraction of the cases is A ahead all the way?

19. Redo Exercise 18 for the election in which A receives four votes and B receives two. [*Ans.* $\frac{1}{3}$ of the cases.]

7. LOGICAL RELATIONS

Until now we have considered statements in isolation. Sometimes, however, we want to consider the relationship between pairs of statements. The most interesting such relation is that one statement (logically) *implies* the other one. If r implies s we also say that s follows from r, or that s is (logically) deducible from r. For instance, in any mathematical theorem the hypothesis implies the conclusion.

If we have listed all logical possibilities, then we shall characterize implication as follows: r implies s if s is true whenever r is true, i.e., if s is true in all the logically possible cases in which r is true.

For compound statements having the same components, truth tables provide a convenient method for testing this relation. In Figure 24 we illustrate this method. Let us take $p \leftrightarrow q$ as our hypothesis r. Since it is true only in the first and fourth cases, and $p \rightarrow q$ is true in both these cases, we see that the statement $p \leftrightarrow q$ implies $p \rightarrow q$. On the other hand, the statement $p \vee q$ is false in the fourth case and hence it is not implied by $p \leftrightarrow q$. Again, a comparison of the last two columns of Figure 24 shows that the statement $p \rightarrow q$ does not imply and is not implied by $p \vee q$.

The relation of implication has a close affinity to the conditional statement, but it is important not to confuse the two. The conditional is a new *statement* compounded from two given statements, while implication is a *relation* between the two statements. The connection is the following: r implies s if and only if the conditional $r \rightarrow s$ is logically true.

That this is the case is shown by a simple argument. The statement r implies the statement s if s is true whenever r is true. This means that there is no case in which r is true and s false, i.e., no case in which $r \rightarrow s$ is false. But this in turn means that $r \rightarrow s$ is logically true. In Exercise 1 this result will be applied to Figure 24.

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$p \vee q$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	T	T	F

Figure 24

Let us now take up the "paradoxes" of the conditional. Conditional statements sound paradoxical when the components are not related. For example, it sounds strange to say that "If it is a nice day then chalk is made of wood" is true on a rainy day. It must be remembered that the conditional statement just quoted means no more and no less than that one of the following holds: (1) It is a nice day and chalk is made of wood, or (2) It is not a nice day and chalk is made of wood, or (3) It is not a nice day and chalk is not made of wood. [See Figure 11b.] And on a rainy day number (3) happens to be correct.

But it is by no means true that "It is a nice day" implies that "Chalk is made of wood." It is logically possible for the former to be true and for the latter to be false (indeed, this is the case on a nice day, with the usual method of chalk manufacture), hence the implication does not hold. Thus, while the conditional quoted in the previous paragraph is true on a given day, it is not logically true.

In common parlance "if . . . then . . ." is usually asserted on logical grounds. Hence any usage in which such an assertion happens to be true, but is not logically true, sounds paradoxical. Similar remarks apply to the common usage of "if and only if."

If the biconditional $r \leftrightarrow s$ is not only true but logically true, then this establishes a relation between r and s. If $r \leftrightarrow s$ is true in every logically possible case, then the statements r and s have the same truth value in every case. We say, under these circumstances, that r and s are (logically) *equivalent*. For compound statements having the same components, the truth table provides a convenient means of testing for equivalence. We merely have to verify that the compounds have the same truth table. Figure 25 establishes that $\sim p \wedge \sim q$ is equivalent

p	q	$\sim p \wedge \sim q$	$\sim(p \vee q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Figure 25

to $\sim(p \vee q)$. This is one of the so-called *De Morgan laws*. (See Exercise 13.)

A third important relationship is that of inconsistency. Statements r and s are *inconsistent* if it is impossible for both of them to be true, in other words, if $r \wedge s$ is a self-contradiction. For example, the statements $p \wedge q$ and $\sim q$ are inconsistent. An important use of logic is to check for inconsistencies in a set of assumptions or beliefs.

EXERCISES

1. Show that $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$ is logically true, but that $(p \leftrightarrow q) \rightarrow (p \vee q)$ is not logically true.

2. Prove that r is equivalent to s just in case r implies s and s implies r.

3. Construct truth tables for the following compounds, and test for implications and equivalences.

(a) $p \wedge q$.
(b) $p \rightarrow \sim q$.
(c) $\sim p \vee \sim q$.
(d) $\sim p \vee q$.
(e) $p \wedge \sim q$. [*Ans.* (b) equiv. (c); (a) impl. (d); (e) impl. (b), (c).]

4. Construct truth tables for the following compounds, and arrange them in order so that each compound implies all the following ones.

(a) $\sim p \leftrightarrow q$.

(b) $p \rightarrow (\sim p \rightarrow q)$.

(c) $\sim[p \rightarrow (q \rightarrow p)]$.

(d) $p \vee q$.

(e) $\sim p \wedge q$. [*Ans.* (c); (e); (a); (d); (b).]

5. Construct a compound equivalent to $p \wedge q$, using only the connectives $\sim$ and $\vee$.

6. Construct a compound equivalent to $p \leftrightarrow q$, using only the connectives $\rightarrow$ and $\wedge$. (Cf. Exercise 2.)

7. Construct a compound statement equivalent to $p \vee q$, using only the connectives $\sim$ and $\wedge$.

8. If p is logically true, prove that

(a) $p \vee q$ is logically true.

(b) $\sim p \wedge q$ is logically false.

(c) $p \wedge q$ is equivalent to q.

(d) $\sim p \vee q$ is equivalent to q.

9. If p and q are logically true and r is logically false, what is the status of $(p \vee \sim q) \wedge \sim r$? [*Ans.* Logically true.]

10. Pick out an inconsistent pair from among the following four compound statements.

r: $p \vee q$.

s: $p \rightarrow q$.

t: $\sim q$.

u: $\sim(q \rightarrow p)$.

11. What implications hold between pairs of statements in Exercise 10? [*Ans.* u implies r and s.]

12. In Exercise 10, is there an inconsistent pair among r, s, and t? Is it possible that all three statements are true?

13. One of the De Morgan laws is established in Figure 25. The other one states that $\sim(p \wedge q)$ is equivalent to $\sim p \vee \sim q$. Prove this.

14. What relation exists between two logically true statements? Between two self-contradictions?

15. Prove that

(a) A logically true statement is implied by every statement, and that a self-contradiction implies every statement.

(b) The conjunction or disjunction of a statement with itself is equivalent to the statement.

(c) The double negation of a statement is equivalent to the statement.
(d) A statement which implies its own negation is a self-contradiction.

16. Using the results of Section 4, Exercises 10 and 11, prove that for any compound statement there is an equivalent compound statement
(a) Whose only connective is $\downarrow$.
(b) Whose only connective is $|$.

17. What is the status of a statement equivalent to its own negation?
[*Ans.* Impossible.]

*8. *A SYSTEMATIC ANALYSIS OF LOGICAL RELATIONS*

The relation of implication is characterized by the fact that it is impossible for the hypothesis to be true and the conclusion to be false. If two statements are equivalent, it is impossible for one to be true and the other to be false. Thus we see that for an implication one truth table case must not occur, and for an equivalence two oi the four truth table cases must not occur. The absence of one or more truth table cases is thus characteristic of logical relations. In this section we shall investigate all conceivable relations that can exist between two statements.

p	q	Case No.
T	T	1
T	F	2
F	T	3
F	F	4

Figure 26

We shall say that two statements are *unrelated* if each of the four truth table cases (see Figure 26) can occur. The two statements are *related* if one or more of the four cases in Figure 26 cannot occur. [Cf. Section 5.]

If p and q are statements such that exactly one of the cases in Figure 26 is excluded, then we say that there is a *onefold* relation between them. Obviously there are four possible onefold relations which we list below. (a) If case 1 is excluded, the two statements cannot both be true. In this case p and q are said to be a pair of *contraries* or are said to be *inconsistent.* (b) If case 2 is excluded, then (cf. Section 7) p implies q. (c) If case 3 is excluded, it is false that q is true and p is false, that is, q implies p. (d) If case 4 is excluded, both statements cannot be false, i.e., at least one of them is true. Such a pair of statements is called a pair of *subcontraries.*

If p and q are statements such that exactly two of the cases in Figure 26 are excluded, then we say that there is a *twofold* relation between them. There are six ways in which two cases can be selected from four, but several of these do not produce interesting relations. For example, suppose cases 1 and 2 are excluded; then p cannot be true, i.e., it is logically false. Similarly, if cases 1 and 3 are excluded, then q is logically false. On the other hand, if cases 3 and 4 are excluded, then p is logically true; and if 2 and 4 are excluded, then q is logically true. Hence we see that these choices do not give us new relations; they merely indicate that one of the two statements is logically true or false. We now have only two alternatives remaining: (A) cases 2 and 3 are excluded, which means that the two statements are equivalent; and (B) cases 1 and 4 are excluded, which means that the two statements cannot both be true and cannot both be false, in other words, one must be true and the other false. We shall then say that p and q are *contradictories*, or a *pair of alternatives*.

It is not hard to see that there are no threefold relations, for if three of the cases in Figure 26 are excluded, then there is only one possibility for each of the two statements, so that each must be either logically true or logically false.

We have already discussed implication and equivalence and have noted their connection to the conditional and the biconditional, respectively. We can do the same for the three remaining relations. If p and q are subcontraries, then they cannot both be false; since this is the only case in which their disjunction is false, we see that p and q are subcontraries if and only if $p \vee q$ is logically true. If p and q are contraries, then they cannot both be true; since this is the only case in which their conjunction is true, we see that p and q are contraries if and only if $p \wedge q$ is logically false. Finally, if p and q are contradictories, then cases 1 and 4 of Figure 26 are excluded, hence $p \leftrightarrow q$ is logically false. (Note also that, if p and q are contradictories, then $p \veebar q$ is logically true.) The table in Figure 27 gives a summary of the relevant facts about the six relations we have derived.

Subcontraries are not of great theoretical importance, but contraries and contradictories are very important. Each of these relations can be generalized to hold for more than two statements. If we have n different statements, not all of which can be true, then we say that they are *inconsistent*. Then the conjunction of these statements must be false. Special cases of inconsistent statements are the following: if $n = 1$,

Case(s) Excluded	Relation	Alternate Definition
T-T	Contraries	$p \wedge q$ logically false
F-F	Subcontraries	$p \vee q$ logically true
T-F	First implies second	$p \rightarrow q$ logically true
F-T	Second implies first	$q \rightarrow p$ logically true
T-F and F-T	Equivalents	$p \leftrightarrow q$ logically true
T-T and F-F	Contradictories	$p \leftrightarrow q$ logically false

Figure 27

then we have a single self-contradictory statement; and if $n = 2$, then we have a pair of inconsistent statements (i.e., a pair of contraries).

If we have n different statements such that one and only one of them can be true, then we say they form a *complete set of alternatives.* Again the special cases are: if $n = 1$, then we have a single logically true statement; and if $n = 2$, then we have a pair of contradictories.

Truth tables again furnish a method for recognizing when relations hold between statements. The examples below show how the method works.

Examples. Consider the five compound statements, all having the same components, which appear in Figure 28. Find all relations which exist between pairs of these statements.

p	q	$p \wedge q$	$\sim p \vee \sim q$	$\sim p \vee q$	$\sim p$	$p \rightarrow q$
T	T	T	F	T	F	T
T	F	F	T	F	F	F
F	T	F	T	T	T	T
F	F	F	T	T	T	T
Statement Number		1	2	3	4	5

Figure 28

First of all we note that statements 3 and 5 have identical truth tables, hence they are equivalent. Therefore we need consider only one of them, say statement 3. Statements 1 and 2 have exactly opposite truth tables, hence they are contradictories. Upon comparing statements 1 and 3 we find no T-F case, so that 1 implies 3. Since numbers 1 and 4 are never both true, they are contraries, while numbers 2 and 3 are never both false, so that they are subcontraries. Finally, upon comparing either 2 or 3 to 4 we find no F-T case and hence both are implied by 4. Thus the six relations we found above are all exemplified in Figure 28. Observe also that statements $\sim p$ and q give an example of a pair of unrelated statements. [Cf. Section 5.]

EXERCISES

1. Construct truth tables for the following four statements and state what relation (if any) holds between each of the six pairs formable.

(a) $\sim p$.
(b) $\sim q$.
(c) $p \wedge \sim q$.
(d) $\sim(\sim p \vee q)$.

[*Ans.* (a) and (b) unrelated; (a) and (c), (d) contraries; (c), (d) imply (b); (c) equiv. (d).]

2. Construct truth tables for each of the following six statements. Give an example of an unrelated pair, and an example of each of the six possible relations among these.

(a) $p \leftrightarrow q$.
(b) $p \rightarrow q$.
(c) $\sim p \wedge \sim q$.
(d) $(p \wedge q) \vee (\sim p \wedge \sim q)$.
(e) $\sim q$.
(f) $p \wedge \sim q$.

3. Prove the following assertions.

(a) The disjunction of two contradictory statements is logically true.
(b) The contradictories of two contraries are subcontraries.

4. What is the relation between the following pair of statements?

(a) $p \rightarrow [p \wedge \sim(q \vee r)]$.
(b) $\sim p \vee (\sim q \wedge \sim r)$. [*Ans.* Equivalent.]

5. At most how many of the following assertions can one person consistently believe?

(a) Joe is smart.
(b) Joe is unlucky.
(c) Joe is lucky but not smart.

(d) If Joe is smart, then he is unlucky.
(e) Joe is smart if and only if he is lucky.
(f) Either Joe is smart, or he is lucky, but not both. [*Ans.* 4.]

6. Prove the following assertions.
(a) The contradictories of two equivalent statements are equivalent.
(b) In a complete set of alternatives any two statements are contraries.
(c) If p and q are subcontraries, and if each implies r, then r is logically true.

7. Pick out a complete set of (four) alternatives from the following.
(a) It is raining but the wind is not blowing.
(b) It rains if and only if the wind blows.
(c) It is not the case that it rains and the wind blows.
(d) It is raining and the wind is blowing.
(e) It is neither raining nor is the wind blowing.
(f) It is not the case that it is raining or the wind is not blowing.
[*Ans.* (a); (d); (e); (f).]

8. What is the relation between $[p \vee \sim(q \vee r) \vee (p \wedge s)]$ and $\sim(p \wedge q \wedge r \wedge s)$? [*Ans.* Subcontraries.]

9. Suppose that p and q are contraries. What is the relation between
(a) p and $\sim q$.
(b) $\sim p$ and q.
(c) $\sim p$ and $\sim q$.
(d) p and $\sim p$.

10. Let p, q, and r be three statements such that any two of them are unrelated. Discuss the possible relations among the three statements. [*Hint:* If we ignore the order of the statements, there are 16 such relations. The relations are at most fourfold. There are two fourfold relations, and 12 relations are found from these by excluding fewer cases. There are two other possible relations.]

11. In Section 5 we considered an example comparing the height of two men. Suppose that we allow for the possibilities: below 5 ft. 9 in., 5 ft. 9 in., 5 ft. 10 in., 5 ft. 11 in., 6 ft. 0 in., above 6 ft. We will, for the purpose of this problem, consider two men of the same height if they fall into the same category according to the above analysis.
(a) Construct the set of all possibilities for a pair of men, Jim and Bill.
(b) Find the cases in which "Jim is taller than Bill" is true.
(c) Find the cases where "Bill is taller than Jim" is true.
(d) Are all four truth table cases present?
(e) What is the relation between the two statements?

12. Construct the set of logical possibilities which classify a person with respect to sex and marital status.

(a) Show that "If the person is a bachelor, then he is unmarried" is logically true.
(b) Show that "If a person is an old maid, then the person is a man" is not logically false.
(c) Find the relation between "The person is a man" and "The person is a bachelor."
(d) Find a simple statement that is a subcontrary of "The person is a man" and is consistent with it.

*9. *VARIANTS OF THE CONDITIONAL*

The conditional of two statements differs from the biconditional and from disjunctions and conjunctions of these two in that it lacks symmetry. Thus $p \vee q$ is equivalent to $q \vee p$, $p \wedge q$ is equivalent to $q \wedge p$, and $p \leftrightarrow q$ is equivalent to $q \leftrightarrow p$; *but* $p \rightarrow q$ *is not equivalent to* $q \rightarrow p$. The latter statement, $q \rightarrow p$, is called the *converse* of $p \rightarrow q$. Many of the most common fallacies in thinking arise from a confusion of a statement with its converse.

It is of interest to consider conditionals formed from the statements p and q and their negations. The truth tables of four such conditionals together with their names are tabulated in Figure 29. We note that

		Conditional	Converse of Conditional	Converse of Contra-positive	Contra-positive
p	q	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Figure 29

$p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$. The latter is called the *contrapositive* of the former. For many arguments the contrapositive is a very useful form of the conditional. In the same manner the statement $\sim p \rightarrow \sim q$ is the converse of the contrapositive. Since the contrapositive is equivalent to $p \rightarrow q$, the converse of the former is equivalent to the converse of the latter as can be seen in Figure 29.

The use of conditionals seems to cause more trouble than the use of the other connectives, perhaps because of the lack of symmetry, but also perhaps because there are so many different ways of expressing conditionals. In many cases only a careful analysis of a conditional statement shows whether the person making the assertion means the given conditional or its converse. Indeed, sometimes he means both of these, i.e., he means the biconditional. (See Exercise 5.)

The statement "I will go for a walk only if the sun shines" is a variant of a conditional statement. A statement of the form "p only if q" is closely related to the statement "If p then q," but just how? Actually the two express the same idea. The statement "p only if q" states that "If $\sim q$ then $\sim p$" and hence is equivalent to "If p then q." Thus the statement at the beginning of the paragraph is equivalent to the statement, "If I go for a walk, then the sun will be shining."

Other phrases, in common use by mathematicians, which indicate a conditional statement are: "a necessary condition" and "a sufficient condition." To say that p is a sufficient condition for q means that if p takes place, then q will also take place. Hence the sentence "p is a sufficient condition for q" is equivalent to the sentence "If p then q."

Similarly, the sentence "p is a necessary condition for q" is equivalent to "q only if p." Since we know that the latter is equivalent to "If q then p," it follows that the assertion of a necessary condition is the converse of the assertion of a sufficient condition.

Finally, if both a conditional statement and its converse are asserted, then effectively the biconditional statement is being asserted. Hence the assertion "p is a necessary and sufficient condition for q" is equivalent to the assertion "p if and only if q."

These various equivalences are summarized in Figure 30.

Basic Statement	Equivalent Forms
If p then q	p only if q p is a sufficient condition for q
If q then p	q only if p p is a necessary condition for q
p if and only if q	p is a necessary and sufficient condition for q

Figure 30

EXERCISES

1. Let p stand for "I will pass this course" and q for "I will do homework regularly." Put the following statements into symbolic form.
   (a) I will pass the course only if I do homework regularly.
   (b) Doing homework regularly is a necessary condition for me to pass this course.
   (c) Passing this course is a sufficient condition for me to do homework regularly.
   (d) I will pass this course if and only if I do homework regularly.
   (e) Doing homework regularly is a necessary and sufficient condition for me to pass this course.

2. Take the statement in part (a) of the previous exercise. Form its converse, its contrapositive, and the converse of the contrapositive. For each of these give both a verbal and a symbolic form.

3. Let p stand for "It snows" and q for "The train is late." Put the following statements into symbolic form.
   (a) Snowing is a sufficient condition for the train to be late.
   (b) Snowing is a necessary and sufficient condition for the train to be late.
   (c) The train is late only if it snows.

4. Take the statement in part (a) of the previous exercise. Form its converse, its contrapositive, and the converse of its contrapositive. Give a verbal form of each of them.

5. Prove that the conjunction of a conditional and its converse is equivalent to the biconditional.

6. To what is the conjunction of the contrapositive and its converse equivalent? Prove it.

7. Prove that
   (a) $\sim\sim p$ is equivalent to p.
   (b) The contrapositive of the contrapositive is equivalent to the original conditional.

8. "For a matrix to have an inverse it is necessary that its determinant be different from zero." Which of the following statements follow from this? [No knowledge of matrices is required.]
   (a) For a matrix to have an inverse it is sufficient that its determinant be zero.
   (b) For its determinant to be different from zero it is sufficient for the matrix to have an inverse.

(c) For its determinant to be zero it is necessary that the matrix have no inverse.
(d) A matrix has an inverse if and only if its determinant is not zero.
(e) A matrix has a zero determinant only if it has no inverse.

[*Ans.* (b); (c); (e).]

9. "A function that is differentiable is continuous." This statement is true for all functions, but its converse is not always true. Which of the following statements are true for all functions? [No knowledge of functions is required.]

(a) A function is differentiable only if it is continuous.
(b) A function is continuous only if it is differentiable.
(c) Being differentiable is a necessary condition for a function to be continuous.
(d) Being differentiable is a sufficient condition for a function to be continuous.
(e) Being differentiable is a necessary and sufficient condition for a function to be differentiable. [*Ans.* (a); (d); (e).]

10. Prove that the negation of "p is a necessary and sufficient condition for q" is equivalent to "p is a necessary and sufficient condition for $\sim q$."

*10. *VALID ARGUMENTS*

One of the most important tasks of a logician is the checking of *arguments.* By an argument we shall mean the assertion that a certain statement (the *conclusion*) follows from other statements (the *premises*). An argument will be said to be *valid* if and only if the conjunction of the premises implies the conclusion, i.e., if the premises are all true, the conclusion *must* also be true.

It is important to realize that the truth of the conclusion is irrelevant as far as the test of the validity of the argument goes. A true conclusion is neither necessary nor sufficient for the validity of the argument. The two examples below show this, and they also show the form in which we shall state arguments, i.e., first we state the premises, then draw a line, and then state the conclusion.

Example 1.

If the United States is a democracy, then its citizens have the right to vote.
Its citizens do have the right to vote.

Therefore the United States is a democracy.

The conclusion is, of course, true. However, the argument is not valid since the conclusion does not follow from the two premises.

Example 2.

To pass this Math course you must be a genius.
Every player on the football team has passed this course.
The captain of the football team is not a genius.
Therefore the captain of the football team does not play on the team.

Here the conclusion is false, but the argument is valid since the conclusion follows from the premises. If we observe that the first premise is false, the paradox disappears. There is nothing surprising in the correct derivation of a false conclusion from false premises.

If an argument is valid, then the conjunction of the premises implies the conclusion. Hence if all the premises are true, then the conclusion is also true. However, if one or more of the premises is false, so that the conjunction of all the premises is false, then the conclusion may be either true or false. In fact, all the premises could be false, the conclusion true, and the argument valid, as the following example shows.

Example 3.

All dogs have two legs.
All two-legged animals are carnivorous.
Therefore, all dogs are carnivorous.

Here the argument is valid and the conclusion is true, but both premises are false!

Each of these examples underlines the fact that neither the truth value nor the content of the statements appearing in an argument affect the validity of the argument. In Figures 31a and 31b are two valid forms of arguments.

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

Figure 31a

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

Figure 31b

The symbol ∴ means "therefore." The truth tables for these argument forms appear in Figure 32.

p	q	$p \to q$	p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	T	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	F	T	F	F	T	T	T

Figure 32

For the argument of Figure 31a, we see in Figure 32 that there is only one case in which both premises are true, namely, the first case, and in this case the conclusion is true, hence the argument is valid. Similarly, in the argument of Figure 31b, both premises are true in the fourth case only, and in this case the conclusion is also true, hence the argument is valid.

An argument that is not valid is called a *fallacy*. Two examples of fallacies are the following argument forms.

$$\begin{array}{l} p \to q \\ \underline{q \quad\quad} \\ \therefore p \end{array} \qquad \textit{Fallacies} \qquad \begin{array}{l} p \to q \\ \underline{\sim p \quad\quad} \\ \therefore \sim q \end{array}$$

In the first fallacy, both premises are true in the first and third cases of Figure 32, but the conclusion is false in the third case, so that the argument is invalid. (This is the form of Example 1.) Similarly, in the second fallacy we see that both premises are true in the last two cases, but the conclusion is false in the third case.

We say that an argument depends only upon its form in that it does not matter what the components of the argument are. The truth tables in Figure 32 show that if both premises are true, then the conclusions of the arguments in Figures 31a and 31b are also true. For the fallacies above, the truth tables show that it is possible to choose both premises true without making the conclusion true, namely, choose a false p and a true q.

Example 4. Consider the following argument.

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

The truth table of the argument appears in Figure 33.

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Figure 33

Both premises are true in the first, fifth, seventh, and eighth rows of the truth table. Since in each of these cases the conclusion is also true, the argument is valid. (Example 3 can be written in this form.)

Once we have discovered that a certain form of argument is valid, we can use it in drawing conclusions. It is then no longer necessary to compute truth tables. Presumably, this is what we do when we reason in everyday life; we apply a variety of valid forms known to us from previous experience. However, the truth table method has one great advantage: It is always applicable and purely automatic. We can even get a computer to test the validity of arguments involving compound statements.

EXERCISES

1. Test the validity of the following arguments.

(a) $$\begin{array}{l} p \leftrightarrow q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

(b) $$\begin{array}{l} p \lor q \\ \underline{\sim p \qquad} \\ \therefore \ q \end{array}$$

(c) $$\begin{array}{l} p \land q \\ \underline{\sim p \to q} \\ \therefore \sim q \end{array}$$

[*Ans.* (a), (b) are valid.]

2. Test the validity of the following arguments.

(a) $$\begin{array}{l} p \to \ q \\ \underline{\sim q \to \sim r} \\ \therefore \ r \to \ p \end{array}$$

(b) $$\begin{array}{l} p \to \ q \\ \underline{\sim r \to \sim q} \\ \therefore \sim r \to \sim p \end{array}$$

[*Ans.* (b) is valid.]

3. Test the validity of the argument

$$\begin{array}{rl} & p \leftrightarrow q \\ & q \vee r \\ & \underline{\sim r} \\ \therefore & \sim p \end{array}$$

[*Ans.* Not valid.]

4. Test the validity of the argument

$$\begin{array}{rl} & p \veebar q \\ & \sim q \to r \\ & \underline{\sim p \vee \sim r} \\ \therefore & \sim p \end{array}$$

5. Test the validity of the argument

$$\begin{array}{rl} & p \to q \\ & \sim p \to \sim q \\ & \underline{p \wedge \sim r} \\ \therefore & s \end{array}$$

6. Given are the premises $\sim p \to q$ and $\sim r \to \sim q$. We wish to find a valid conclusion involving p and r (if there is any).

(a) Construct truth tables for the two premises.

(b) Note the cases in which the conclusion must be true.

(c) Construct a truth table for a combination of p and r only, filling in T wherever necessary.

(d) Fill in the remainder of the truth table, making sure that you do not end up with a logically true statement.

(e) What combination of p and r has this truth table? This is a valid conclusion. [*Ans.* $p \vee r$.]

7. Translate the following argument into symbolic form, and test its validity.

If this is a good course, then it is worth taking.
Either the grading is lenient, or the course is not worth taking.
But the grading is not lenient.
Therefore, this is not a good course. [*Ans.* Valid.]

8. Write the following argument in symbolic form, and test its validity.

"For the candidate to win, it is sufficient that he carry New York. He will carry New York only if he takes a strong stand on civil rights. He will not take a strong stand on civil rights. Therefore, he will not win."

9. Write the following argument in symbolic form and test its validity.

"Father praises me only if I can be proud of myself. Either I do well in sports or I cannot be proud of myself. If I study hard, then I cannot do well in sports. Therefore, if father praises me, then I do not study hard."

10. Supply a conclusion to the following argument, making it a valid argument. [Adapted from Lewis Carroll.]

"If he goes to a party, he does not fail to brush his hair.
To look fascinating it is necessary to be tidy.
If he is an opium eater, then he has no self-command.
If he brushes his hair, he looks fascinating.
He wears white kid gloves only if he goes to a party.
Having no self-command is sufficient to make one look untidy.
Therefore . . ."

SUPPLEMENTARY EXERCISES

11. Show that the following method may be used for testing the validity of an argument: Find the cases in which the conclusion is false, and show that in each case at least one premise is false.

12. Use the method of Exercise 11 to test Example 4.

13. Redo Exercise 1 using the method of Exercise 11.

14. Redo Exercise 4 using the method of Exercise 11.

15. Draw a valid conclusion from the following premises.

Either he is a man or a mouse.
He has no skill in athletics.
To be a man it is necessary to command respect.
A man can command respect only if he has some athletic skill.

16. Draw a valid conclusion from the following premises.

Either he will go to graduate school, or he will be drafted.
If he does not go to graduate school, he will get married.
If he gets married, he will need a good income.
He will not have a good income in the Army.

*11. *THE INDIRECT METHOD OF PROOF*

A proof is an argument which shows that a conditional statement of the form $p \rightarrow q$ is logically true. (Namely, p is the conjunction of the premises, and q is the conclusion of the argument.) Sometimes it is more convenient to show that an equivalent conditional statement is logically true.

Example 1. Let x and y be positive integers.

Theorem. If xy is an odd number, then x and y are both odd.

Proof. Suppose, on the contrary, that they are not both odd. Then one of them is even, say $x = 2z$. Then $xy = 2zy$ is an even number, contrary to hypothesis. Hence we have proved our theorem.

Example 2. "He did not know the first name of the president of the Jones Corporation, hence he cannot be an employee of that firm. Why? Because every employee of that firm calls the boss by his first name (behind his back). Therefore, if he were really an employee of Jones, then he would know Jones's first name."

These are simple examples of a very common form of argument, frequently used both in mathematics and in everyday discussions. Let us try to unravel the form of the argument.

Given:	xy is an odd number.	He doesn't know Jones's first name.	p
To prove:	x and y are both odd numbers.	He doesn't work for Jones.	q
Suppose:	x and y are not both odd numbers.	He does work for Jones.	$\sim q$
Then:	xy is an even number.	He must know what Jones's first name is.	$\sim p$

In each case we assume the contradictory to the conclusion and derive, by a valid argument, a result contradictory to the hypothesis. This is one form of the *indirect* method of proof.

To restate, what we want to do is to show that the conditional $p \to q$ is logically true; what we actually show is that the contrapositive

$$\text{(1)} \qquad \sim q \to \sim p$$

is logically true. Since these two statements are equivalent our procedure is valid. (See Section 9.)

There are several other important variants of this method of proof.

It is easy to check that the following statements have the same truth table as (are equivalent to) the conditional $p \rightarrow q$.

(2) $(p \wedge \sim q) \rightarrow \sim p$.
(3) $(p \wedge \sim q) \rightarrow q$.
(4) $(p \wedge \sim q) \rightarrow (r \wedge \sim r)$.

Statement (2) shows that in the indirect method of proof we may make use of the original hypothesis in addition to the contradictory assumption $\sim q$. Statement (3) shows that we may also use this double hypothesis in the direct proof of the conclusion q. Statement (4) shows that if, from the double hypothesis p and $\sim q$ we can arrive at a contradiction of the form $r \wedge \sim r$, then the proof of the original statement is complete. This last form of the method is often referred to as *reductio ad absurdum*.

These last forms of the method are very useful for the following reasons: First of all we see that we can always take $\sim q$ as a hypothesis in addition to p. Second we see that besides q there are two other conclusions ($\sim p$ or a contradiction) which are just as good.

EXERCISES

1. Construct indirect proofs for the following assertions.
(a) If x^2 is odd, then x is odd (x an integer).
(b) If I am to pass this course, I must do homework regularly.
(c) If he earns a great deal of money (more than $30,000), he is not a college professor.

2. Give a symbolic analysis of the following argument.

"If he is to succeed, he must be both competent and lucky. Because, if he is not competent, then it is impossible for him to succeed. If he is not lucky, something is sure to go wrong."

3. Construct indirect proofs for the following assertions.
(a) If $p \vee q$ and $\sim q$, then p.
(b) If $p \leftrightarrow q$ and $q \rightarrow \sim r$ and r, then $\sim p$.

4. Give a symbolic analysis of the following argument.

"If Jones is the murderer, then he knows the exact time of death and the murder weapon. Therefore, if he does not know the exact time or does not know the weapon, then he is not the murderer."

5. Verify that forms (2), (3), and (4) given above are equivalent to $p \rightarrow q$.

6. Give an example of an indirect proof of some statement in which from p and $\sim q$ a contradiction is derived.

7. Give a statement equivalent to $(p \wedge q) \rightarrow r$, which is in terms of $\sim p$, $\sim q$, and $\sim r$. Show how this can be used in a proof where there are two hypotheses given.

8. Use the indirect method to establish the validity of the following argument.

$$\begin{array}{rl} p \veebar & q \\ \sim p \rightarrow & r \\ r \rightarrow & s \\ \underline{q \rightarrow} & \underline{\sim s} \\ \therefore p & \end{array}$$

9. Use the indirect method on Exercise 7 of Section 10.

*12. APPLICATIONS TO SWITCHING CIRCUITS

The theory of compound statements has many applications to subjects other than pure mathematics. As an example we shall develop a theory of simple switching networks.

A switching network is an arrangement of wires and switches which connect together two terminals T_1 and T_2. Each switch can be either "open" or "closed." An open switch prevents the flow of current, while a closed switch permits flow. The problem that we want to solve is the following: Given a network and given the knowledge of which switches are closed, determine whether or not current will flow from T_1 to T_2.

Figure 34 shows the simplest kind of a network in which the terminals are connected by a single wire containing a switch P. If P is closed, then current will flow between the terminals; otherwise it does not.

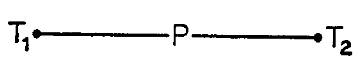


Figure 34

Figure 35

The network in Figure 35 has two switches P and Q in "series." Here the current flows only if both P and Q are closed.

To see how our logical analysis can be used to solve the problem stated above, let us associate a statement with each switch. Let p be the statement "Switch P is closed" and let q be the statement "Switch

Q is closed." Then in Figure 34 current will flow if and only if p is true. Similarly, in Figure 35 the current will flow if and only if both p and q are true, that is, if and only if $p \wedge q$ is true. Thus the first circuit is represented by p and the second by $p \wedge q$.

In Figure 36 is shown a network with switches P and Q in "parallel." In this case the current flows if either of the switches is closed, so the circuit is represented by the statement $p \vee q$.

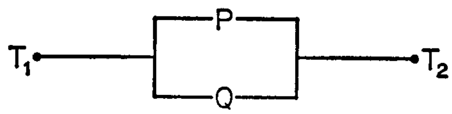


Figure 36

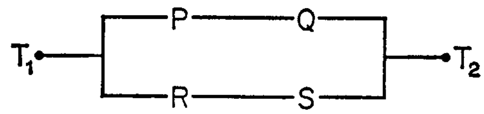


Figure 37

The network in Figure 37 combines the series and parallel types of connections. The upper branch of the network is represented by the statement $p \wedge q$ and the lower by $r \wedge s$; hence the entire circuit is represented by $(p \wedge q) \vee (r \wedge s)$. Since there are four switches and each one can be either open or closed, there are $2^4 = 16$ possible settings for these switches. Similarly, the statement $(p \wedge q) \vee (r \wedge s)$ has four variables, so that its truth table has 16 rows in it. The switch settings for which current flows correspond to the entries in the truth table for which the above compound statement is true.

Switches need not always act independently of each other. It is possible to couple two or more switches together so that they open and close simultaneously, and we shall indicate this in diagrams by giving all such switches the same letter. It is also possible to couple two switches together, so that if one is closed, the other is open. We shall indicate this by giving the first switch the letter P and the second the letter P′. Then the statement "P is closed" is true if and only if the statement "P′ is closed" is false. Therefore if p is the statement "P is closed," then $\sim p$ is equivalent to the statement "P′ is closed."

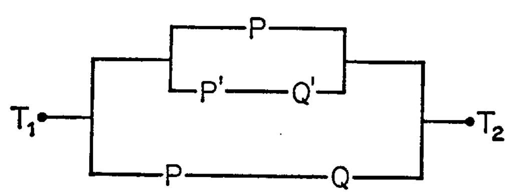


Figure 38

Such a circuit is illustrated in Figure 38. The associated compound statement is $[p \vee (\sim p \wedge \sim q)] \vee [p \wedge q]$. Since this statement is false only if p is false and q is true, the current will flow unless P is open and Q is closed. We can also check directly. If P is closed, current will flow through the top branch regardless of Q's setting. If both switches are open, then P′ and Q′ will be closed, so that current will flow through the middle branch. But if P is open and Q is closed, none of the branches will pass current.

Notice that we never had to consider current flow through the bottom branch. The logical counterpart of this fact is that the statement associated with the network is equivalent to $[p \vee (\sim p \wedge \sim q)]$ whose associated network is just the upper two branches of Figure 38. Thus the electrical properties of the circuit of Figure 38 would be the same if the lower branch were omitted.

As a last problem, we shall consider the design of a switching network having certain specified properties. An equivalent problem, which we solved in Section 4, is that of constructing a compound statement having a given truth table. As in that section, we shall limit ourselves to statements having three variables, although our methods could easily be extended.

In Section 4 we developed a general method for finding a statement having a given truth table not consisting entirely of F's. (The circuit which corresponds to a statement whose truth table consists entirely of F's is one in which current never flows, and hence is not of interest.) Each such statement could be constructed as a disjunction of basic conjunctions. Since the basic conjunctions were of the form $p \wedge q \wedge r$, $p \wedge q \wedge \sim r$, etc., each will be represented by a circuit consisting of three switches in series and will be called a *basic series circuit*. The disjunction of certain of these basic conjunctions will then be represented by the circuit obtained by putting several basic series circuits in parallel. The resulting network will not, in general, be the simplest possible such network fulfilling the requirements, but the method always suffices to find one.

Example. A three-man committee wishes to employ an electric circuit to record a secret simple majority vote. Design a circuit so that each member can push a button for his "yes" vote (not push it for a "no" vote), and so that a signal light will go on if a majority of the committee members vote yes.

Let p be the statement "Committee member 1 votes yes," let q be the statement "Member 2 votes yes," and let r be "Member 3 votes yes." The truth table of the statement "Majority of the members

p	q	r	Desired Truth Value	Corresponding Basic Conjunction
T	T	T	T	$p \wedge q \wedge r$
T	T	F	T	$p \wedge q \wedge \sim r$
T	F	T	T	$p \wedge \sim q \wedge r$
T	F	F	F	$p \wedge \sim q \wedge \sim r$
F	T	T	T	$\sim p \wedge q \wedge r$
F	T	F	F	$\sim p \wedge q \wedge \sim r$
F	F	T	F	$\sim p \wedge \sim q \wedge r$
F	F	F	F	$\sim p \wedge \sim q \wedge \sim r$

Figure 39

vote yes" appears in Figure 39. From that figure we can read off the desired compound statement as

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r).$$

The circuit desired for the voting procedure appears in Figure 40.

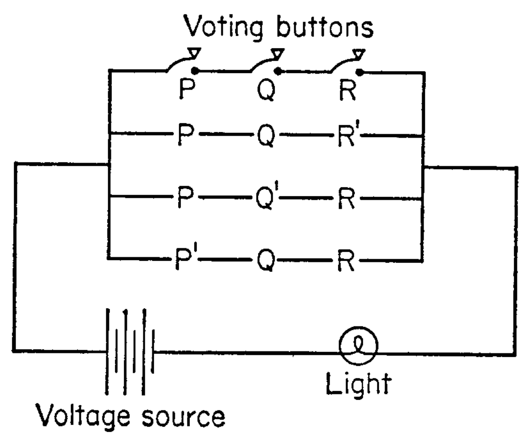


Figure 40

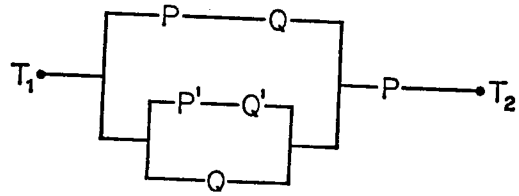
EXERCISES

1. What kind of a circuit has a logically true statement assigned to it? Give an example.

2. Construct a network corresponding to

$$[(p \wedge \sim q) \vee (\sim p \wedge q)] \vee (\sim p \wedge \sim q).$$

3. What compound statement represents the following circuit?



4. Work out the truth table of the statement in Exercise 3. What does this tell us about the circuit?

5. Design a simpler circuit than the one in Exercise 3, having the same properties.

6. Construct a network corresponding to

$$[(p \vee q) \wedge \sim r] \vee [(\sim p \wedge r) \vee q].$$

7. Design a circuit for an electrical version of the game of matching pennies: At a given signal each of the two players either opens or closes a switch under his control. If they both do the same, A wins; if they do the opposite, then B wins. Design the circuit so that a light goes on if A wins.

8. In a large hall it is desired to turn the lights on or off from any one of four switches on the four walls. This can be accomplished by designing a circuit which turns the light on if an even number of switches are closed, and off if an odd number are closed. (Why does this solve the problem?) Design such a circuit.

9. A committee has five members. It takes a majority vote to carry a measure, except that the chairman has a veto (i.e., the measure carries only if he votes for it). Design a circuit for the committee, so that each member votes for a measure by pressing a button, and the light goes on if and only if the measure is carried.

10. A group of candidates is asked to take a true-false exam, with four questions. Design a circuit such that a candidate can push the buttons of those questions to which he wants to answer "true," and that the circuit will indicate the number of correct answers. [*Hint:* Have five lights, corresponding to 0, 1, 2, 3, 4 correct answers, respectively.]

11. Devise a scheme for working truth tables by means of switching circuits.

12. Figure 40 uses 12 switches. Find a circuit that accomplishes the same goal with only 5 switches. Check that the corresponding truth table agrees with Figure 39.

SUGGESTED READING

Tarski, A., *Introduction to Logic*, Oxford University Press, New York, 2d rev. ed., 1946, Chapters I, II.

Suppes, P., *Introduction to Logic*, D. Van Nostrand, Princeton, 1957, Chapters 1, 2, 3.

Cohen, M. R., and E. Nagel, *An Introduction to Logic and Scientific Method*, Harcourt, Brace, New York, 1934.

Church, A., *Introduction to Mathematical Logic*, Volume I, Princeton University Press, 1956.

Hilbert, D., and W. Ackermann, *Principles of Mathematical Logic*, Chelsea, New York, 2d ed., 1950.

Allendoerfer, C. B., and C. O. Oakley, *Principles of Mathematics*, McGraw-Hill, New York, 1955, Chapter I.

Johnstone, H. W., Jr., *Elementary Deductive Logic*, Crowell, New York, 1954. Parts One and Two.

Hohn, Franz, "Some Mathematical Aspects of Switching," *The American Mathematical Monthly*, 62 (1955), pp. 75–90.