IV

Probability theory

1. INTRODUCTION

We often hear statements of the following kind: "It is likely to rain today," "I have a fair chance of passing this course," "There is an even chance that a coin will come up heads," etc. In each case our statement refers to a situation in which we are not certain of the outcome, but we express some degree of confidence that our prediction will be verified. The theory of probability provides a mathematical framework for such assertions.

Consider an experiment whose outcome is not known. Suppose that someone makes an assertion p about the outcome of the experiment, and we want to assign a probability to p. When statement p is considered in isolation, we usually find no natural assignment of probabilities. Rather, we look for a method of assigning probabilities to all conceivable statements concerning the outcome of the experiment. At first this might seem to be a hopeless task, since there is no end to the statements we can make about the experiment. However we are aided by a basic principle:

Fundamental assumption. Any two equivalent statements will be assigned the same probability.

As long as there are a finite number of logical possibilities, there are only a finite number of truth sets, and hence the process of assigning probabilities is a finite one. We proceed in three steps: (1) we first determine $\mathcal{U}$, the possibility set, that is, the set of all logical possibilities, (2) to each subset X of $\mathcal{U}$ we assign a number called the measure $m(X)$, (3) to each statement p we assign $m(p)$, the measure of its truth set, as a probability. The probability of statement p is denoted by $\Pr[p]$.

The first step, that of determining the set of logical possibilities, is one that we considered in the previous chapters. It is important to recall that there is no unique method for analyzing logical possibilities. In a given problem we may arrive at a very fine or a very rough analysis of possibilities, causing $\mathcal{U}$ to have many or few elements.

Having chosen $\mathcal{U}$, the next step is to assign a number to each subset X of $\mathcal{U}$, which will in turn be taken to be the probability of any statement having truth set X. We do this in the following way.

Assignment of a measure. Assign a positive number (weight) to each element of $\mathcal{U}$, so that the sum of the weights assigned is 1. Then the measure of a set is the sum of the weights of its elements. The measure of the set $\mathcal{E}$ is 0.

In applications of probability to scientific problems, the analysis of the logical possibilities and the assignment of measures may depend upon factual information and hence can best be done by the scientist making the application.

Once the weights are assigned, to find the probability of a particular statement we must find its truth set and find the sum of the weights assigned to elements of the truth set. This problem, which might seem easy, can often involve considerable mathematical difficulty. The development of techniques to solve this kind of problem is the main task of probability theory.

Example 1. An ordinary die is thrown. What is the probability that the number which turns up is less than four? Here the possibility set is $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$. The symmetry of the die suggests that each face should have the same probability of turning up. To make this so, we assign weight $\frac{1}{6}$ to each of the outcomes. The truth set of the statement "The number which turns up is less than four" is $\{1, 2, 3\}$. Hence the probability of this statement is $\frac{3}{6} = \frac{1}{2}$, the sum of the weights of the elements in its truth set.

Example 2. A man attends a race involving three horses A, B, and C. He feels that A and B have the same chance of winning but that A (and hence also B) is twice as likely to win as C is. What is the probability that A or C wins? We take as $\mathcal{U}$ the set $\{A, B, C\}$. If we were to assign weight a to the outcome C, then we would assign weight $2a$ to each of the outcomes A and B. Since the sum of the weights must be 1, we have $2a + 2a + a = 1$, or $a = \frac{1}{5}$. Hence we assign weights $\frac{2}{5}, \frac{2}{5}, \frac{1}{5}$ to the outcomes A, B, and C, respectively. The truth set of the statement "Horse A or C wins" is $\{A, C\}$. The sum of the weights of the elements of this set is $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$. Hence the probability that A or C wins is $\frac{3}{5}$.

EXERCISES

1. Assume that there are n possibilities for the outcome of a given experiment. How should the weights be assigned if it is desired that all outcomes be assigned the same weight?

2. Let $\mathcal{U} = \{a, b, c\}$. Assign weights to the three elements so that no two have the same weight, and find the measures of the eight subsets of $\mathcal{U}$.

3. In an election Jones has probability $\frac{1}{2}$ of winning, Smith has probability $\frac{1}{3}$, and Black has probability $\frac{1}{6}$.

(a) Construct $\mathcal{U}$.
(b) Assign weights.
(c) Find the measures of the eight subsets.
(d) Give a pair of nonequivalent predictions which have the same probability.

4. Give the possibility set $\mathcal{U}$ for each of the following experiments.

(a) An election between candidates A and B is to take place.
(b) A number from 1 to 5 is chosen at random.
(c) A two-headed coin is thrown.
(d) A student is asked for the day of the year on which his birthday falls.

5. For which of the cases in Exercise 4 might it be appropriate to assign the same weight to each outcome?

6. Suppose that the following probabilities have been assigned to the possible results of putting a penny in a certain defective peanut-vending machine: The probability that nothing comes out is $\frac{1}{2}$. The probability that either you get your money back or you get peanuts (but not both) is $\frac{1}{3}$.

(a) What is the probability that you get your money back and also get peanuts? [*Ans.* $\frac{1}{6}$.]

(b) From the information given, is it possible to find the probability that you get peanuts? [*Ans.* No.]

7. A die is loaded in such a way that the probability of each face is proportional to the number of dots on that face. (For instance, a 6 is three times as probable as a 2.) What is the probability of getting an even number in one throw? [*Ans.* $\frac{4}{7}$.]

8. If a coin is thrown three times, list the eight possibilities for the outcomes of the three successive throws. A typical outcome can be written (HTH). Determine a probability measure by assigning an equal weight to each outcome. Find the probabilities of the following statements.

(r) The number of heads that occur is greater than the number of tails. [*Ans.* $\frac{1}{2}$.]

(s) Exactly two heads occur. [*Ans.* $\frac{3}{8}$.]

(t) The same side turns up on every throw. [*Ans.* $\frac{1}{4}$.]

9. For the statements given in Exercise 8, which of the following equalities are true?

(a) $$\Pr[r \vee s] = \Pr[r] + \Pr[s]$$

(b) $$\Pr[s \vee t] = \Pr[s] + \Pr[t]$$

(c) $$\Pr[r \vee \sim r] = \Pr[r] + \Pr[\sim r]$$

(d) $$\Pr[r \vee t] = \Pr[r] + \Pr[t]$$

10. Which of the following pairs of statements (see Exercise 8) are inconsistent? (Recall that two statements are inconsistent if their truth sets have no element in common.)

(a) r, s (b) s, t

(c) $r, \sim r$ (d) r, t [*Ans.* (b) and (c).]

11. State a theorem suggested by Exercises 9 and 10.

12. An experiment has three possible outcomes, a, b, and c. Let p be the statement "the outcome is a or b," and q be the statement "the outcome is b or c." Assume that weights have been assigned to the three outcomes so that $\Pr[p] = \frac{2}{3}$ and $\Pr[q] = \frac{5}{6}$. Find the weights. [*Ans.* $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$.]

13. Repeat Exercise 12 if $\Pr[p] = \frac{1}{2}$ and $\Pr[q] = \frac{3}{8}$.

2. *PROPERTIES OF A PROBABILITY MEASURE*

Before studying special probability measures, we shall consider some general properties of such measures which are useful in computations and in the general understanding of probability theory.

Three basic properties of a probability measure are

(A) $m(X) = 0$ if and only if $X = \mathcal{E}$.

(B) $0 \leq m(X) \leq 1$ for any set X.

(C) For two sets X and Y,

$$m(X \cup Y) = m(X) + m(Y)$$

if and only if X and Y are disjoint, i.e., have no elements in common.

The proofs of properties (A) and (B) are left as an exercise (see Exercise 16). We shall prove (C).

We observe first that $m(X) + m(Y)$ is the sum of the weights of the elements of X added to the sum of the weights of Y. If X and Y are disjoint, then the weight of every element of $X \cup Y$ is added once and only once, and hence $m(X) + m(Y) = m(X \cup Y)$.

Assume now that X and Y are not disjoint. Here the weight of every element contained in both X and Y, i.e., in $X \cap Y$, is added twice in the sum $m(X) + m(Y)$. Thus this sum is greater than $m(X \cup Y)$ by an amount $m(X \cap Y)$. By (A) and (B), if $X \cap Y$ is not the empty set, then $m(X \cap Y) > 0$. Hence in this case we have $m(X) + m(Y) > m(X \cup Y)$. Thus if X and Y are not disjoint, the equality in (C) does not hold. Our proof shows that in general we have

(C′) For any two sets X and Y,

$$m(X \cup Y) = m(X) + m(Y) - m(X \cap Y).$$

Since the probabilities for statements are obtained directly from the probability measure $m(X)$, any property of $m(X)$ can be translated into a property about the probability of statements. For example, the above properties become, when expressed in terms of statements,

(a) $\Pr[p] = 0$ if and only if p is logically false.

(b) $0 \leq \Pr[p] \leq 1$ for any statement p.

(c) The equality

$$\Pr[p \vee q] = \Pr[p] + \Pr[q]$$

holds if and only if p and q are inconsistent.

(c′) For any two statements p and q,

$$\Pr[p \vee q] = \Pr[p] + \Pr[q] - \Pr[p \wedge q].$$

Another property of a probability measure which is often useful in computation is

(D) $\quad m(\tilde{X}) = 1 - m(X)$,

or, in the language of statements,

(d) $\quad \Pr[\sim p] = 1 - \Pr[p]$.

The proofs of (D) and (d) are left as an exercise (see Exercise 17).

It is important to observe that our probability measure assigns probability 0 only to statements which are logically false, i.e., which are false for every logical possibility. Hence, a prediction that such a statement will be true is certain to be wrong. Similarly, a statement is assigned probability 1 only if it is true in every case, i.e., logically true. Thus the prediction that a statement of this type will be true is certain to be correct. (While these properties of a probability measure seem quite natural, it is necessary, when dealing with infinite possibility sets, to weaken them slightly. We consider in this book only the finite possibility sets.)

We shall now discuss the interpretation of probabilities that are not 0 or 1. We shall give only some intuitive ideas that are commonly held concerning probabilities. While these ideas can be made mathematically more precise, we offer them here only as a guide to intuitive thinking.

Suppose that, relative to a given experiment, a statement has been assigned probability p. From this it is often inferred that if a sequence of such experiments is performed under identical conditions, the fraction of experiments which yield outcomes making the statement true would be approximately p. The mathematical version of this is the "law of large numbers" of probability theory (which will be treated in Section 10). In cases where there is no natural way to assign a probability measure, the probability of a statement is estimated experimentally. A sequence of experiments is performed and the fraction of the experiments which make the statement true is taken as the approximate probability for the statement.

A second and related interpretation of probabilities is concerned with betting. Suppose that a certain statement has been assigned probability p. We wish to offer a bet that the statement will in fact turn out to be true. We agree to give r dollars if the statement does not turn out to be true, provided that we receive s dollars if it does turn out to be true. What should r and s be to make the bet fair? If it were true that in a large number of such bets we would win s a fraction p of the times and lose r a fraction $1 - p$ of the time, then our average winning per bet would be $sp - r(1 - p)$. To make the bet fair we should make this average winning 0. This will be the case if $sp = r(1 - p)$ or if $r/s = p/(1 - p)$. Notice that this determines only the ratio of r and s. Such a ratio, written $r{:}s$, is said to give *odds* in favor of the statement.

DEFINITION. The *odds* in favor of an outcome are $r{:}s$ (r *to* s), if the probability of the outcome is p, and $r/s = p/(1 - p)$. Any two numbers having the required ratio may be used in place of r and s. Thus 6:4 odds are the same as 3:2 odds.

Example. Assume that a probability of $\frac{3}{4}$ has been assigned to a certain horse winning a race. Then the odds for a fair bet would be $\frac{3}{4}{:}\frac{1}{4}$. These odds could be equally well written as 3:1, 6:2 or 12:4, etc. A fair bet would be to agree to pay $3 if the horse loses and receive $1 if the horse wins. Another fair bet would be to pay $6 if the horse loses and win $2 if the horse wins.

EXERCISES

1. Let p and q be statements such that $\Pr[p \wedge q] = \frac{1}{4}$, $\Pr[\sim p] = \frac{1}{3}$, and $\Pr[q] = \frac{1}{2}$. What is $\Pr[p \vee q]$? [*Ans.* $\frac{11}{12}$.]

2. Using the result of Exercise 1, find $\Pr[\sim p \wedge \sim q]$.

3. Let p and q be statements such that $\Pr[p] = \frac{1}{2}$ and $\Pr[q] = \frac{2}{3}$. Are p and q consistent? [*Ans.* Yes.]

4. Show that, if $\Pr[p] + \Pr[q] > 1$, then p and q are consistent.

5. A student is worried about his grades in English and Art. He estimates that the probability of passing English is .4, that he will pass at least one course with probability .6, but that he has only probability .1 of passing both courses. What is the probability that he will pass Art? [*Ans.* .3.]

6. Given that a school has grades A, B, C, D, and F, and that a student has probability .9 of passing a course, and .6 of getting a grade lower than B, what is the probability that he will get a C or D? [*Ans.* $\frac{1}{2}$.]

7. What odds should a person give on a bet that a six will turn up when a die is thrown?

8. Referring to Example 2 of Section 1, what odds should the man be willing to give for a bet that either A or B will come in first?

9. Prove that if the odds in favor of a given statement are $r{:}s$, then the probability that the statement will be true is $r/(r + s)$.

10. Using the result of Exercise 9 and the definition of "odds," show that if the odds are $r:s$ that a statement is true, then the odds are $s:r$ that it is false.

11. A man is willing to give 5:4 odds that the Dodgers will win the World Series. What must the probability of a Dodger victory be for this to be a fair bet? [*Ans.* $\frac{5}{9}$.]

12. A man has found through long experience that if he washes his car, it rains the next day 85 per cent of the time. What odds should he give that this will occur next time?

13. A man offers 1:3 odds that A will occur, 1:2 odds that B will occur. He knows that A and B cannot both occur. What odds should he give that A or B will occur? [*Ans.* 7:5.]

14. A man offers 3:1 odds that A will occur, 2:1 odds that B will occur. He knows that A and B cannot both occur. What odds should he give that A or B will occur?

15. Show from the definition of a probability measure that $m(X) = 1$ if and only if $X = \mathcal{U}$.

16. Show from the definition of a probability measure that properties (A), (B) of the text are true.

17. Prove property (D) of the text. Why does property (d) follow from this property?

18. Prove that if R, S, and T are three sets that have no element in common,

$$m(R \cup S \cup T) = m(R) + m(S) + m(T).$$

19. If X and Y are two sets such that X is a subset of Y, prove that $m(X) \leq m(Y)$.

20. If p and q are two statements such that p implies q, prove that $\Pr[p] \leq \Pr[q]$.

21. Suppose that you are given n statements and each has been assigned a probability less than or equal to r. Prove that the probability of the disjunction of these statements is less than or equal to nr.

22. The following is an alternative proof of property (C′) of the text. Give a reason for each step.

(a) $X \cup Y = (X \cap \tilde{Y}) \cup (X \cap Y) \cup (Y \cap \tilde{X})$.
(b) $m(X \cup Y) = m(X \cap \tilde{Y}) + m(X \cap Y) + m(\tilde{X} \cap Y)$.
(c) $m(X \cup Y) = m(X) + m(Y) - m(X \cap Y)$.

23. If X, Y, and Z are any three sets, prove that, for any probability measure,

$$m(X \cup Y \cup Z) = m(X) + m(Y) + m(Z) - m(X \cap Y) - m(Y \cap Z) - m(X \cap Z) + m(X \cap Y \cap Z).$$

24. Translate the result of Exercise 23 into a result concerning three statements p, q, and r.

25. A man offers to bet "dollars to doughnuts" that a certain event will take place. Assuming that a doughnut costs a nickel, what must the probability of the event be for this to be a fair bet? [*Ans.* $\frac{20}{21}$.]

26. Show that the inclusion-exclusion formula [Chapter III, Section 8, formula (2)] is true if n is replaced by m. Apply this result to

$$\Pr(p_1 \vee p_2 \vee \ldots \vee p_n).$$

3. *THE EQUIPROBABLE MEASURE*

We have already seen several examples where it was natural to assign the same weight to all possibilities in determining the appropriate probability measure. The probability measure determined in this manner is called the *equiprobable measure.* The measure of sets in the case of the equiprobable measure has a very simple form. In fact, if $\mathcal{U}$ has n elements and if the equiprobable measure has been assigned, then for any set X, $m(X)$ is r/n, where r is the number of elements in the set X. This is true since the weight of each element in X is $1/n$, and hence the sum of the weights of elements of X is r/n.

The particularly simple form of the equiprobable measure makes it easy to work with. In view of this, it is important to observe that a particular choice for the set of possibilities in a given situation may lead to the equiprobable measure, while some other choice will not. For example, consider the case of two throws of an ordinary coin. Suppose that we are interested in statements about the number of heads which occur. If we take for the possibility set the set $\mathcal{U} = \{\text{HH, HT, TH, TT}\}$ then it is reasonable to assign the same weight to each outcome, and we are led to the equiprobable measure. If, on the other hand, we were to take as possible outcomes the set $\mathcal{U} = \{\text{no H, one H, two H}\}$, it would not be natural to assign the same weight to each outcome, since one head can occur in two different ways, while each of the other possibilities can occur in only one way.

Example 1. Suppose that we throw two ordinary dice. Each die can turn up a number from 1 to 6; hence there are $6 \cdot 6$ possibilities. We assign weight $\frac{1}{36}$ to each possibility. A prediction that is true in j cases will then have probability $j/36$. For example, "The sum of the dice is 5" will be true if we get $1 + 4$, $2 + 3$, $3 + 2$, or $4 + 1$.

Hence the probability that the sum of the dice is 5 is $\frac{4}{36} = \frac{1}{9}$. The sum can be 12 in only one way, $6 + 6$. Hence the probability that the sum is 12 is $\frac{1}{36}$.

Example 2. Suppose that two cards are drawn successively from a deck of cards. What is the probability that both are hearts? There are 52 possibilities for the first card, and for each of these there are 51 possibilities for the second. Hence there are $52 \cdot 51$ possibilities for the result of the two draws. We assign the equiprobable measure. The statement "The two cards are hearts" is true in $13 \cdot 12$ of the $52 \cdot 51$ possibilities. Hence the probability of this statement is $13 \cdot 12/52 \cdot 51 = \frac{1}{17}$.

Example 3. Assume that, on the basis of a predictive index applied to students A, B, and C when entering college, it is predicted that after four years of college the scholastic record of A will be the highest, C the second highest, and B the lowest of the three. Suppose, in fact, that these predictions turn out to be exactly correct. If the predictive index has no merit at all and hence the predictions amount simply to guessing, what is the probability that such a prediction will be correct? There are $3! = 6$ orders in which the men might finish. If the predictions were really just guessing, then we would assign an equal weight to each of the six outcomes. In this case the probability that a particular prediction is true is $\frac{1}{6}$. Since this probability is reasonably large, we would hesitate to conclude that the predictive index is in fact useful, on the basis of this one experiment. Suppose, on the other hand, it predicted the order of six men correctly. Then a similar analysis would show that, by guessing, the probability is $\dfrac{1}{6!} = \dfrac{1}{720}$ that such a prediction would be correct. Hence, we might conclude here that there is strong evidence that the index has some merit.

EXERCISES

1. A letter is chosen at random from the word "random." What is the probability that it is an n? That it is a vowel? [*Ans.* $\frac{1}{6}$; $\frac{1}{3}$.]

2. An integer between 3 and 12 inclusive is chosen at random. What is the probability that it is an even number? That it is even and divisible by three?

3. A card is drawn at random from a pack of playing cards.

(a) What is the probability that it is either a heart or the king of clubs? [*Ans.* $\frac{7}{26}$.]

(b) What is the probability that it is either the queen of hearts or an honor card (i.e., ten, jack, queen, king, or ace)? [*Ans.* $\frac{5}{13}$.]

4. A word is chosen at random from the set of words $\mathfrak{U}$ = {men, bird, ball, field, book}. Let p, q, and r be the statements:

p: The word has two vowels.
q: The first letter of the word is b.
r: The word rhymes with cook.

Find the probability of the following statements.

(a) p.
(b) q.
(c) r.
(d) $p \wedge q$.
(e) $(p \vee q) \wedge \sim r$.
(f) $p \rightarrow q$. [*Ans.* $\frac{4}{5}$.]

5. A single die is thrown. Find the probability that

(a) An odd number turns up.
(b) The number which turns up is greater than two.
(c) A seven turns up.

6. In the primary voting example of Chapter II, Section 1, assume that all 36 possibilities in the elections are equally likely. Find

(a) The probability that candidate A wins more states than either of his rivals. [*Ans.* $\frac{7}{18}$.]
(b) That all the states are won by the same candidate. [*Ans.* $\frac{1}{36}$.]
(c) That every state is won by a different candidate. [*Ans.* 0.]

7. A single die is thrown twice. What value for the sum of the two outcomes has the highest probability? What value or values of the sum has the lowest probability of occurring?

8. Two boys and two girls are placed at random in a row for a picture. What is the probability that the boys and girls alternate in the picture? [*Ans.* $\frac{1}{3}$.]

9. A certain college has 500 students and it is known that

300 read French.
200 read German.
50 read Russian.
20 read French and Russian.
30 read German and Russian.
20 read German and French.
10 read all three languages.

If a student is chosen at random from the school, what is the probability that the student

(a) Reads two and only two languages?
(b) Reads at least one language?

10. Suppose that three people enter a restaurant which has a row of six seats. If they choose their seats at random, what is the probability that they sit with no seats between them? What is the probability that there is at least one empty seat between any two of them?

11. Find the probability of obtaining each of the following poker hands. (A poker hand is a set of five cards chosen at random from a deck of 52 cards.)

(a) Royal flush (ten, jack, queen, king, ace in a single suit). [*Ans.* $4/\binom{52}{5} = .0000015$.]
(b) Straight flush (five in a sequence in a single suit, but not a royal flush). [*Ans.* $(40 - 4)/\binom{52}{5} = .000014$.]
(c) Four of a kind (four cards of the same face value). [*Ans.* $624/\binom{52}{5} = .00024$.]
(d) Full house (one pair and one triple of the same face value). [*Ans.* $3744/\binom{52}{5} = .0014$.]
(e) Flush (five cards in a single suit but not a straight or royal flush). [*Ans.* $(5148 - 40)/\binom{52}{5} = .0020$.]
(f) Straight (five cards in a row, not all of the same suit). [*Ans.* $(10{,}240 - 40)/\binom{52}{5} = .0039$.]
(g) Straight or better. [*Ans.* .0076.]

12. If ten people are seated at a circular table at random, what is the probability that a particular pair of people are seated next to each other? [*Ans.* $\frac{2}{9}$.]

13. A room contains a group of n people who are wearing badges numbered from 1 to n. If two people are selected at random, what is the probability that the larger badge number is a 3? Answer this problem assuming that $n = 5, 4, 3, 2$. [*Ans.* $\frac{1}{5}$; $\frac{1}{3}$; $\frac{2}{3}$; 0.]

14. In Exercise 13, suppose that we observe two men leaving the room and that the larger of their badge numbers is 3. What might we guess as to the number of people in the room?

15. Find the probability that a bridge hand will have suits of

(a) 5, 4, 3, and 1 cards. [*Ans.* $\dfrac{4!\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} \cong .129$.]
(b) 6, 4, 2, and 1 cards. [*Ans.* .047.]
(c) 4, 4, 3, and 2 cards. [*Ans.* .216.]
(d) 4, 3, 3, and 3 cards. [*Ans.* .105.]

16. There are $\binom{52}{13} = 6.35 \times 10^{11}$ possible bridge hands. Find the probability that a bridge hand dealt at random will be all of one suit. Estimate *roughly* the number of bridge hands dealt in the entire country in a year. Is it likely that a hand of all one suit will occur sometime during the year in the United States?

SUPPLEMENTARY EXERCISES

17. Find the probability of *not* having a pair in a hand of poker.

18. Find the probability of a "bust" hand in poker. [*Hint:* A hand is a "bust" if there is no pair, and it is neither a straight nor a flush.]
[*Ans.* .5012.]

19. In poker, find the probability of having
(a) Exactly one pair. [*Ans.* .4226.]
(b) Two pairs. [*Ans.* .0475.]
(c) Three of a kind. [*Ans.* .0211.]

20. Verify from Exercises 11, 18, and 19 that the probabilities for all possible poker hands add up to one (within a rounding error).

21. A certain French professor announces that he will select three out of eight pages of text to put on an examination and that each student can choose one of these three pages to translate. What is the maximum number of pages that a student should prepare in order to be certain of being able to translate a page that he has studied?

Smith decides to study only four of the eight pages. What is the probability that one of these four pages will appear on the examination?

*4. TWO NONINTUITIVE EXAMPLES

There are occasions in probability theory when one finds a problem for which the answer, based on probability theory, is not at all in agreement with one's intuition. It is usually possible to arrange a few wagers that will bring one's intuition into line with the mathematical theory. A particularly good example of this is provided by the matching birthdays problem.

Assume that we have a room with r people in it and we propose the bet that there are at least two people in the room having the same birthday, i.e., the same month and day of the year. We ask for the value of r which will make this a fair bet. Few people would be willing to bet even money on this wager unless there were at least 100 people in the room. Most people would suggest 150 as a reasonable number. However, we shall see that with 150 people the odds are approximately 4,100,000,000,000,000, to 1 in favor of two people having the same birthday, and that one should be willing to bet even money with as few as 23 people in the room.

Let us first find the probability that in a room with r people, no two have the same birthday. There are 365 possibilities for each person's birthday (neglecting February 29). There are then 365^r possibilities for the birthdays of r people. We assume that all these possibilities are equally likely. To find the probability that no two have the same birthday we must find the number of possibilities for the birthdays which have *no* day represented twice. The first person can have any of 365 days for his birthday. For each of these, if the second person is to have a different birthday, there are only 364 possibilities for his birthday. For the third man, there are 363 possibilities if he is to have a different birthday than the first two, etc. Thus the probability that no two people have the same birthday in a group of r people is

$$q_r = \frac{365 \cdot 364 \cdot \ldots \cdot (365 - r + 1)}{365^r}.$$

The probability that at least two people have the same birthday is then $p_r = 1 - q_r$. In Figure 1 the values of p_r and the odds for a fair bet, $p_r:(1 - p_r)$, are given for several values of r.

We consider now a second problem in which intuition does not lead to the correct answer. A hat-check girl has checked n hats, but they have become hopelessly scrambled. She hands back the hats at random. What is the probability that at least one man gets his own hat? For this problem some people's intuition would lead them to guess that for a large number of hats this probability should be small, while others guess that it should be large. Few people guess that the probability is neither large nor small and essentially independent of the number of hats involved.

Let p_j be the statement "the jth man gets his own hat back." We wish to find $\Pr [p_1 \vee p_2 \vee \ldots \vee p_n]$. We know from Section 2, Exercise 26, that a probability of this form can be found from the inclusion-exclusion formula. We must add all probabilities of the form $\Pr [p_i]$, then subtract the sum of all probabilities of the form $\Pr [p_i \wedge p_j]$, then add the sum of all probabilities of the form $\Pr [p_i \wedge p_j \wedge p_k]$, etc.

Number of people in the room	Probability of at least two with same birthday	Approximate odds for a fair bet
5	.027	
10	.117	
15	.253	
20	.411	70:100
21	.444	80:100
22	.476	91:100
23	.507	103:100
24	.538	117:100
25	.569	132:100
30	.706	241:100
40	.891	819:100
50	.970	33:1
60	.994	170:1
70		1,200:1
80		12,000:1
90		160,000:1
100		3,300,000:1
125		31,000,000,000:1
150		4,100,000,000,000,000:1

Figure 1

However, each of these probabilities represents the probability that a particular set of men get their own hats back. These probabilities are very easy to compute.

Let us find the probability that out of n men some particular m of them get back their own hats. There are $n!$ ways that the hats can be returned. If a particular m of them are to get their own hats there are only $(n - m)!$ ways that it can be done. Hence the probability that a particular m men get their own hats back is

$$\frac{(n-m)!}{n!}.$$

There are $\binom{n}{m}$ different ways we can choose m men out of n. Hence the mth group of terms contributes

$$\binom{n}{m} \cdot \frac{(n-m)!}{n!} = \frac{1}{m!}$$

to the alternating sum. Thus

$$\Pr[p_1 \vee p_2 \vee \ldots \vee p_n] = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \cdots \pm \frac{1}{n!},$$

where the + sign is chosen if n is odd and the − sign if n is even. In Figure 2, these numbers are given for the first few values of n.

Number of hats	Probability p_n that at least one man gets his own hat
2	.500000
3	.666667
4	.625000
5	.633333
6	.631944
7	.632143
8	.632118

Figure 2

It can be shown that, as the number of hats increases, the probabilities approach a number $1 - (1/e) = .632121\ldots$, where the number $e = 2.71828\ldots$ is a number that plays an important role in many branches of mathematics.

EXERCISES

1. What odds should you be willing to give on a bet that at least two people in the United States Senate have the same birthday?

[*Ans.* 3,300,000:1.]

2. What is the probability that in the House of Representatives at least two men have the same birthday?

3. What odds should you be willing to give on a bet that at least two of the Presidents of the United States have had the same birthday? Would you win the bet?

[*Ans.* More than 4:1; Yes. Polk and Harding were born on Nov. 2.]

4. What odds should you be willing to give on the bet that at least two of the Presidents of the United States have died on the same day of the year? Would you win the bet?

[*Ans.* More than 2.7:1; Yes. Jefferson, Adams, and Monroe all died on July 4.]

5. Four men check their hats. Assuming that the hats are returned at random, what is the probability that *exactly* four men get their own hats? Calculate the answer for 3, 2, 1, 0 men. [*Ans.* $\frac{1}{24}$; 0; $\frac{1}{4}$; $\frac{1}{3}$; $\frac{3}{8}$.]

6. A group of 50 men and their wives attend a dance. The partners for a dance are chosen by lot. What is the approximate probability that no man dances with his wife?

7. Show that the probability that, in a group of r people, *exactly* one pair has the same birthday is

$$t_r = \binom{r}{2}\frac{365\cdot 364\ldots(365-r+2)}{365^r}.$$

8. Show that $t_r = \binom{r}{2}\frac{q_r}{366-r}$, where t_r is defined in Exercise 7, and q_r is the probability that no pair has the same birthday.

9. Using the result of Exercise 8 and the results given in Figure 1, find the probability of exactly one pair of people with the same birthday in a group of r people, for $r = 15, 20, 25, 30, 40$, and 50.

[*Ans.* .22; .32; .38; .38; .26; .12.]

10. What is the approximate probability that there has been exactly one pair of Presidents with the same birthday?

SUPPLEMENTARY EXERCISES

11. Find a formula for the probability of having more than one coincidence of birthdays among n people, i.e., of having at least two pairs of identical birthdays, or of three or more people having the same birthday. [*Hint:* Take the probability of at least one coincidence, and subtract the probability of having exactly one pair.]

12. Compute the probability of having more than one coincidence of birthdays when there are 20, 25, 30, 40, or 50 people in the room.

13. What is the smallest number of people you need in order to have a better than even chance of finding more than one coincidence of birthdays?

[*Ans.* 36.]

14. Is it very surprising that there was more than one coincidence of birthdays among the dates on which Presidents died?

15. A game of solitaire is played as follows: A deck of cards is shuffled, and then the player turns the cards up one at a time. As he turns the cards, he calls out the names of the cards in a standard order—say "two of clubs," "three of clubs," etc. The object of the game is to go through the entire deck without once calling out the name of the card one turns up. What is the probability of winning? How does the probability change if one uses a single suit in place of a whole deck?

5. CONDITIONAL PROBABILITY

Suppose that we have a given $\mathcal{U}$ and that measures have been assigned to all subsets of $\mathcal{U}$. A statement p will have probability $\Pr[p] = m(P)$. Suppose we now receive some additional information, say that statement q is true. How does this additional information alter the probability of p?

The probability of p after the receipt of the information q is called its *conditional probability*, and it is denoted by $\Pr[p|q]$, which is read "the probability of p given q." In this section we will construct a method of finding this conditional probability in terms of the measure m.

If we know that q is true, then the original possibility set $\mathcal{U}$ has been reduced to Q and therefore we must define our measure on the subsets of Q instead of on the subsets of $\mathcal{U}$. Of course, every subset X of Q is a subset of $\mathcal{U}$, and hence we know $m(X)$, its measure before q was discovered. Since q cuts down on the number of possibilities, its new measure $m'(X)$ should be larger.

The basic idea on which the definition of m' is based is that, while we know that the possibility set has been reduced to Q, we have no new information about subsets of Q. If X and Y are subsets of Q, and $m(X) = 2 \cdot m(Y)$, then we will want $m'(X) = 2 \cdot m'(Y)$. This will be the case if the measures of subsets of Q are simply increased by a proportionality factor $m'(X) = k \cdot m(X)$, and all that remains is to determine k. Since we know that $1 = m'(Q) = k \cdot m(Q)$, we see that $k = 1/m(Q)$ and our new measure on subsets of $\mathcal{U}$ is determined by the formula

$$(1) \qquad m'(X) = \frac{m(X)}{m(Q)}.$$

How does this affect the probability of p? First of all, the truth set of p has been reduced. Because all elements of $\tilde{Q}$ have been eliminated, the new truth set of p is $P \cap Q$ and therefore

$$\text{(2)} \qquad \Pr[p|q] = m'(P \cap Q) = \frac{m(P \cap Q)}{m(Q)} = \frac{\Pr[p \wedge q]}{\Pr[q]}.$$

Note that if the original measure m is the equiprobable measure, then the new measure m' will also be the equiprobable measure on the set Q.

We must take care that the denominators in (1) and (2) be different from zero. Observe that $m(Q)$ will be zero if Q is the empty set, which happens only if q is self-contradictory. This is also the only case in which $\Pr[q] = 0$, and hence we make the obvious assumption that our information q is not self-contradictory.

Example 1. In an election, candidate A has a .4 chance of winning, B has .3 chance, C has .2 chance, and D has .1 chance. Just before the election, C withdraws. Now what are the chances of the other three candidates? Let q be the statement that C will not win, i.e., that A or B or D will win. Observe that $\Pr[q] = .8$, hence all the other probabilities are increased by a factor of $1/.8 = 1.25$. Candidate A now has .5 chance of winning, B has .375, and D has .125.

Example 2. A family is chosen at random from the set of all families having exactly two children (not twins). What is the probability that the family has two boys, if it is known that there is a boy in the family? Without any information being given, we would assign the equiprobable measure on the set $\mathfrak{U} = \{BB, BG, GB, GG\}$, where the first letter of the pair indicates the sex of the younger child and the second that of the older. The information that there is a boy causes $\mathfrak{U}$ to change to $\{BB, BG, GB\}$, but the new measure is still the equiprobable measure. Thus the conditional probability that there are two boys given that there is a boy is $\frac{1}{3}$. If, on the other hand, we know that the first child is a boy, then the possibilities are reduced to $\{BB, BG\}$ and the conditional probability is $\frac{1}{2}$.

A particularly interesting case of conditional probability is that in which $\Pr[p|q] = \Pr[p]$. That is, the information that q is true has no effect on our prediction for p. If this is the case, we note that

$$\text{(3)} \qquad \Pr[p \wedge q] = \Pr[p] \Pr[q].$$

And the case $\Pr[q|p] = q$ leads to the same equation. Whenever equation (3) holds, we say that p and q are *independent*. Thus if q is not a self-contradiction, p and q are independent if and only if $\Pr[p|q] = \Pr[p]$.

Example 3. Consider three throws of an ordinary coin, where we consider the eight possibilities to be equally likely. Let p be the statement "A head turns up on the first throw" and q be the statement, "A tail turns up on the second throw." Then $\Pr[p] = \Pr[q] = \frac{1}{2}$ and $\Pr[p \wedge q] = \frac{1}{4}$ and therefore p and q are independent statements.

While we have an intuitive notion of independence, it can happen that two statements, which may not seem to be independent, are in fact independent. For example, let r be the statement "The same side turns up all three times." Let s be the statement "At most one head occurs." Then r and s are independent statements (see Exercise 10).

An important use of conditional probabilities arises in the following manner. We wish to find the probability of a statement p. We observe that there is a complete set of alternatives $q_1, q_2, \ldots, q_n$ such that the probability $\Pr[q_i]$ as well as the conditional probabilities $\Pr[p|q_i]$ can be found for every i. Then in terms of these we can find $\Pr[p]$ by

$$\Pr[p] = \Pr[q_1]\Pr[p|q_1] + \Pr[q_2]\Pr[p|q_2] + \ldots + \Pr[q_n]\Pr[p|q_n].$$

The proof of this assertion is left as an exercise (see Exercise 13).

Example 4. A psychology student once studied the way mathematicians solve problems and contended that at times they try too hard to look for symmetry in a problem. To illustrate this she asked a number of mathematicians the following problem: Fifty balls (25 white and 25 black) are to be put in two urns, not necessarily the same number of balls in each. How should the balls be placed in the urns so as to maximize the chance of drawing a black ball, if an urn is chosen at random and a ball drawn from this urn? A quite surprising number of mathematicians answered that you could not do any better than $\frac{1}{2}$, by the symmetry of the problem. In fact one can do a good deal better by putting one black ball in urn 1, and all the 49 other balls in urn 2. To find the probability in this case let p be the statement "A black ball is drawn," q_1 the statement "Urn 1 is drawn" and q_2 the statement "Urn 2 is drawn." Then q_1 and q_2 are a complete set of alternatives so

$$\Pr[p] = \Pr[q_1]\Pr[p|q_1] + \Pr[q_2]\Pr[p|q_2].$$

But $\Pr[q_1] = \Pr[q_2] = \frac{1}{2}$ and $\Pr[p|q_1] = 1$, $\Pr[p|q_2] = \frac{25}{49}$. Thus

$$\Pr[p] = \tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot\tfrac{25}{49} = \tfrac{74}{98} \approx .755.$$

When told the answer, a number of the mathematicians that had said $\frac{1}{2}$ replied that they thought there had to be the same number of balls in each urn. However, since this had been carefully stated not to be necessary, they also had fallen into the trap of assuming too much symmetry.

EXERCISES

1. A card is drawn at random from a pack of playing cards. What is the probability that it is a 5, given that it is between 2 and 7 inclusive?

2. A die is loaded in such a way that the probability of a given number turning up is proportional to that number (e.g., a 6 is three times as likely to turn up as a 2).

(a) What is the probability of rolling a 3 given that an odd number turns up? [*Ans.* $\frac{1}{3}$.]

(b) What is the probability of rolling an even number given that a number greater than three turns up? [*Ans.* $\frac{2}{3}$.]

3. A die is thrown twice. What is the probability that the sum of the faces which turn up is greater than 10, given that one of them is a 6? Given that the first throw is a 6? [*Ans.* $\frac{3}{11}$; $\frac{1}{3}$.]

4. Referring to Chapter IV, Section 3, Exercise 9, what is the probability that the man selected studies German if

(a) He studies French?

(b) He studies French and Russian?

(c) He studies neither French nor Russian?

5. In the primary voting example of Chapter II, Section 1, assuming that the equiprobable measure has been assigned, find the probability that A wins at least two primaries, given that B drops out of the Wisconsin primary? [*Ans.* $\frac{7}{9}$.]

6. If $\Pr[\sim p] = \frac{1}{4}$ and $\Pr[q|p] = \frac{1}{2}$, what is $\Pr[p \wedge q]$? [*Ans.* $\frac{3}{8}$.]

7. A student takes a five-question true-false exam. What is the probability that he will get all answers correct if

(a) He is only guessing?

(b) He knows that the instructor puts more true than false questions on his exams?

(c) He also knows that the instructor never puts three questions in a row with the same answer?

(d) He also knows that the first and last questions must have the opposite answer?

(e) He also knows that the answer to the second problem is "false"?

8. Three persons, A, B, and C, are placed at random in a straight line. Let r be the statement "B is to the right of A" and let s be the statement "C is to the right of A."

(a) What is the Pr $[r \wedge s]$? [*Ans.* $\frac{1}{3}$.]

(b) Are r and s independent? [*Ans.* No.]

9. Let a deck of cards consist of the jacks and queens chosen from a bridge deck, and let two cards be drawn from the new deck. Find

(a) The probability that the cards are both jacks, given that one is a jack. [*Ans.* $\frac{3}{11} = .27$.]

(b) The probability that the cards are both jacks, given that one is a red jack. [*Ans.* $\frac{5}{13} = .38$.]

(c) The probability that the cards are both jacks, given that one is the jack of hearts. [*Ans.* $\frac{3}{7} = .43$.]

10. Prove that statements r and s in Example 3 are independent.

11. The following example shows that r may be independent of p and q without being independent of $p \wedge q$ and $p \vee q$. We throw a coin twice. Let p be "The first toss comes out heads," q be "The second toss comes out heads," and r be "The two tosses come out the same." Compute Pr $[r]$, Pr $[r|p]$, Pr $[r|q]$, Pr $[r|p \wedge q]$, Pr $[r|p \vee q]$. [*Ans.* $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, 1, $\frac{1}{3}$.]

12. Prove that for any two statements p and q,

$$\Pr[p] = \Pr[p \wedge q] + \Pr[p \wedge \sim q].$$

13. Let p be any statement and q_1, q_2, q_3 be a complete set of alternatives. Prove that

$$\Pr[p] = \Pr[q_1]\Pr[p|q_1] + \Pr[q_2]\Pr[p|q_2] + \Pr[q_3]\Pr[p|q_3].$$

14. Prove that the procedure given in Example 4 does maximize the chance of getting a black ball. [*Hint:* Show that you can assume that one urn contains more black balls than white balls and then consider what is the best that could be achieved, first in the urn with more black than white balls, and then in the urn with more white than black balls.]

SUPPLEMENTARY EXERCISES

15. Assume that p and q are independent statements relative to a given measure. Prove that each of the following pairs of statements are independent relative to this same measure.

(a) p and $\sim q$.
(b) $\sim q$ and p.
(c) $\sim p$ and $\sim q$.

16. Prove that for any three statements p, q, and r,

$$\Pr[p \wedge q \wedge r] = \Pr[p] \cdot \Pr[q|p] \cdot \Pr[r|p \wedge q].$$

17. A coin is thrown twice. Let p be the statement "Heads turns up on the first toss" and q the statement "Heads turns up on the second toss." Show that it is possible to assign a measure to the possibility space {HH, HT, TH, TT} so that these statements are *not* independent.

18. A multiple-choice test question lists five alternative answers, of which just one is correct. If a student has done his homework, then he is certain to identify the correct answer; otherwise, he chooses an answer at random. Let p be the statement "A student does his homework" and q the statement "He answers the question correctly." Let $\Pr[p] = a$.
(a) Find a formula for $\Pr[p|q]$ in terms of a.
(b) Show that $\Pr[p|q] \geq \Pr[p]$ for all values of a. When does the equality hold?

19. A coin is weighted so that heads has probability .7, tails has probability .2, and it stands on edge with probability .1. What is the probability that it does not come up heads, given that it does not come up tails? [*Ans.* $\frac{1}{8}$.]

20. A card is drawn at random from a deck of playing cards. Are the following pairs of statements independent?
(a) p: A jack is drawn.
q: A black card is drawn.
(b) p: An even numbered heart is drawn.
q: A red card smaller than a five is drawn.

21. A simple genetic model for the color of a person's eyes is the following: There are two kinds of color-determining genes, B and b, and each person has two color-determining genes. If both are b, he has blue eyes; otherwise he has brown eyes. Assume that one-quarter of the people have two B genes, one-quarter of the people have two b genes, and the rest have one B gene and one b gene.
(a) If a man has brown eyes, what is the probability that he has two B genes?

Assume that a man has brown eyes and that his wife has blue eyes. A child born to this couple will get one gene from the man and one from his wife, the selection in each case being a random selection from the parent's two genes.
(b) What is the probability that the child will have brown eyes?
(c) If the child has brown eyes, what is the probability that the father has two B genes? [*Ans.* (c) $\frac{1}{2}$.]

22. Three red, three green, and three blue balls are to be put into three urns, with at least two balls in each urn. Then an urn is selected at random and two balls withdrawn.

(a) How should the balls be put in the urns in order to maximize the probability of drawing two balls of different color? What is the probability? [*Partial Ans.* 1.]

(b) How should the balls be put in the urns in order to maximize the probability of withdrawing a red and a green ball? What is the maximum probability? [*Partial Ans.* $\frac{7}{10}$.]

6. *FINITE STOCHASTIC PROCESSES*

We consider here a very general situation which we will specialize in later sections. We deal with a sequence of experiments where the outcome on each particular experiment depends on some chance element. Any such sequence is called a *stochastic process*. (The Greek word "stochos" means "guess.") We shall assume a finite number of experiments and a finite number of possibilities for each experiment. We assume that, if all the outcomes of the experiments which precede a given experiment were known, then both the possibilities for this experiment and the probability that any particular possibility will occur would be known. We wish to make predictions about the process as a whole. For example, in the case of repeated throws of an ordinary coin we would assume that on any particular experiment we have two outcomes, and the probabilities for each of these outcomes is one-half regardless of any other outcomes. We might be interested, however, in the probabilities of statements of the form, "More than two-thirds of the throws result in heads," or "The number of heads and tails which occur is the same," etc. These are questions which can be answered only when a probability measure has been assigned to the process as a whole. In this section we show how probability measure can be assigned, using the given information. In the case of coin tossing, the probabilities (hence also the possibilities) on any given experiment do not depend upon the previous results. We will not make any such restriction here since the assumption is not true in general.

We shall show how the probability measure is constructed for a particular example, and the procedure in the general case is similar.

We assume that we have a sequence of three experiments, the possibilities for which are indicated in Figure 3. The set of all possible out-for the third are $\{c, f\}$. We denote by p_a the probability that the first experiment results in outcome a, and by p_b the probability that outcome b occurs in the first experiment. We denote by $p_{b,d}$ the probability that outcome d occurs on the second experiment, which is the probability computed on the assumption that outcome b occurred on the first experiment. Similarly for $p_{b,a}, p_{b,e}, p_{a,a}, p_{a,c}$. We denote by $p_{bd,c}$ the probability that outcome c occurs on the third experiment, the latter probability being computed on the assumption that outcome b occurred on the first experiment and d on the second. Similarly for $p_{ba,c}, p_{ba,f}$, etc. We have assumed that these numbers are given and the fact that they are probabilities assigned to possible outcomes would mean that they are positive and that

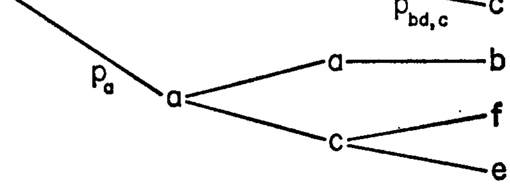


Figure 3

$$p_a + p_b = 1, \quad p_{b,a} + p_{b,e} + p_{b,d} = 1, \quad \text{and} \quad p_{bd,a} + p_{bd,e} = 1, \text{ etc.}$$

It is convenient to associate each probability with the branch of the tree that connects the branch point representing the predicted outcome. We have done this in Figure 3 for several branches. The sum of the numbers assigned to branches from a particular branch point is one, e.g.,

$$p_{b,a} + p_{b,e} + p_{b,d} = 1.$$

A possibility for the sequence of three experiments is indicated by a path through the tree. We define now a probability measure on the set of all paths. We call this a *tree measure*. To the path corresponding to outcome b on the first experiment, d on the second, and c on the third, we assign the weight $p_b \cdot p_{b,d} \cdot p_{bd,c}$. That is the *product* of the probabilities associated with each branch along the path being considered. We find the probability for each path through the tree.

Before showing the reason for this choice, we must first show that it determines a probability measure, in other words, that the weights are positive and the sum of the weights is one. The weights are products of positive numbers and hence positive. To see that their sum is one we first find the sum of the weights of all paths corresponding to a particular outcome, say b, on the first experiment and a particular outcome, say d, on the second. We have

$$p_b \cdot p_{b,d} \cdot p_{bd,a} + p_b \cdot p_{b,d} \cdot p_{bd,c} = p_b \cdot p_{b,d}[p_{bd,a} + p_{bd,c}] = p_b \cdot p_{b,d}.$$

For any other first two outcomes we would obtain a similar result. For example, the sum of the weights assigned to paths corresponding to outcome a on the first experiment and c on the second is $p_a \cdot p_{a,c}$. Notice that when we have verified that we have a probability measure, this will be the probability that the first outcome results in a and the second experiment results in c.

Next we find the sum of the weights assigned to all the paths corresponding to the cases where the outcome of the first experiment is b. We find this by adding the sums corresponding to the different possibilities for the second experiment. But by our preceding calculation this is

$$p_b \cdot p_{b,a} + p_b \cdot p_{b,c} + p_b \cdot p_{b,d} = p_b[p_{b,a} + p_{b,c} + p_{b,d}] = p_b.$$

Similarly, the sum of the weights assigned to paths corresponding to the outcome a on the first experiment is p_a. Thus the sum of all weights is $p_a + p_b = 1$. Therefore we do have a probability measure. Note that we have also shown that the probability that the outcome of the first experiment is a has been assigned probability p_a in agreement with our given probability.

To see the complete connection of our new measure with the given probabilities, let $X_j = z$ be the statement "The outcome of the jth experiment was z." Then the statement $[X_1 = b \wedge X_2 = d \wedge X_3 = c]$ is a compound statement that has been assigned probability $p_b \cdot p_{b,d} \cdot p_{bd,c}$. The statement $[X_1 = b \wedge X_2 = d]$ we have noted has been assigned probability $p_b \cdot p_{b,d}$ and the statement $[X_1 = b]$ has been assigned probability p_b. Thus

$$\Pr [X_3 = c | X_2 = d \wedge X_1 = b] = \frac{p_b \cdot p_{b,d} \cdot p_{bd,c}}{p_b \cdot p_{b,d}} = p_{bd,c}$$

$$\Pr [X_2 = d | X_1 = b] = \frac{p_b \cdot p_{b,d}}{p_b} = p_{b,d}.$$

Thus we see that our probabilities, computed under the assumption that previous results were known, become the corresponding conditional probabilities when computed with respect to the tree measure. It can be shown that the tree measure which we have assigned is the only one which will lead to this agreement. We can now find the probability of any statement concerning the stochastic process from our tree measure.

Example 1. Suppose that we have two urns. Urn 1 contains two black balls and three white balls. Urn 2 contains two black balls and one white ball. An urn is chosen at random and a ball chosen from this urn at random. What is the probability that a white ball is chosen? A hasty answer might be $\frac{1}{2}$, since there are an equal number of black and white balls involved and everything is done at random. However, it is hasty answers like this (which is wrong) which show the need for a more careful analysis.

We are considering two experiments. The first consists in choosing the urn and the second in choosing the ball. There are two possibilities for the first experiment, and we assign $p_1 = p_2 = \frac{1}{2}$ for the probabilities of choosing the first and the second urn, respectively. We then assign $p_{1,W} = \frac{3}{5}$ for the probability that a white ball is chosen, under the assumption that urn 1 is chosen. Similarly we assign $p_{1,B} = \frac{2}{5}$, $p_{2,W} = \frac{1}{3}$, $p_{2,B} = \frac{2}{3}$. We indicate these probabilities on the possibility tree in Figure 4. The probability that a white ball is drawn is then found from the tree measure as the sum of the weights assigned to paths which lead to a choice of a white ball. This is $\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{1}{3} = \frac{7}{15}$.

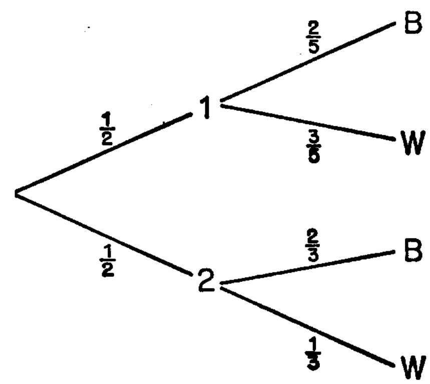


Figure 4

Example 2. Suppose that a man leaves a bar which is on a corner which he knows to be one block from his home. He is unable to remember which street leads to his home. He proceeds to try each of the streets at random without ever choosing the same street twice until he goes on the one which leads to his home. What possibilities are there for his trip home, and what is the probability for each of these possible trips? We label the streets A, B, C, and Home. The possibili-

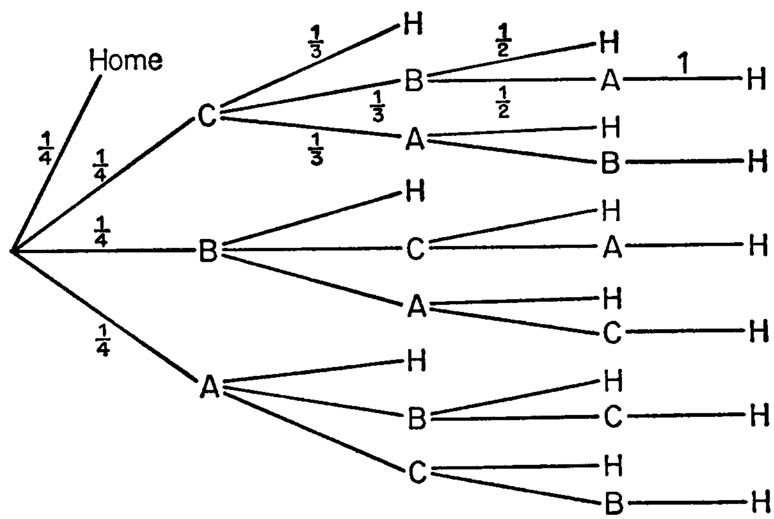


Figure 5

ties together with typical probabilities are given in Figure 5. The probability for any particular trip, or path, is found by taking the product of the branch probabilities.

Example 3. Assume that you are presented with two slot machines, A and B. Each machine pays the same fixed amount when it pays off. Machine A pays off each time with probability $\frac{1}{2}$, and machine B with probability $\frac{1}{4}$. You are not told which machine is A. Suppose that you choose a machine at random and win. What is the probability that you chose machine A? We first construct the tree (Figure 6) to show the possibilities and assign branch probabilities to determine a tree measure. Let p be the statement "Machine A was chosen" and q be the statement "The machine chosen paid off." Then we are asked for

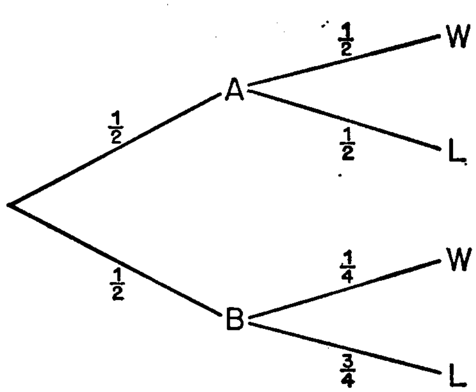


Figure 6

$$\Pr[p|q] = \frac{\Pr[p \land q]}{\Pr[q]}.$$

The truth set of the statement $p \land q$ consists of a single path which has been assigned weight $\frac{1}{4}$. The truth set of the statement q consists of two paths, and the sum of the weights of these paths is $\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8}$.

Thus Pr $[p|q] = \frac{2}{3}$. Thus if we win, it is more likely that we have machine A than B and this suggests that next time we should play the same machine. If we lose, however, it is more likely that we have machine B than A, and hence we would switch machines before the next play. (See Exercise 9.)

EXERCISES

1. The fractions of Republicans, Democrats, and Independent voters in cities A and B are

City A: .30 Republican, .40 Democratic, .30 Independent;
City B: .40 Republican, .50 Democratic, .10 Independent.

A city is chosen at random and two voters are chosen successively and at random from the voters of this city. Construct a tree measure and find the probability that two Democrats are chosen. Find the probability that the second voter chosen is an Independent voter. [*Ans.* .205; .2.]

2. A coin is thrown. If a head turns up, a die is rolled. If a tail turns up, the coin is thrown again. Construct a tree measure to represent the two experiments and find the probability that the die is thrown and a six turns up.

3. A man wins a certain tournament if he can win two consecutive games out of three played alternately with two opponents A and B. A is a better player than B. The probability of winning a game when B is the opponent is $\frac{2}{3}$. The probability of winning a game when A is his opponent is only $\frac{1}{3}$. Construct a tree measure for the possibilities for three games, assuming that he plays alternately but plays A first. Do the same assuming that he plays B first. In each case find the probability that he will win two consecutive games. Is it better to play two games against the strong player or against the weaker player? [*Ans.* $\frac{10}{27}$; $\frac{8}{27}$; better to play strong player twice.]

4. Construct a tree measure to represent the possibilities for four throws of an ordinary coin. Assume that the probability of a head on any toss is $\frac{1}{2}$ regardless of any information about other throws.

5. A student claims to be able to distinguish beer from ale. He is given a series of three tests. In each test he is given two cans of beer and one of ale and asked to pick out the ale. If he gets two or more correct we will admit his claim. Draw a tree to represent the possibilities (either right or wrong) for his answers. Construct the tree measure which would correspond to guessing and find the probability that his claim will be established if he guesses on every trial.

6. A box contains three defective light bulbs and seven good ones. Construct a tree to show the possibilities if three consecutive bulbs are drawn at random from the box (they are not replaced after being drawn). Assign a tree measure and find the probability that at least one good bulb is drawn out. Find the probability that all three are good if the first bulb is good.

[*Ans.* $\frac{119}{120}$; $\frac{5}{12}$.]

7. In Example 2 above, find the probability that the man reaches home after trying at most one wrong street.

8. In Example 3, find the probability that machine A was chosen, given that the player lost.

9. In Example 3, assume that the player makes two plays. Find the probability that he wins at least once under the assumption

(a) That he plays the same machine twice. [*Ans.* $\frac{19}{32}$.]

(b) That he plays the same machine the second time if and only if he won the first time. [*Ans.* $\frac{20}{32}$]

10. A chess player plays three successive games of chess. His psychological makeup is such that the probability of his winning a given game is $(\frac{1}{2})^{k+1}$, where k is the number of games he has won so far. (For instance, the probability of his winning the first game is $\frac{1}{2}$, the probability of his winning the second game *if he has already won the first game* is $\frac{1}{4}$, etc.) What is the probability that he will win at least two of the three games?

11. Before a political convention, a political expert has assigned the following probabilities. The probability that the President will be willing to run again is $\frac{1}{2}$. If he is willing to run, he and his Vice President are sure to be nominated and have probability $\frac{3}{5}$ of being elected again. If the President does not run, the present Vice President has probability $\frac{1}{10}$ of being nominated, and any other presidential candidate has probability $\frac{1}{2}$ of being elected. What is the probability that the present Vice President will be re-elected?

[*Ans.* $\frac{13}{40}$.]

12. There are two urns, A and B. Urn A contains one black and one red ball. Urn B contains two black and three red balls. A ball is chosen at random from urn A and put into urn B. A ball is then drawn at random from urn B.

(a) What is the probability that both balls drawn are of the same color?

[*Ans.* $\frac{7}{12}$.]

(b) What is the probability that the first ball drawn was red, given that the second ball drawn was black? [*Ans.* $\frac{2}{5}$.]

SUPPLEMENTARY EXERCISES

13. Assume that in the World Series each team has probability one-half of winning each game, independently of the outcomes of any other game. Assign a tree measure. (See Chapter 1, Section 6 for the tree.) Find the probability that the series ends in four, five, six, and seven games, respectively.

14. Assume that in the World Series one team is stronger than the other and has probability .6 for winning each of the games. Assign a tree measure and find the following probabilities.

(a) The probability that the stronger team wins in 4, 5, 6, and 7 games, respectively.

(b) The probability that the weaker team wins in 4, 5, 6, and 7 games, respectively.

(c) The probability that the series ends in 4, 5, 6, and 7 games, respectively. [*Ans.* .16; .27; .30; .28.]

(d) The probability that the strong team wins the series. [*Ans.* .71.]

15. Redo Exercise 14 for the case of two poorly matched teams, where the better team has probability .9 of winning a game.

[*Ans.* (c) .66; .26; .07; .01.]
[*Ans.* (d) .997.]

16. In the World Series from 1905 through 1965 (excluding series of more than seven games) there were 11 four-game, 14 five-game, 13 six-game, and 20 seven-game series. Which of the assumptions in Exercises 13–15 comes closest to predicting these results? Is it a good fit? [*Ans.* .6; no.]

17. Consider the following assumption concerning World Series: Ninety per cent of the time the two teams are evenly matched, while 10 per cent of the time they are poorly matched, with the better team having probability .9 of winning a game. Show that this assumption comes closer to predicting the actual outcomes than those considered in Exercise 16.

18. We are given three coins. Coin A is fair while coins B and C are loaded: B has probability .6 of heads and C has probability .4 of heads. A game is played by tossing a coin twice starting with coin B. If a head is obtained, B is tossed again, otherwise the second coin to be tossed is chosen at random from A and C.

(a) Draw the tree for this game, assigning branch and path weights.

(b) Let p be the statement "The first toss results in heads" and let q be the statement "The second toss results in heads." Find Pr $[p]$, Pr $[q]$, Pr $[q|p]$. [*Ans.* (b) .6; .54; .6.]

19. A and B play a series of games for which they are evenly matched. A player wins the series either by winning two games in a row, or by winning a total of three games. Construct the tree and the tree measure.

(a) What is the probability that A wins the series?

(b) What is the probability that more than three games need to be played?

20. In a room there are three chests, each chest contains two drawers, and each drawer contains one coin. In one chest each drawer contains a gold coin; in the second chest each drawer contains a silver coin; and in the last chest one drawer contains a gold coin and the other contains a silver coin. A chest is picked at random and then a drawer is picked at random from that chest. When the drawer is opened, it is found to contain a gold coin. What is the probability that the other drawer of that same chest will also contain a gold coin? [*Ans.* $\frac{2}{3}$.]

7. BAYES' PROBABILITIES

The following situation often occurs. Measures have been assigned in a possibility space $\mathcal{U}$. A complete set of alternatives, $p_1, p_2, \ldots, p_n$ has been singled out. Their probabilities are determined by the assigned measure. (Recall that a complete set of alternatives is a set of statements such that for any possible outcome one and only one of the statements is true.) We are now given that a statement q is true. We wish to compute the new probabilities for the alternatives relative to this information. That is, we wish the conditional probabilities $\Pr[p_j|q]$ for each p_j. We shall give two different methods for obtaining these probabilities.

The first is by a general formula. We illustrate this formula for the case of four alternatives: p_1, p_2, p_3, p_4. Consider $\Pr[p_2|q]$. From the definition of conditional probability,

$$\Pr[p_2|q] = \frac{\Pr[p_2 \wedge q]}{\Pr[q]}.$$

But since p_1, p_2, p_3, p_4, are a complete set of alternatives,

$$\Pr[q] = \Pr[p_1 \wedge q] + \Pr[p_2 \wedge q] + \Pr[p_3 \wedge q] + \Pr[p_4 \wedge q].$$

Thus

$$\Pr[p_2|q] = \frac{\Pr[p_2 \wedge q]}{\Pr[p_1 \wedge q] + \Pr[p_2 \wedge q] + \Pr[p_3 \wedge q] + \Pr[p_4 \wedge q]}.$$

Since $\Pr[p_j \wedge q] = \Pr[p_j]\Pr[q|p_j]$, we have the desired formula

$$\Pr[p_2|q] = \frac{\Pr[p_2]\cdot\Pr[q|p_2]}{\Pr[p_1]\cdot\Pr[q|p_1]+\Pr[p_2]\cdot\Pr[q|p_2]+\Pr[p_3]\cdot\Pr[q|p_3]+\Pr[p_4]\cdot\Pr[q|p_4]}.$$

Similar formulas apply for the other alternatives, and the formula generalizes in an obvious way to any number of alternatives. In its most general form it is called *Bayes' theorem.*

Example 1. Suppose that a freshman must choose among mathematics, physics, chemistry, and botany as his science course. On the basis of the interest he expressed, his adviser assigns probabilities of .4, .3, .2, and .1 to his choosing each of the four courses, respectively. His adviser does not hear which course he actually chose, but at the end of the term the adviser hears that he received A in the course chosen. On the basis of the difficulties of these courses the adviser estimates the probability of the student getting an A in mathematics to be .1, in physics .2, in chemistry .3, and in botany .9. How can the adviser revise his original estimates as to the probabilities of the student taking the various courses? Using Bayes' theorem we get

$$\Pr[\text{He took math}|\text{He got an A}] = \frac{(.4)(.1)}{(.4)(.1)+(.3)(.2)+(.2)(.3)+(.1)(.9)} = \frac{4}{25} = .16.$$

Similar computations assign probabilities of .24, .24, and .36 to the other three courses. Thus the new information, that he received an A, had little effect on the probability of his having taken physics or chemistry, but it has made it much less likely that he took mathematics, and much more likely that he took botany.

It is important to note that knowing the conditional probabilities of q relative to the alternatives is not enough. Unless we also know the probabilities of the alternatives at the start, we cannot apply Bayes' theorem. However, in some situations it is reasonable to assume that the alternatives are equally probable at the start. In this case the factors $\Pr[p_1], \ldots, \Pr[p_4]$ cancel from our basic formula, and we get the special form of the theorem:

If $\Pr[p_1] = \Pr[p_2] = \Pr[p_3] = \Pr[p_4]$, then

$$\Pr[p_2|q] = \frac{\Pr[q|p_2]}{\Pr[q|p_1]+\Pr[q|p_2]+\Pr[q|p_3]+\Pr[q|p_4]}.$$

Example 2. In a sociological experiment the subjects are handed four sealed envelopes, each containing a problem. They are told to open one envelope and try to solve the problem in ten minutes. From past experience, the experimenter knows that the probability of their being able to solve the hardest problem is .1. With the other problems, they have probabilities of .3, .5, and .8. Assume the group succeeds within the allotted time. What is the probability that they selected the hardest problem? Since they have no way of knowing which problem is in which envelope, they choose at random, and we assign equal probabilities to the selection of the various problems. Hence the above simple formula applies. The probability of their having selected the hardest problem is

$$\frac{.1}{.1 + .3 + .5 + .8} = \frac{1}{17}.$$

The second method of computing Bayes' probabilities is to draw a tree, and then to redraw the tree in a different order. This is illustrated in the following example.

Example 3. There are three urns. Each urn contains one white ball. In addition, urn I contains one black ball, urn II contains two, and urn III contains 3. An urn is selected and one ball is drawn. The probability for selecting the three urns is $\frac{1}{6}$, $\frac{1}{2}$, and $\frac{1}{3}$, respectively. If we know that a white ball is drawn, how does this alter the probability that a given urn was selected?

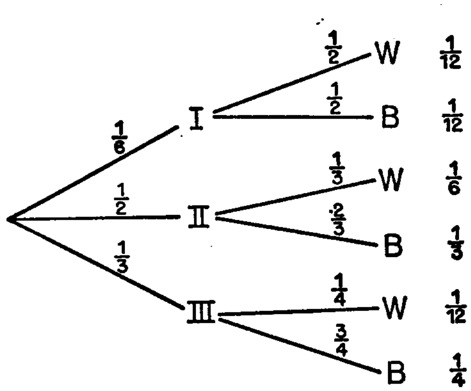


Figure 7

First we construct the ordinary tree and tree measure, in Figure 7.

Next we redraw the tree, using the ball drawn as stage 1, and the urn selected as stage 2. We have the same paths as before, but in a different order. So the path weights are read off from the previous tree. The probability of drawing a white ball is

$$\frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}.$$

This leaves the branch weights of the second stage to be computed (see Figure 8). But this is simply a matter of division. For example, the branch weights for the branches starting at "W" must be $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ to yield the correct path weights. Thus, if a white ball is drawn, the probability of having selected urn I has increased to $\frac{1}{4}$, the probability of having picked urn III has fallen to $\frac{1}{4}$, while the probability of having chosen urn II is unchanged (see Figure 9).

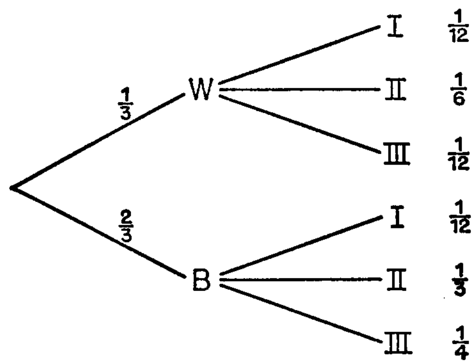


Figure 8

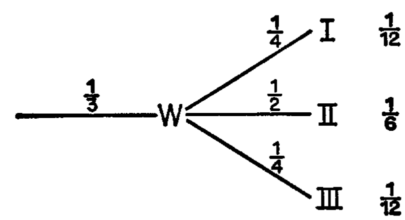


Figure 9

This method is particularly useful when we wish to compute all the conditional probabilities. We will apply the method next to Example 1. The tree and tree measure for this example in the natural order is shown in Figure 10. In that figure the letters M, P, C, and B stand for mathematics, physics, chemistry, and botany, respectively.

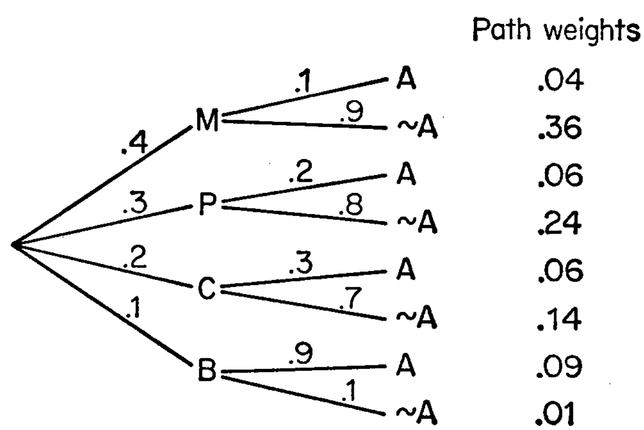


Figure 10

The tree drawn in reverse order is shown in Figure 11.

Each path in this tree corresponds to one of the paths in the original tree. Therefore the path weights for this new tree are the same as the weights assigned to the corresponding paths in the first tree. The two branch weights at the first level represent the probability that the student

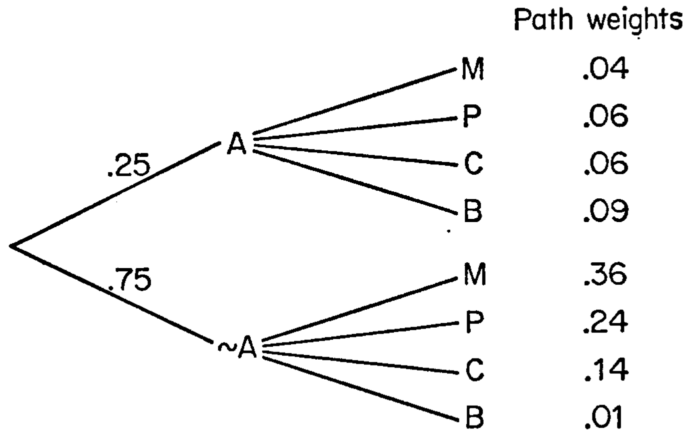


Figure 11

receives an A or that he does not receive an A. These probabilities are also easily obtained from the first tree. In fact,

$$\Pr[A] = .04 + .06 + .06 + .09 = .25$$

and

$$\Pr[\sim A] = 1 - .25 = .75.$$

We have now enough information to obtain the branch weights at the second level, since the product of the branch weights must be the path weights. For example, to obtain $p_{A,M}$ we have

$$.25 \cdot p_{A,M} = .04 \quad \text{or} \quad p_{A,M} = .16.$$

But $p_{A,M}$ is also the conditional probability that the student took math given that he got an A. Hence this is one of the new probabilities for the alternatives in the event that the student received an A. The other branch probabilities are found in the same way and represent the prob-

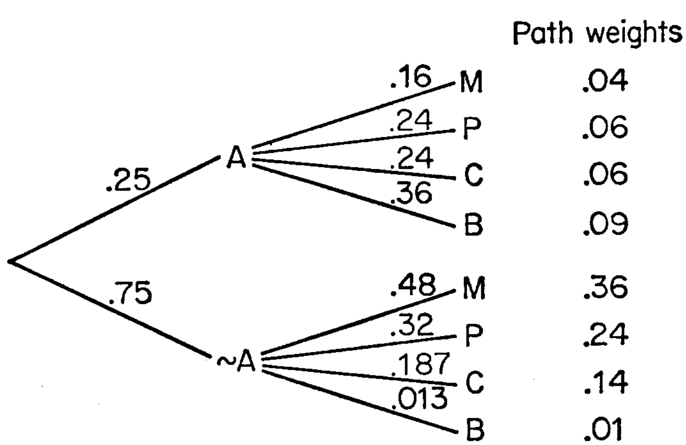


Figure 12

abilities for the other alternatives. By this method we obtain the new probabilities for all alternatives under the hypothesis that the student receives an A as well as the hypothesis that the student does not receive an A. The results are shown in the completed tree in Figure 12.

EXERCISES

1. Urn I contains 7 red and 3 black balls and urn II contains 6 red and 4 black balls. An urn is chosen at random and two balls are drawn from it in succession without replacement. The first ball is red and the second black. Show that it is more probable that urn II was chosen than urn I.

2. A gambler is told that one of three slot machines pays off with probability $\frac{1}{2}$, while each of the other two pays off with probability $\frac{1}{3}$.

(a) If the gambler selects one at random and plays it twice, what is the probability that he will lose the first time and win the second? [*Ans.* $\frac{25}{108}$.]

(b) If he loses the first time and wins the second, what is the probability he chose the favorable machine? [*Ans.* $\frac{9}{25}$.]

3. During the month of May the probability of a rainy day is .2. The Dodgers win on a clear day with probability .7, but on a rainy day only with probability .4. If we know that they won a certain game in May, what is the probability that it rained on that day? [*Ans.* $\frac{1}{8}$.]

4. Construct a diagram to represent the truth sets of various statements occurring in the previous exercise.

5. On a multiple-choice exam there are four possible answers for each question. Therefore, if a student knows the right answer, he has probability one of choosing correctly; if he is guessing, he has probability $\frac{1}{4}$ of choosing correctly. Let us further assume that a good student will know 90 per cent of the answers, a poor student only 50 per cent. If a good student has the right answer, what is the probability that he was only guessing? Answer the same question about a poor student, if the poor student has the right answer. [*Ans.* $\frac{1}{37}$, $\frac{1}{5}$.]

6. Three economic theories are proposed at a given time, which appear to be equally likely on the basis of existing evidence. The state of the American economy is observed the following year, and it turns out that its actual development had probability .6 of happening according to the first theory; and probabilities .4 and .2 according to the others. How does this modify the probabilities of correctness of the three theories?

7. Let p_1, p_2, p_3, and p_4 be a set of equally likely alternatives. Let $\Pr[q|p_1] = a$, $\Pr[q|p_2] = b$, $\Pr[q|p_3] = c$, $\Pr[q|p_4] = d$. Show that if $a + b + c + d = 1$, then the revised probabilities of the alternatives relative to q are a, b, c, and d, respectively.

8. In poker, Smith holds a very strong hand and bets a considerable amount. The probability that his opponent, Jones, has a better hand is .05. With a better hand Jones would raise the bet with probability .9, but with a poorer hand Jones would raise only with probability .2. Suppose that Jones raises, what is the new probability that he has a winning hand? [*Ans.* $\frac{9}{47}$.]

9. A rat is allowed to choose one of five mazes at random. If we know that the probabilities of his getting through the various mazes in three minutes are .6, .3, .2, .1, .1, and we find that the rat escapes in three minutes, how probable is it that he chose the first maze? The second maze? [*Ans.* $\frac{6}{13}$, $\frac{3}{13}$.]

10. Three men, A, B, and C, are in jail, and one of them is to be hanged the next day. The jailor knows which man will hang, but must not announce it. Man A says to the jailor, "Tell me the name of one of the other two who will not hang. If both are to go free, just toss a coin to decide which to say. Since I already know that at least one of them will go free, you are not giving away the secret." The jailor thinks a moment and then says, "No, this would not be fair to you. Right now you think the probability that you will hang is $\frac{1}{3}$; but if I tell you the name of one of the others who is to go free, your probability of hanging increases to $\frac{1}{2}$. You would not sleep as well tonight." Was the jailor's reasoning correct? [*Ans.* No.]

11. One coin in a collection of 8 million coins has two heads. The rest are fair coins. A coin chosen at random from the collection is tossed ten times and comes up heads every time. What is the probability that it is the two-headed coin?

12. Referring to Exercise 11, assume that the coin is tossed n times and comes up heads every time. How large does n have to be to make the probability approximately $\frac{1}{2}$ that you have the two-headed coin? [*Ans.* 23.]

13. A man will accept job a with probability $\frac{1}{2}$, job b with probability $\frac{1}{3}$, and job c with probability $\frac{1}{6}$. In each case he must decide whether to rent or buy a house. The probabilities of his buying are $\frac{1}{3}$ if he takes job a, $\frac{2}{3}$ if he takes job b, and 1 if he takes job c. Given that he buys a house, what are the probabilities of his having taken each job? [*Ans.* .3; .4; .3.]

14. Assume that chest X-rays for detecting tuberculosis have the following properties. For people having tuberculosis the test will detect the disease 90 out of every 100 times. For people not having the disease the test will in 1 out of every 100 cases diagnose the patient incorrectly as having the disease.

Assume that the incidence of tuberculosis is 5 persons per 10,000. A person is selected at random, given the X-ray test, and the radiologist reports the presence of tuberculosis. What is the probability that the person in fact has the disease?

8. *INDEPENDENT TRIALS WITH TWO OUTCOMES*

In the preceding section we developed a way to determine a probability measure for any sequence of chance experiments where there are only a finite number of possibilities for each experiment. While this provides the framework for the general study of stochastic processes, it is too general to be studied in complete detail. Therefore, in probability theory we look for simplifying assumptions which will make our probability measure easier to work with. It is desired also that these assumptions be such as to apply to a variety of experiments which would occur in practice. In this book we shall limit outselves to the study of two types of processes. The first, the independent trials process, will be considered in the present section. This process was the first one to be studied extensively in probability theory. The second, the Markov chain process, is a process that is finding increasing application, particularly in the social and biological sciences, and will be considered in Section 13.

A process of independent trials applies to the following situation. Assume that there is a sequence of chance experiments, each of which consists of a repetition of a single experiment, carried out in such a way that the results of any one experiment in no way affect the results in any other experiment. We label the possible outcome of a single experiment by $a_1, \ldots, a_r$. We assume that we are also given probabilities $p_1, \ldots, p_r$ for each of these outcomes occurring on any single experiment, the probabilities being independent of previous results. The tree representing the possibilities for the sequence of experiments will have the same outcomes from each branch point, and the branch probabilities will be assigned by assigning probability p_j to any branch leading to outcome a_j. The tree measure determined in this way is the measure of an *independent trials process*. In this section we shall consider the important case of two outcomes for each experiment. The more general case is studied in Section 11.

In the case of two outcomes we arbitrarily label one outcome "success" and the other "failure." For example, in repeated throws of a coin we might call heads success, and tails failure. We assume there is given a probability p for success and a probability $q = 1 - p$ for failure. The tree measure for a sequence of three such experiments is shown in Figure 13. The weights assigned to each path are indicated at the end of the path.

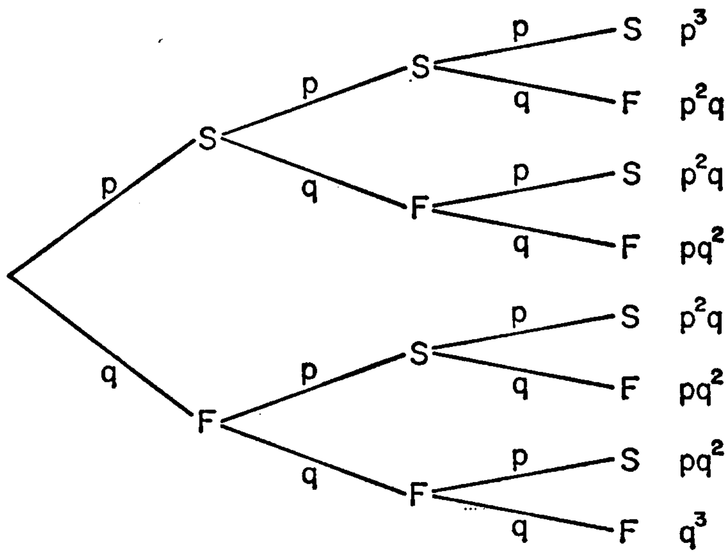


Figure 13

The question which we now ask is the following. Given an independent trials process with two outcomes, what is the probability of *exactly* x successes in n experiments? We denote this probability by $f(n, x; p)$ to indicate that it depends upon n, x, and p.

Assume that we had a tree for this general situation, similar to the tree in Figure 13 for three experiments, with the branch points labeled S for success and F for failure. Then the truth set of the statement "Exactly x successes occur" consists of all paths which go through x branch points labeled S and $n - x$ labeled F. To find the probability of this statement we must add the weights for all such paths. We are helped first by the fact that our tree measure assigns the same weight to any such path, namely $p^x q^{n-x}$. The reason for this is that every branch leading to an S is assigned probability p, and every branch leading to F is assigned probability q, and in the product there will be x p's and $(n - x)$ q's. To find the desired probability we need only find the number of paths in the truth set of the statement "Exactly x successes occur." To each such path we make correspond an ordered partition of the integers from 1 to n which has two cells, x elements in the first and $n - x$ in the second. We do this by putting the numbers of the experiments on which success occurred in the first cell and those for which failure occurred in the second cell. Since there are $\binom{n}{x}$ such partitions there are also this number of paths in the truth set of the statement considered. Thus we have proved:

In an independent trials process with two outcomes the probability of exactly x successes in n experiments is given by

$$f(n, x; p) = \binom{n}{x} p^x q^{n-x}.$$

Example 1. Consider n throws of an ordinary coin. We label heads "success" and tails "failure," and we assume that the probability is $\frac{1}{2}$ for heads on any one throw independently of the outcome of any other throw. Then the probability that exactly x heads will turn up is

$$f(n, x; \tfrac{1}{2}) = \binom{n}{x}\left(\frac{1}{2}\right)^n.$$

For example, in 100 throws the probability that exactly 50 heads will turn up is

$$f(100, 50; \tfrac{1}{2}) = \binom{100}{50}\left(\frac{1}{2}\right)^{100},$$

which is approximately .08. Thus we see that it is quite unlikely that exactly one-half of the tosses will result in heads. On the other hand, suppose that we ask for the probability that nearly one-half of the tosses will be heads. To be more precise, let us ask for the probability that the number of heads which occur does not deviate by more than 10 from 50. To find this we must add

$$f(100, x; \tfrac{1}{2}) \text{ for } x = 40, 41, \ldots, 60.$$

If this is done, we obtain a probability of approximately .96. Thus, while it is unlikely that exactly 50 heads will occur, it is very likely that the number of heads which occur will not deviate from 50 by more than 10.

Example 2. Assume that we have a machine which, on the basis of data given, is to predict the outcome of an election as either a Republican victory or a Democratic victory. If two identical machines are given the same data, they should predict the same result. We assume, however, that any such machine has a certain probability q of reversing the prediction that it would ordinarily make, because of a mechanical or electrical failure. To improve the accuracy of our prediction we give the same data to r identical machines, and choose the answer which the majority of the machines give. To avoid ties we assume that r is odd. Let us see how this decreases the probability of an error due to a faulty machine.

Consider r experiments, where the jth experiment results in success if the jth machine produces the prediction which it would make when operating without any failure of parts. The probability of success is then $p = 1 - q$. The majority decision will agree with that of a perfectly operating machine if we have more than $r/2$ successes. Suppose, for example, that we have five machines, each of which has a probability of .1 of reversing the prediction because of a parts failure. Then the probability for success is .9, and the probability that the majority decision will be the desired one is

$$f(5, 3; 0.9) + f(5, 4; 0.9) + f(5, 5; 0.9)$$

which is found to be approximately .991 (see Exercise 3).

Thus the above procedure decreases the probability of error due to machine failure from .1 in the case of one machine to .009 for the case of five machines.

EXERCISES

1. Compute for $n = 4$, $n = 8$, $n = 12$, and $n = 16$ the probability of obtaining exactly $\frac{1}{2}$ heads when an ordinary coin is thrown.

[*Ans.* .375; .273; .226; .196.]

2. Compute for $n = 4$, $n = 8$, $n = 12$, and $n = 16$ the probability that the fraction of heads deviates from $\frac{1}{2}$ by less than $\frac{1}{5}$.

[*Ans.* .375; .711; .854; .923.]

3. Verify that the probability .991 given in Example 2 is correct.

4. Assume that Peter and Paul match pennies four times. (In matching pennies, Peter wins a penny with probability $\frac{1}{2}$, and Paul wins a penny with probability $\frac{1}{2}$.) What is the probability that Peter wins more than Paul? Answer the same for five throws. For the case of 12,917 throws.

[*Ans.* $\frac{5}{16}$; $\frac{1}{2}$; $\frac{1}{2}$.]

5. If an ordinary die is thrown four times, what is the probability that exactly two sixes will occur?

6. In a ten-question true-false exam, what is the probability of getting 70 per cent or better by guessing? [*Ans.* $\frac{11}{64}$.]

7. Assume that, every time a batter comes to bat, he has probability .3 for getting a hit. Assuming that his hits form an independent trials process and that the batter comes to bat four times, what fraction of the games would he expect to get at least two hits? At least three hits? Four hits? [*Ans.* .348; .084; .008.]

8. A coin is to be thrown eight times. What is the most probable number of heads that will occur? What is the number having the highest probability, given that the first four throws resulted in heads?

9. A small factory has ten workers. The workers eat their lunch at one of two diners, and they are just as likely to eat in one as in the other. If the proprietors want to be more than .95 sure of having enough seats, how many seats must each of the diners have? [*Ans.* Eight seats.]

10. Suppose that five people are chosen at random and asked if they favor a certain proposal. If only 30 per cent of the people favor the proposal, what is the probability that a majority of the five people chosen will favor the proposal?

11. In Example 2, if the probability for a machine reversing its answer due to a parts failure is .2, how many machines would have to be used to make the probability greater than .89 that the answer obtained would be that which a machine with no failure would give? [*Ans.* Three machines.]

12. Assume that it is estimated that a torpedo will hit a ship with probability $\frac{1}{3}$. How many torpedoes must be fired if it is desired that the probability for at least one hit should be greater than .9?

13. A student estimates that, if he takes four courses, he has probability .8 of passing each course. If he takes five courses he has probability .7 of passing each course, and if he takes six courses he has probability .5 for passing each course. His only goal is to pass at least four courses. How many courses should he take for the best chance of achieving his goal? [*Ans.* 5.]

SUPPLEMENTARY EXERCISES

14. In a certain board game players move around the board, and each turn consists of a player's rolling a pair of dice. If a player is on the square *Park Bench*, he must roll a seven or doubles before he is allowed to move out.

(a) What is the probability that a player stuck on *Park Bench* will be allowed to move out on his next turn?

(b) How many times must a player stuck on *Park Bench* roll before the chances of his getting out exceed $\frac{3}{4}$? [*Ans.* (a) $\frac{1}{3}$; (b) 4.]

15. A restaurant orders five pieces of apple pie and five pieces of cherry pie. Assume that the restaurant has ten customers, and the probability that a customer will ask for apple pie is $\frac{3}{4}$ and for cherry pie is $\frac{1}{4}$.

(a) What is the probability that the ten customers will all be able to have their first choice?

(b) What number of each kind of pie should the restaurant order if it wishes to order ten pieces of pie and wants to maximize the probability that the ten customers will all have their first choice?

16. Show that it is more probable to get at least one ace with 4 dice than at least one double ace in 24 throws of two dice.

17. A thick coin, when tossed, will land "heads" with a probability of $\frac{5}{12}$, "tails" with a probability of $\frac{5}{12}$, and will land on edge with a probability of $\frac{1}{6}$. If it is tossed six times, what is the probability that it lands on edge exactly two times? [*Ans.* .2009.]

18. Without actually computing the probabilities, find the value of x for which $f(20, x; .3)$ is largest.

19. A certain team has probability $\frac{2}{3}$ of winning whenever it plays.

(a) What is the probability the team will win exactly four out of five games?

(b) What is the probability the team will win at most four out of five games?

(c) What is the probability the team will win exactly four games out of five if it has already won the first two games of the five?

[*Ans.* (a) $\frac{80}{243}$; (b) $\frac{211}{243}$; (c) $\frac{4}{9}$.]

*9. *A PROBLEM OF DECISION*

In the preceding sections we have dealt with the problem of calculating the probability of certain statements based on the assumption of a given probability measure. In a statistics problem, one is often called upon to make a decision in a case where the decision would be relatively easy to make if we could assign probabilities to certain statements, but we do not know how to assign these probabilities. For example, if a vaccine for a certain disease is proposed, we may be called upon to decide whether or not the vaccine should be used. We may decide that we could make the decision if we could compare the probability that a person vaccinated will get the disease with the probability that a person not vaccinated will get the disease. Statistical theory develops methods to obtain from experiments some information which will aid in estimating these probabilities, or will otherwise help in making the required decision. We shall illustrate a typical procedure.

Smith claims that he has the ability to distinguish ale from beer and has bet Jones a dollar to that effect. Now Smith does not mean that he can distinguish beer from ale with 100 per cent accuracy, but rather that he believes that he can distinguish them a proportion of the time which is significantly greater than $\frac{1}{2}$.

Assume that it is possible to assign a number p which represents the probability that Smith can pick out the ale from a pair of glasses, one containing ale and one beer. We identify $p = \frac{1}{2}$ with his having no ability, $p > \frac{1}{2}$ with his having some ability, and $p < \frac{1}{2}$ with his being able to distinguish, but having the wrong idea which is the ale. If we knew the value of p, we would award the dollar to Jones if p were $\leq \frac{1}{2}$, and to Smith if p were $> \frac{1}{2}$. As it stands, we have no knowledge of p and thus cannot make a decision. We perform an experiment and make a decision as follows.

Smith is given a pair of glasses, one containing ale and the other beer, and is asked to identify which is the ale. This procedure is repeated ten times, and the number of correct identifications is noted. If the number correct is at least eight, we award the dollar to Smith, and if it is less than eight, we award the dollar to Jones.

We now have a definite procedure and shall examine this procedure both from Jones' and Smith's points of view. We can make two kinds of errors. We may award the dollar to Smith when in fact the appropriate value of p is $\leq \frac{1}{2}$, or we may award the dollar to Jones when the appropriate value for p is $> \frac{1}{2}$. There is no way that these errors can be completely avoided. We hope that our procedure is such that each of the bettors will be convinced that, if he is right, he will very likely win the bet.

Jones believes that the true value of p is $\frac{1}{2}$. We shall calculate the probability of Jones winning the bet if this is indeed true. We assume that the individual tests are independent of each other and all have the same probability $\frac{1}{2}$ for success. (This assumption will be unreasonable if the glasses are too large.) We have then an independent trials process with $p = \frac{1}{2}$ to describe the entire experiment. The probability that Jones will win the bet is the probability that Smith gets fewer than eight correct. From the table in Figure 14 we compute that this probability is approximately .945. Thus Jones sees that, if he is right, it is very likely that he will win the bet.

Table of Values of $f(10, x; p)$

x \ p	0.1	0.25	0.50	0.75	0.90
0	.349	.056	.001	.000	.000
1	.387	.188	.010	.000	.000
2	.194	.282	.044	.000	.000
3	.057	.250	.117	.003	.000
4	.011	.146	.205	.016	.000
5	.001	.058	.246	.058	.001
6	.000	.016	.205	.146	.011
7	.000	.003	.117	.250	.057
8	.000	.000	.044	.282	.194
9	.000	.000	.010	.188	.387
10	.000	.000	.001	.056	.349

Figure 14

Smith, on the other hand, believes that p is significantly greater than $\frac{1}{2}$. If he believes that p is as high as .9, we see from Figure 14 that the probability of his getting eight or more correct is .930. Then both men will be satisfied by the bet.

Suppose, however, that Smith thinks the value of p is only about .75. Then the probability that he will get eight or more correct and thus win the bet is .526. There is then only an approximately even chance that the experiment will discover his abilities, and he probably will not be satisfied with this. If Smith really thinks his ability is represented by a p value of about $\frac{3}{4}$, we would have to devise a different method of awarding the dollar. We might, for example, propose that Smith win the bet if he gets seven or more correct. Then, if he has probability $\frac{3}{4}$ of being correct on a single trial, the probability that he will win the bet is approximately .776. If $p = \frac{1}{2}$, the probability that Jones will win the bet is about .828 under this new arrangement. Jones' chances of winning are thus decreased, but Smith may be able to convince him that it is a fairer arrangement than the first procedure.

In the above example, it was possible to make two kinds of errors. The probability of making these errors depended on the way we designed the experiment and the method we used for the required decision. In some cases we are not too worried about the errors and can make a relatively simple experiment. In other cases, errors are very important, and the experiment must be designed with that fact in mind. For example, the possibility of error is certainly important in the case that a vaccine for a given disease is proposed, and the statistician is asked to help in deciding whether or not it should be used. In this case it might be assumed that there is a certain probability p that a person will get the disease if not vaccinated, and a probability r that he will get it if he is vaccinated. If we have some knowledge of the approximate value of p, we are then led to construct an experiment to decide whether r is greater than p, equal to p, or less than p. The first case would be interpreted to mean that the vaccine actually tends to produce the disease, the second that it has no effect, and the third that it prevents the disease; so that we can make three kinds of errors. We could recommend acceptance when it is actually harmful, we could recommend acceptance when it has no effect, or finally we could reject it when it actually is effective. The first and third might result in the loss of lives, the second in the loss of time and money of those administering the test. Here it would certainly be important that the probability of the first and third kinds of errors be made small. To see how it is possible to make the probability of both errors small, we return to the case of Smith and Jones.

Suppose that, instead of demanding that Smith make at least eight correct identifications out of ten trials, we insist that he make at least 60 correct identifications out of 100 trials. (The glasses must now be very small.) Then, if $p = \frac{1}{2}$, the probability that Jones wins the bet is about .98; so that we are extremely unlikely to give the dollar to Smith when in fact it should go to Jones. (If $p < \frac{1}{2}$, it is even more likely that Jones will win.) If $p > \frac{1}{2}$, we can also calculate the probability that

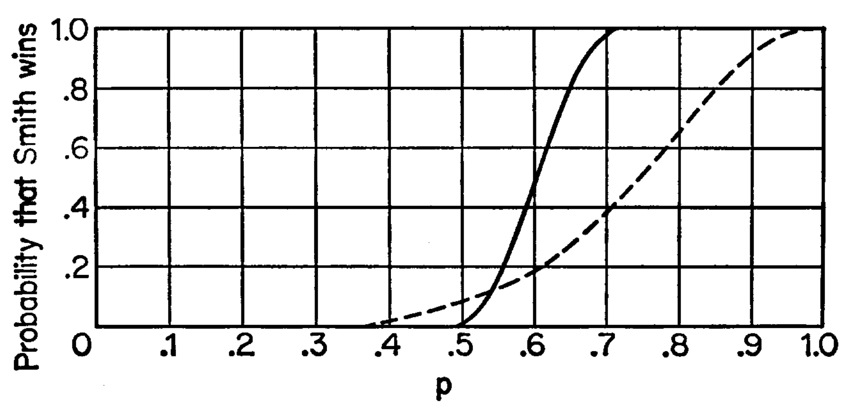


Figure 15

Smith will win the bet. These probabilities are shown in the graph in Figure 15. The dashed curve gives for comparison the corresponding probabilities for the test requiring eight out of ten correct. Note that with 100 trials, if p is $\frac{3}{4}$, the probability that Smith wins the bet is nearly 1, while in the case of eight out of ten, it was only about $\frac{1}{2}$. Thus in the case of 100 trials, it would be easy to convince both Smith and Jones that whichever one is correct is very likely to win the bet.

Thus we see that the probability of both types of errors can be made small at the expense of having a large number of experiments.

EXERCISES

1. Assume that in the beer and ale experiment Jones agrees to pay Smith if Smith gets at least nine out of ten correct.

(a) What is the probability of Jones paying Smith even though Smith cannot distinguish beer and ale, and guesses? [*Ans.* .011.]

(b) Suppose that Smith can distinguish with probability .9. What is the probability of his not collecting from Jones? [*Ans.* .264.]

2. Suppose that in the beer and ale experiment Jones wishes the probability to be less than .1 that Smith will be paid if, in fact, he guesses. How many of ten trials must he insist that Smith get correct to achieve this?

3. In the analysis of the beer and ale experiment, we assume that the various trials were independent. Discuss several ways that error can enter, because of the nonindependence of the trials, and how this error can be eliminated. (For example, the glasses in which the beer and ale were served might be distinguishable.)

4. Consider the following two procedures for testing Smith's ability to distinguish beer from ale.

(a) Four glasses are given at each trial, three containing beer and one ale, and he is asked to pick out the one containing ale. This procedure is repeated ten times. He must guess correctly seven or more times.

(b) Ten glasses are given him, and he is told that five contain beer and five ale, and he is asked to name the five which he believes contain ale. He must choose all five correctly.

In each case, find the probability that Smith establishes his claim by guessing. Is there any reason to prefer one test over the other?

[*Ans.* (a) .003; (b) .004.]

5. A testing service claims to have a method for predicting the order in which a group of freshmen will finish in their scholastic record at the end of college. The college agrees to try the method on a group of five students, and says that it will adopt the method if, for these five students, the prediction is either exactly correct or can be changed into the correct order by interchanging one pair of *adjacent* men in the predicted order. If the method is equivalent to simply guessing, what is the probability that it will be accepted? [*Ans.* $\frac{1}{24}$.]

6. The standard treatment for a certain disease leads to a cure in $\frac{1}{4}$ of the cases. It is claimed that a new treatment will result in a cure in $\frac{3}{4}$ of the cases. The new treatment is to be tested on ten people having the disease. If seven or more are cured, the new treatment will be adopted. If three or fewer people are cured, the treatment will not be considered further. If the number cured is four, five, or six, the results will be called inconclusive, and a further study will be made. Find the probabilities for each of these three alternatives under the assumption first, that the new treatment has the same effectiveness as the old, and second, under the assumption that the claim made for the treatment is correct.

7. Three students debate the intelligence of blonde dates. One claims that blondes are mostly (say 90 per cent of them) intelligent. A second claims that very few (say 10 per cent) blondes are intelligent, while a third one claims that a blonde is just as likely to be intelligent as not. They administer an intelligence test to ten blondes, classifying them as intelligent or not. They agree that the first man wins the bet if eight or more are intelligent, the second if two or fewer, the third in all other cases. For each man, calculate the probability that he wins the bet, if he is right. [*Ans.* .930, .930, .890.]

8. Ten men take a test with ten problems. Each man on each question has probability $\frac{1}{2}$ of being right, if he does not cheat. The instructor determines the number of students who get each problem correct. If he finds on four or more problems there are fewer than three or more than seven correct, he considers this convincing evidence of communication between the students. Give a justification for the procedure. [*Hint:* The table in Figure 14 must be used twice, once for the probability of fewer than three or more than seven correct answers on a given problem, and the second time to find the probability of this happening on four or more problems.]

10. *THE LAW OF LARGE NUMBERS*

In this section we shall study some further properties of the independent trials process with two outcomes. In Section 8 we saw that the probability for x successes in n trials is given by

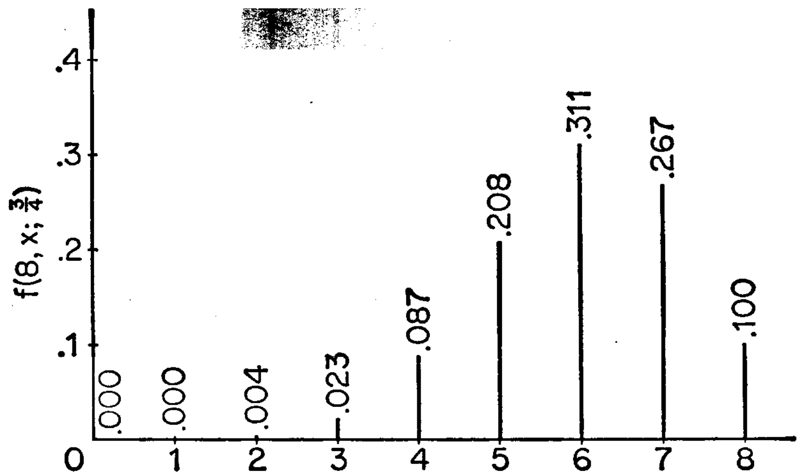


Figure 16

$$f(n, x; p) = \binom{n}{x} p^x q^{n-x}.$$

In Figure 16 we show these probabilities graphically for $n = 8$ and $p = \frac{3}{4}$. In Figure 17 we have done similarly for the case of $n = 7$ and $p = \frac{3}{4}$.

We see in the first case that the values increase up to a maximum value at $x = 6$ and then decrease. In the second case the values increase up to a maximum value at $x = 5$, have the same value for $x = 6$, and then decrease. These two cases are typical of what can happen in general.

Consider the ratio of the probability of $x + 1$ successes in n trials to the probability of x successes in n trials, which is

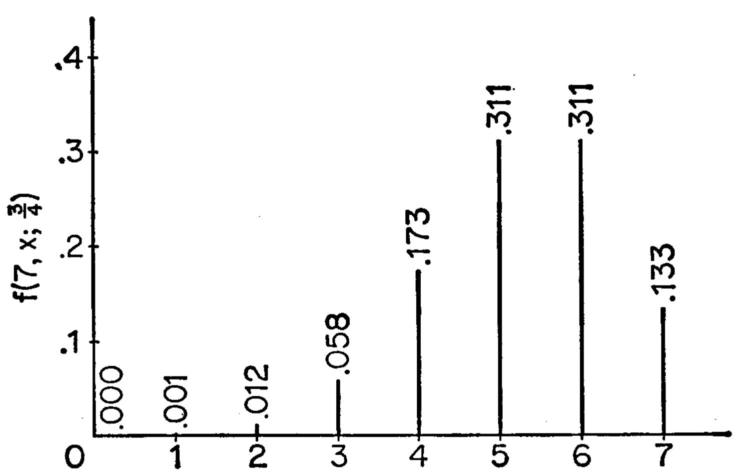


Figure 17

$$\frac{\binom{n}{x+1} p^{x+1} q^{n-x-1}}{\binom{n}{x} p^x q^{n-x}} = \frac{n-x}{x+1} \cdot \frac{p}{q}.$$

This ratio will be greater than one as long as $(n - x)p > (x + 1)q$ or as long as $x < np - q$. If $np - q$ is not an integer, the values $\binom{n}{x} p^x q^{n-x}$ increase up to a maximum value, which occurs at the first integer greater than $np - q$, and then decrease. In case $np - q$ is an integer, the values $\binom{n}{x} p^x q^{n-x}$ increase up to $x = np - q$, are the same for $x = np - q$ and $x = np - q + 1$, and then decrease.

Thus we see that, in general, values near np will occur with the largest probability. It is not true that one particular value near np is highly likely to occur, but only that it is relatively more likely than a value further from np. For example, in 100 throws of a coin, $np = 100 \cdot \frac{1}{2} = 50$. The probability of exactly 50 heads is approximately .08. The probability of exactly 30 is approximately .00002.

More information is obtained by studying the probability of a given deviation of the proportion of successes x/n from the number p; that is, by studying for $\epsilon > 0$,

$$\Pr\left[\left|\frac{x}{n} - p\right| < \epsilon\right].$$

For any fixed n, p, and ϵ, the latter probability can be found by adding all the values of $f(n, x; p)$ for values of x for which the inequality $p - \epsilon < x/n < p + \epsilon$ is true. In Figure 18 we have given these probabilities for the case $p = .3$ with various values for ϵ and n. In the first column we have the case $\epsilon = .1$. We observe that as n increases, the probability that the fraction of successes deviates from .3 by less than .1 tends to the value 1. In fact to four decimal places the answer is 1 after $n = 400$. In column two we have the same probabilities for the smaller value of $\epsilon = .05$. Again the probabilities are tending to 1 but not so fast. In the third column we have given these probabilities for the case $\epsilon = .02$. We see now that even after 1000 trials there is still a reasonable chance that the fraction x/n is not within .02 of the value of $p = .3$. It is natural to ask if we can expect these probabilities also to tend to 1 if we increase n sufficiently. The answer is yes and this is assured by one

$\Pr\left[\left|\frac{x}{n} - p\right| < \epsilon\right]$ for $p = .3$ and $\epsilon = .1, .05, .01.$

n	$\Pr\left[\left\|\frac{x}{n} - .3\right\| < .10\right]$	$\Pr\left[\left\|\frac{x}{n} - .3\right\| < .05\right]$	$\Pr\left[\left\|\frac{x}{n} - .3\right\| < .02\right]$
20	.5348	.1916	.1916
40	.7738	.3945	.1366
60	.8800	.5184	.3269
80	.9337	.6068	.2853
100	.9626	.6740	.2563
200	.9974	.8577	.4107
300	.9998	.9326	.5116
400	1.0000	.9668	.5868
500	1.0000	.9833	.6461
600	1.0000	.9915	.6944
700	1.0000	.9956	.7345
800	1.0000	.9977	.7683
900	1.0000	.9988	.7970
1000	1.0000	.9994	.8216

Figure 18

of the fundamental theorems of probability called the *law of large numbers*. This theorem asserts that, for any $\epsilon > 0$,

$$\Pr\left[\left|\frac{x}{n} - p\right| < \epsilon\right]$$

tends to 1 as n increases indefinitely.

It is important to understand what this theorem says and what it does not say. Let us illustrate its meaning in the case of coin tossing.

We are going to toss a coin n times and we want the probability to be very high, say greater than .99, that the fraction of heads which turn up will be very close, say within .001 of the value .5. The law of large numbers assures us that we can have this if we simply choose n large enough. The theorem itself gives us no information about how large n must be. Let us however consider this question.

To say that the fraction of the times success results is near p is the same as saying that the actual number of successes x does not deviate too much from the expected number np. To see the kind of deviations which might be expected we can study the value of $\Pr\,[|x - np| \geq d]$.

A table of these values for $p = .3$ and various values of n and d are given in Figure 19. Let us ask how large d must be before a deviation as large as d could be considered surprising. For example, let us see for each n the value of d which makes $\Pr[|x - np| \geq d]$ about .04. From the table, we see that d should be 7 for $n = 50$, 9 for $n = 80$, 10 for $n = 100$, etc. To see deviations which might be considered more typical we look for the values of d which make $\Pr[|x - np| \geq d]$ approximately $\frac{1}{3}$. Again from the table, we see that d should be 3 or 4 for $n = 50$, 4 or 5 for $n = 80$, 5 for $n = 100$, etc. The answers to these two questions are given in the last two columns of the table. An examination of these numbers shows us that deviations which we would consider surprising are approximately $\sqrt{n}$ while those which are more typical are about one half as large or $\sqrt{n}/2$.

This suggests that $\sqrt{n}$, or a suitable multiple of it, might be taken as a unit of measurement for deviations. Of course, we would also have to study how $\Pr\left[\left|\frac{x}{n} - p\right| \geq d\right]$ depends on p. When this is done, one finds that $\sqrt{npq}$ is a natural unit; it is called a *standard deviation.* It can be shown that for large n the following approximations hold.

$$\Pr[|x - np| \geq \sqrt{npq}] \approx .3174$$
$$\Pr[|x - np| \geq 2\sqrt{npq}] \approx .0455$$
$$\Pr[|x - np| \geq 3\sqrt{npq}] \approx .0027$$

That is, a deviation from the expected value of one standard deviation is rather typical, while a deviation of as much as two standard deviations is quite surprising and three very surprising. For values of p not too near 0 or 1, the value of $\sqrt{pq}$ is approximately $\frac{1}{2}$. Thus these approximations are consistent with the results we observed from our table.

For large n, $\Pr[x - np \geq k\sqrt{npq}]$ or $\Pr[x - np \leq -k\sqrt{npq}]$ can be shown to be approximately the same. Hence these probabilities can be estimated for $k = 1$, 2, and 3 by taking $\frac{1}{2}$ the values given above.

Example 1. In throwing an ordinary coin 10,000 times, the expected number of heads is 5000, and the standard deviation for the number of heads is $\sqrt{10{,}000(\frac{1}{2})(\frac{1}{2})} = 50$. Thus the probability that the number of heads which turn up deviates from 5000 by as much as one standard deviation, or 50, is approximately .317. The probability of a deviation

$p = .3; \qquad \Pr[|x - np| \geq d]$.

n \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	Pr near to .04	Pr near to $\frac{1}{3}$
50	.878	.644	.441	.280	.164	.088	.043	.020	.008									7	3–4
80	.903	.715	.542	.393	.272	.179	.112	.066	.037	.020	.010							9	4–5
100	.913	.744	.586	.445	.326	.230	.155	.101	.063	.037	.021	.012						10	5
120	.921	.765	.619	.486	.370	.273	.195	.135	.090	.058	.036	.022	.012					11	5–6
140	.927	.782	.645	.519	.407	.310	.230	.166	.116	.079	.052	.033	.021	.012				12	6
170	.933	.802	.676	.558	.451	.357	.276	.209	.154	.111	.078	.054	.036	.024	.015	.009		13	6
200	.939	.817	.700	.589	.488	.396	.316	.247	.189	.142	.105	.076	.053	.037	.025	.017	.011	14	7

Figure 19

of as much as two standard deviations, or 100, is approximately .046. The probability of a deviation of as much as three standard deviations, or 150, is approximately .003.

Example 2. Assume that in a certain large city, 900 people are chosen at random and asked if they favor a certain proposal. Of the 900 asked, 550 say they favor the proposal and 350 are opposed. If, in fact, the people in the city are equally divided on the issue, would it be unlikely that such a large majority would be obtained in a sample of 900 of the citizens? If the people were equally divided, we would assume that the 900 people asked would form an independent trials process with probability $\frac{1}{2}$ for a "yes" answer and $\frac{1}{2}$ for a "no" answer. Then the standard deviation for the number of "yes" answers in 900 trials is $\sqrt{900(\frac{1}{2})(\frac{1}{2})} = 15$. Then it would be very unlikely that we would obtain a deviation of more than 45 from the expected number of 450. The fact that the deviation in the sample from the expected number was 100, then, is evidence that the hypothesis that the voters were equally divided is incorrect. The assumption that the true proportion is any value less than $\frac{1}{2}$ would also lead to the fact that a number as large as 550 favoring in a sample of 900 is very unlikely. Thus we are led to suspect that the true proportion is greater than $\frac{1}{2}$. On the other hand, if the number who favored the proposal in the sample of 900 were 465, we would have only a deviation of one standard deviation, under the assumption of an equal division of opinion. Since such a deviation is not unlikely, we could not rule out this possibility on the evidence of the sample.

Example 3. A certain Ivy League college would like to admit 800 students in their freshman class. Experience has shown that if they admit 1250 students they will have acceptances from approximately 800. If they admit as many as 50 too many students they will have to provide additional dormitory space. Let us find the probability that this will happen assuming that the acceptances of the students can be considered to be an independent trials process. We take as our estimate for the probability of an acceptance $p = \frac{800}{1250} = .64$. Then the expected number of acceptances is 800 and the standard deviation for the number of acceptances is $\sqrt{1250 \times .64 \times .36} \approx 17$. The probability that the number accepted is three standard deviations or 51 from the mean is approximately .0027. This probability takes into account a deviation above the mean or below the mean. Since in this case we are only interested in a deviation above the mean, the probability we desire is half of this or approximately .0013. Thus we see that it is highly unlikely that the college will have to have new dormitory space under the assumptions we have made.

We finish this discussion of the law of large numbers with some final remarks about the interpretation of this important theorem.

Of course no matter how large n is we cannot prevent the coin from coming up heads every time. If this were the case we would observe a fraction of heads equal to 1. However, this is not inconsistent with the theorem, since the probability of this happening is $(\frac{1}{2})^n$ which tends to 0 as n increases. Thus a fraction of 1 is always possible, but becomes increasingly unlikely.

The law of large numbers is often misinterpreted in the following manner. Suppose that we plan to toss the coin 1000 times and after 500 tosses we have already obtained 400 heads. Then we must obtain less than one half heads in the remaining 500 tosses to have the fraction come out near $\frac{1}{2}$. It is tempting to argue that the coin therefore owes us some tails and it is more likely that tails will occur in the last 500 tosses. Of course this is nonsense, since the coin has no memory. The point is that something very unlikely has already happened in the first 500 tosses. The final result can therefore also be expected to be a result not predicted before the tossing began.

We could also argue that perhaps the coin is a biased coin but this would make us predict more heads than tails in the future. Thus the law of averages, or the law of large numbers, should not give you great comfort if you have had a series of very bad hands dealt you in your last 100 poker hands. If the dealing is fair, you have the same chance as ever of getting a good hand.

Early attempts to define the probability p that success occurs on a single experiment sounded like this. If the experiment is repeated indefinitely, the fraction of successes obtained will tend to a number p, and this number p is called the probability of success on a single experiment. While this fails to be satisfactory as a definition of probability, the law of large numbers captures the spirit of this frequency concept of probability.

EXERCISES

1. If an ordinary die is thrown 20 times, what is the expected number of times that a six will turn up? What is the standard deviation for the number of sixes that turn up? [*Ans.* $\frac{10}{3}$; $\frac{5}{3}$.]

2. Suppose that an ordinary die is thrown 450 times. What is the expected number of throws that result in either a three or a four? What is the standard deviation for the number of such throws?

3. In 16 tosses of an ordinary coin, what is the expected number of heads that turn up? What is the standard deviation for the number of heads that occur? [*Ans.* 8; 2.]

4. In 16 tosses of a coin, find the exact probability that the number of heads that turn up differs from the expected number by (a) as much as one standard deviation, and (b) by more than one standard deviation. Do the same for the case of two standard deviations, and for the case of three standard deviations. Show that the approximations given for large n lie between the values obtained, but are not very accurate for so small an n.
[*Ans.* .454; .210; .077; .021; .004; .001.]

5. Consider n independent trials with probability p for success. Let r and s be numbers such that $p < r < s$. What does the law of large numbers say about

$$\Pr\left[r < \frac{x}{n} < s\right]$$

as we increase n indefinitely? Answer the same question in the case that $r < p < s$.

6. A drug is known to be effective in 20 per cent of the cases where it is used. A new agent is introduced, and in the next 900 times the drug is used it is effective 250 times. What can be said about the effectiveness of the drug?

7. In a large number of independent trials with probability p for success, what is the approximate probability that the number of successes will deviate from the expected number by more than one standard deviation but less than two standard deviations? [*Ans.* .272.]

8. What is the approximate probability that, in 10,000 throws of an ordinary coin, the number of heads which turn up lies between 4850 and 5150? What is the probability that the number of heads lies in the same interval, given that in the first 1900 throws there were 1600 heads?

9. Suppose that it is desired that the probability be approximately .95 that the fraction of sixes that turn up when a die is thrown n times does not deviate by more than .01 from the value $\frac{1}{6}$. How large should n be?

[*Ans.* Approximately 5555.]

10. Suppose that for each roll of a fair die you lose \$1 when an odd number comes up and win \$1 when an even number comes up. Then after 10,000 rolls you can, with approximately 84 per cent confidence, expect to have lost not more than \$______.

11. Assume that 10 per cent of the people in a certain city have cancer. If 900 people are selected at random from the city, what is the expected number which will have cancer? What is the standard deviation? What is the approximate probability that more than 108 of the 900 chosen have cancer?

[*Ans.* 90; 9; .023.]

12. Suppose that in Exercise 11, the 900 people are chosen at random from those people in the city who smoke. Under the hypothesis that smoking has no effect on the incidence of cancer, what is the expected number in the 900 chosen that have cancer? Suppose that more than 120 of the 900 chosen have cancer, what might be said concerning the hypothesis that smoking has no effect on the incidence of cancer?

13. In Example 2, we made the assumption in our calculations that, if the true proportion of voters in favor of the proposal were p, then the 900 people chosen at random represented an independent trials process with probability p for a "yes" answer, and $1 - p$ for a "no" answer. Give a method for choosing the 900 people which would make this a reasonable assumption. Criticize the following methods.

(a) Choose the first 900 people in the list of registered Republicans.
(b) Choose 900 names at random from the telephone book.
(c) Choose 900 houses at random and ask one person from each house, the houses being visited in the mid-morning.

14. For n throws of an ordinary coin, let t_n be such that

$$\Pr\left[-t_n < \frac{x}{n} - \frac{1}{2} < t_n\right] = .997,$$

where x is the number of heads that turn up. Find t_n for $n = 10^4$, $n = 10^6$, and $n = 10^{20}$. [*Ans.* .015; .0015; .000,000,000,15.]

15. Assume that a calculating machine carries out a million operations to solve a certain problem. In each operation the machine gives the answer 10^{-5} too small, with probability $\frac{1}{2}$, and 10^{-5} too large, with probability $\frac{1}{2}$. Assume that the errors are independent of one another. What is a reasonable accuracy to attach to the answer? What if the machine carries out 10^{10} operations?

[*Ans.* $\pm$.01; $\pm$1.]

16. The Dartmouth Computer tossed a coin 1 million times (see Chapter VII, Section 10). It obtained 499,588 heads. Is this number reasonable?

*11. INDEPENDENT TRIALS WITH MORE THAN TWO OUTCOMES

By extending the results of Section 8, we shall study the case of independent trials in which we allow more than two outcomes. We assume that we have an independent trials process where the possible outcomes are $a_1, a_2, \ldots, a_k$, occurring with probabilities $p_1, p_2, \ldots, p_k$, respectively. We denote by

$$f(r_1, r_2, \ldots, r_k; p_1, p_2, \ldots, p_k)$$

the probability that, in

$$n = r_1 + r_2 + \ldots + r_k$$

such trials, there will be r_1 occurrences of a_1, r_2 of a_2, etc. In the case of two outcomes this notation would be $f(r_1, r_2; p_1, p_2)$. In Section 8 we wrote this as $f(n, r_1; p_1)$ since r_2 and p_2 are determined from n, r_1, and p_1. We shall indicate how this probability is found in general, but carry out the details only for a special case. We choose $k = 3$, and $n = 5$ for purposes of illustration. We shall find $f(1, 2, 2; p_1, p_2, p_3)$.

We show in Figure 20 enough of the tree for this process to indicate the branch probabilities for a path (heavy lined) corresponding to the outcomes a_2, a_3, a_1, a_2, a_3. The tree measure assigns weight $p_2 \cdot p_3 \cdot p_1 \cdot p_2 \cdot p_3 = p_1 \cdot p_2^2 \cdot p_3^2$ to this path.

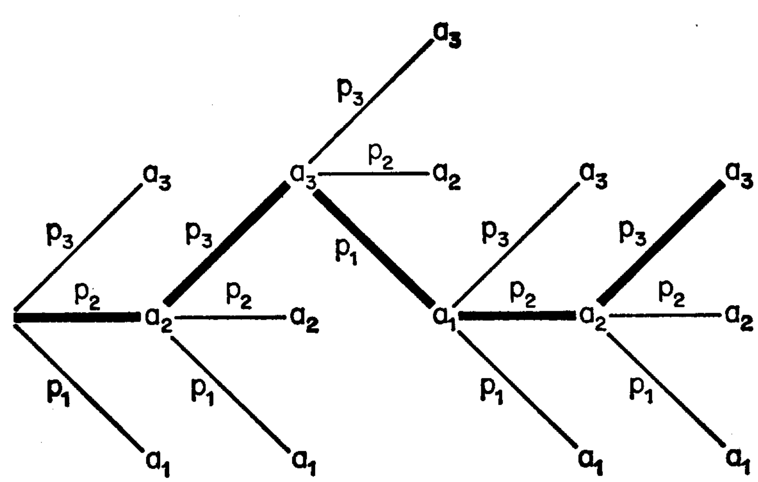


Figure 20

There are, of course, other paths through the tree corresponding to one occurrence of a_1, two of a_2, and two of a_3. However, they would all be assigned the same weight, $p_1 \cdot p_2^2 \cdot p_3^2$, by the tree measure. Hence to find $f(1, 2, 2; p_1, p_2, p_3)$, we must multiply this weight by the number of paths having the specified number of occurrences of each outcome.

We note that the path a_2, a_3, a_1, a_2, a_3 can be specified by the three-cell partition $[\{3\}, \{1, 4\}, \{2, 5\}]$ of the numbers from 1 to 5. Here the first cell shows the experiment which resulted in a_1, the second cell shows the two that resulted in a_2, and the third shows the two that resulted in a_3. Conversely, any such partition of the numbers from 1 to 5 with one element in the first cell, two in the second, and two in the third corresponds to a unique path of the desired kind. Hence the number of paths is the number of such partitions. But this is

$$\binom{5}{1, 2, 2} = \frac{5!}{1!\,2!\,2!}$$

(see Chapter III, Section 4), so that the probability of one occurrence of a_1, two of a_2, and two of a_3 is

$$\binom{5}{1, 2, 2} \cdot p_1 \cdot p_2^2 \cdot p_3^2.$$

The above argument carried out in general leads, for the case of independent trials with outcomes $a_1, a_2, \ldots, a_k$ occurring with probabilities $p_1, p_2, \ldots, p_k$, to the following.

The probability for r_1 occurrences of a_1, r_2 occurrences of a_2, etc., is given by

$$f(r_1, r_2, \ldots, r_k; p_1, p_2, \ldots, p_k) = \binom{n}{r_1, r_2, \ldots, r_k} p_1^{r_1} \cdot p_2^{r_2} \cdots p_k^{r_k}.$$

Example 1. A die is thrown 12 times. What is the probability that each number will come up twice? Here there are six outcomes, 1, 2, 3, 4, 5, 6 corresponding to the six sides of the die. We assign each outcome probability $\frac{1}{6}$. We are then asked for

$$f(2, 2, 2, 2, 2, 2; \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}),$$

which is

$$\binom{12}{2, 2, 2, 2, 2, 2} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 = .0034\ldots.$$

Example 2. Suppose that we have an independent trials process with four outcomes a_1, a_2, a_3, a_4 occurring with probability p_1, p_2, p_3, p_4, respectively. It might be that we are interested only in the probability that r_1 occurrences of a_1 and r_2 occurrences of a_2 will take place with no specification about the number of each of the other possible outcomes. To answer this question we simply consider a new experiment where the outcomes are a_1, a_2, $\bar{a}_3$. Here $\bar{a}_3$ corresponds to an occurrence of either a_3 or a_4 in our original experiment. The corresponding probabilities would be p_1, p_2, and $\bar{p}_3$ with $\bar{p}_3 = p_3 + p_4$. Let $\bar{r}_3 = n - (r_1 + r_2)$. Then our question is answered by finding the probability in our new experiment for r_1 occurrences of a_1, r_2 of a_2, and $\bar{r}_3$ of $\bar{a}_3$, which is

$$\binom{n}{r_1, r_2, \bar{r}_3} p_1^{r_1} \cdot p_2^{r_2} \cdot \bar{p}_3^{\bar{r}_3}.$$

The same procedure can be carried out for experiments with any number of outcomes where we specify the number of occurrences of such particular outcomes. For example, if a die is thrown ten times the probability that a one will occur exactly twice and a three exactly three times is given by

$$\binom{10}{2, 3, 5}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^3\left(\frac{4}{6}\right)^5 = .043\ldots.$$

EXERCISES

1. Suppose that in a city 60 per cent of the population are Democrats, 30 per cent are Republicans, and 10 per cent are Independents. What is the probability that if three people are chosen at random there will be one Republican, one Democrat, and one Independent voter? [*Ans.* .108.]

2. Three horses, A, B, and C, compete in four races. Assuming that each horse has an equal chance in each race, what is the probability that A wins two races and B and C win one each? What is the probability that the same horse wins all four races? [*Ans.* $\frac{4}{27}$; $\frac{1}{27}$.]

3. Assume that in a certain large college 40 per cent of the students are freshmen, 30 per cent are sophomores, 20 per cent are juniors, and 10 per cent are seniors. A committee of eight is chosen at random from the student body. What is the probability that there are equal numbers from each class on the committee?

4. Let us assume that when a batter comes to bat, he has probability .6 of being put out, .1 of getting a walk, .2 of getting a single, .1 of getting an extra base hit. If he comes to bat five times in a game, what is the probability that

(a) He get two walks and three singles? [*Ans.* .0008.]

(b) He gets a walk, a single, an extra base hit (and is out twice)? [*Ans.* .043.]

(c) He has a perfect day (i.e., never out)? [*Ans.* .010.]

5. Assume that a single torpedo has a probability $\frac{1}{2}$ of sinking a ship, probability $\frac{1}{4}$ of damaging it, and probability $\frac{1}{4}$ of missing. Assume further that two damaging shots sink the ship. What is the probability that four torpedoes will succeed in sinking the ship? [*Ans.* $\frac{251}{256}$.]

6. Jones, Smith, and Green live in the same house. The mailman has observed that Jones and Smith receive the same amount of mail on the average, but that Green receives twice as much as Jones (and hence also twice as much as Smith). If he has four letters for this house, what is the probability that each man receives at least one letter?

7. If three dice are thrown, find the probability that there is one six and two fives, given that all the outcomes are greater than three. [*Ans.* $\frac{1}{9}$.]

8. A man plays a tournament consisting of three games. In each game he has probability $\frac{1}{2}$ for a win, $\frac{1}{4}$ for a loss, and $\frac{1}{4}$ for a draw, independently of the outcomes of other games. To win the tournament he must win more games than he loses. What is the probability that he wins the tournament?

9. Assume that in a certain course the probability that a student chosen at random will get an A is .1, that he will get a B is .2, that he will get a C is .4, that he will get a D is .2, and that he will get an E is .1. What distribution of grades is most likely in the case of four students? [*Ans.* One B, two C's, one D.]

10. Let us assume that in a World Series game a batter has probability $\frac{1}{4}$ of getting no hits, $\frac{1}{2}$ for getting one hit, $\frac{1}{4}$ for getting two hits, assuming that the probability of getting more than two hits is negligible. In a four-game World Series, find the probability that the batter gets

(a) Exactly two hits.

(b) Exactly three hits.

(c) Exactly four hits.

(d) Exactly five hits.

(e) Fewer than two hits or more than five.

[*Ans.* $\frac{7}{64}$; $\frac{7}{32}$; $\frac{35}{128}$; $\frac{7}{32}$; $\frac{23}{128}$.]

11. Gypsies sometimes toss a thick coin for which heads and tails are equally likely, but which also has probability $\frac{1}{5}$ of standing on edge (i.e., neither heads nor tails). What is the probability of exactly one head and four tails in five tosses of a gypsy coin?

12. A family car is driven by the father, two sons, and the mother. The fenders have been dented four times, three times while the mother was driving. Is it fair to say that the mother is a worse driver than the men?

12. EXPECTED VALUE

In this section we shall discuss the concept of expected value. Although it originated in the study of gambling games, it enters into almost any detailed probabilistic discussion.

DEFINITION. If in an experiment the possible outcomes are numbers, $a_1, a_2, \ldots, a_k$, occurring with probability $p_1, p_2, \ldots, p_k$, then the *expected value* is defined to be

$$E = a_1p_1 + a_2p_2 + \ldots + a_kp_k.$$

The term "expected value" is not to be interpreted as the value that will necessarily occur on a single experiment. For example, if a person bets \$1 that a head will turn up when a coin is thrown, he may either win \$1 or lose \$1. His expected value is $(1)(\frac{1}{2}) + (-1)(\frac{1}{2}) = 0$, which is not one of the possible outcomes. The term, expected value, had its origin in the following consideration. If we repeat an experiment with expected value E a large number of times, and if we expect a_1 a fraction p_1 of the time, a_2 a fraction p_2 of the time, etc., then the average that we expect per experiment is E. In particular, in a gambling game E is interpreted as the average winning expected in a large number of plays. Here the expected value is often taken as the value of the game to the player. If the game has a positive expected value, the game is said to be favorable; if the game has expected value zero it is said to be fair; and if it has negative expected value it is described as unfavorable. These terms are not to be taken too literally, since many people are quite happy to play games that, in terms of expected value, are unfavorable. For instance, the buying of life insurance may be considered an unfavorable game which most people choose to play.

Example 1. For the first example of the application of expected value we consider the game of roulette as played at Monte Carlo. There are several types of bets which the gambler can make, and we consider two of these.

The wheel has the number 0 and the numbers from 1 to 36 marked on equally spaced slots. The wheel is spun and a ball comes to rest in one of these slots. If the player puts a stake, say of \$1, on a given number, and the ball comes to rest in this slot, then he receives from the croupier 36 times his stake, or \$36. The player wins \$35 with probability $\frac{1}{37}$ and loses \$1 with probability $\frac{36}{37}$. Hence his expected winnings are

$$\tfrac{35}{37} - 1\cdot\tfrac{36}{37} = -\tfrac{1}{37} = -.027.$$

This can be interpreted to mean that in the long run he can expect to lose about 2.7 per cent of his stakes.

A second way to play is the following. A player may bet on "red" or "black." The numbers from 1 to 36 are evenly divided between the two colors. If a player bets on "red," and a red number turns up, he receives twice his stake. If a black number turns up, he loses his stake. If 0 turns up, then the wheel is spun until it stops on a number different from 0. If this is black, the player loses; but if it is red, he receives only his original stake, not twice it. For this type of play, the gambler wins \$1 with probability $\frac{18}{37}$, breaks even with probability $\frac{1}{2}\cdot\frac{1}{37} = \frac{1}{74}$, and loses \$1 with probability $\frac{18}{37} + \frac{1}{2}\cdot\frac{1}{37} = \frac{37}{74}$. Hence his expected winning is

$$1\cdot\tfrac{18}{37} + 0\cdot\tfrac{1}{74} - 1\cdot\tfrac{37}{74} = -.0135.$$

In this case the player can expect to lose about 1.35 per cent of his stakes in the long run. Thus the expected loss in this case is only half as great as in the previous case.

Example 2. A player rolls a die and receives a number of dollars corresponding to the number of dots on the face which turns up. What should the player pay for playing, to make this a fair game? To answer this question, we note that the player wins 1, 2, 3, 4, 5 or 6 dollars, each with probability $\frac{1}{6}$. Hence, his expected winning is

$$1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + 5(\tfrac{1}{6}) + 6(\tfrac{1}{6}) = 3\tfrac{1}{2}.$$

Thus if he pays \$3.50, his expected winnings will be zero.

Example 3. What is the expected number of successes in the case of four independent trials with probability $\frac{1}{3}$ for success? We know that the probability of x successes is $\binom{4}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{4-x}$. Thus

$$E = 0\cdot\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + 1\cdot\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + 2\cdot\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$
$$+ 3\cdot\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 + 4\cdot\binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0$$
$$= 0 + \frac{32}{81} + \frac{48}{81} + \frac{24}{81} + \frac{4}{81} = \frac{108}{81} = \frac{4}{3}.$$

In general, it can be shown that in n trials with probability p for success, the expected number of successes is np.

Example 4. In the game of craps a pair of dice is rolled by one of the players. If the sum of the spots shown is 7 or 11, he wins. If it is 2, 3, or 12, he loses. If it is another sum, he must continue rolling the dice until he either repeats the same sum or rolls a 7. In the former case he wins, in the latter he loses. Let us suppose that he wins or loses \$1. Then the two possible outcomes are $+1$ and -1. We will compute the expected value of the game. First we must find the probability that he will win.

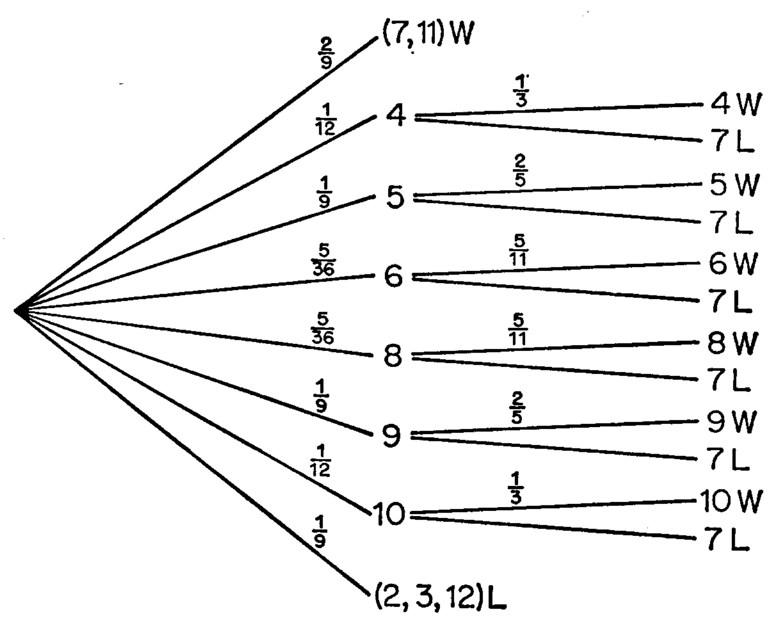


Figure 21

We represent the possibilities by a two-stage tree shown in Figure 21. While it is theoretically possible for the game to go on indefinitely, we do not consider this possibility. This means that our analysis applies only to games which actually stop at some time.

The branch probabilities at the first stage are determined by thinking of the 36 possibilities for the throw of the two dice as being equally likely and taking in each case the fraction of the possibilities which correspond to the branch as the branch probability. The probabilities for the branches at the second level are obtained as follows. If, for example, the first outcome was a 4, then when the game ends, a 4 or 7 must have occurred. The possible outcomes for the dice were

$$\{(3, 1), (1, 3), (2, 2), (4, 3), (3, 4), (2, 5), (5, 2), (1, 6), (6, 1)\}.$$

Again we consider these possibilities to be equally likely and assign to the branch considered the fraction of the outcomes which correspond to this branch. Thus to the 4 branch we assign a probability $\frac{3}{9} = \frac{1}{3}$. The other branch probabilities are determined in a similar way. Having the tree measure assigned, to find the probability of a win we must simply add the weights of all paths leading to a win. If this is done, we obtain $\frac{244}{495}$. Thus the player's expected value is

$$1 \cdot (\tfrac{244}{495}) + (-1) \cdot (\tfrac{251}{495}) = -\tfrac{7}{495} = -.0141.$$

Hence he can expect to lose 1.41 per cent of his stakes in the long run. It is interesting to note that this is just slightly less favorable than his losses in betting on "red" in roulette.

EXERCISES

1. Suppose that A tosses two coins and receives \$2 if two heads appear, \$1 if one head appears, and nothing if no heads appear. What is the expected value of the game to him? [*Ans.* \$1.]

2. Smith and Jones are matching coins. If the coins match, Smith gets \$1, and if they do not, Jones get \$1.

(a) If the game consists of matching twice, what is the expected value of the game for Smith?

(b) Suppose that if Smith wins the first round he quits, and if he loses the first he plays the second. Jones is not allowed to quit. What is the expected value of the game for Smith?

3. If five coins are thrown, what is the expected number of heads that will turn up? [*Ans.* $\frac{5}{2}$.]

4. A coin is thrown until the first time a head comes up or until three tails in a row occur. Find the expected number of times the coin is thrown.

5. A man wishes to purchase a five cent newspaper. He has in his pocket one dime and five pennies. The newsman offers to let him have the paper in exchange for one coin drawn at random from the customer's pocket.

(a) Is this a fair proposition and, if not, to whom is it favorable?

[*Ans.* Favorable to man.]

(b) Answer the same questions as in (a) assuming that the newsman demands two coins drawn at random from the customer's pocket.

[*Ans.* Fair proposition.]

6. A bets 50 cents against B's x cents that, if two cards are dealt from a shuffled pack of ordinary playing cards, both cards will be of the same color. What value of x will make this bet fair?

7. Prove that if the expected value of a given experiment is E, and if a constant c is added to each of the outcomes, the expected value of the new experiment is $E + c$.

8. Prove that, if the expected value of a given experiment is E, and if each of the possible outcomes is multiplied by a constant k, the expected value of the new experiment is $k \cdot E$.

9. A man plays the following game: He draws a card from a bridge deck; if it is an ace he wins \$5; if it is a jack, a queen or a king, he wins \$2; for any other card he loses \$1. What is his expected winning per play?

10. An urn contains two black and three white balls. Balls are successively drawn from the urn without replacement until a black ball is obtained. Find the expected number of draws required.

11. Using the result of Exercises 13 and 14 of Section 6, find the expected number of games in the World Series (a) under the assumption that each team has probability $\frac{1}{2}$ of winning each game and (b) under the assumption that the stronger team has probability .6 of winning each game.

[*Ans.* 5.81; 5.75.]

12. Suppose that we modify the game of craps as follows: On a 7 or 11 the player wins \$2, on a 2, 3, or 12 he loses \$3; otherwise the game is as usual. Find the expected value of the new game, and compare it with the old value.

13. Suppose that in roulette at Monte Carlo we place 50 cents on "red" and 50 cents on "black." What is the expected value on the game? Is this better or worse than placing \$1 on "red"?

14. Betting on "red" in roulette can be described roughly as follows. We win with probability .49, get our money back with probability .01, and lose with probability .50. Draw the tree for three plays of the game, and compute (to three decimals) the probability of each path. What is the probability that we are ahead at the end of three bets? [*Ans.* .485.]

15. Assume that the odds are $r:s$ that a certain statement will be true. If a man receives s dollars if the statement turns out to be true, and gives r dollars if not, what is his expected winning?

16. Referring to Exercise 9 of Section 3, find the expected number of languages that a student chosen at random reads.

17. Referring to Exercise 5 of Section 4, find the expected number of men who get their own hats. [*Ans.* 1.]

18. A pair of dice is rolled. Each die has the number 1 on two opposite faces, the number 2 on two opposite faces, and the number 3 on two opposite faces. The "roller" wins a dollar if

(i) the sum of four occurs on the first roll; *or*

(ii) the sum of three or five occurs on the first roll and the same sum occurs on a subsequent roll before the sum of four occurs.

Otherwise he loses a dollar.

(a) What is the probability that the person rolling the dice wins?

(b) What is the expected value of the game? [*Ans.* (a) $\frac{23}{45}$; (b) $\frac{1}{45}$.]

13. MARKOV CHAINS

In this section we shall study a more general kind of process than the ones considered in the last three sections.

We assume that we have a sequence of experiments with the following properties. The outcome of each experiment is one of a finite number of possible outcomes $a_1, a_2, \ldots, a_r$. It is assumed that the probability of outcome a_j on any given experiment is not necessarily independent of the outcomes of previous experiments but depends at most upon the outcome of the immediately preceding experiment. We assume that there are given numbers p_{ij} which represent the probability of outcome a_j on any given experiment, given that outcome a_i occurred on the preceding experiment. The outcomes $a_1, a_2, \ldots, a_r$ are called *states*, and the numbers p_{ij} are called *transition probabilities*. If we assume that the process begins in some particular state, then we have enough information to determine the tree measure for the process and can calculate probabilities of statements relating to the over-all sequence of experiments. A process of the above kind is called a *Markov chain process*.

The transition probabilities can be exhibited in two different ways. The first way is that of a square array. For a Markov chain with states a_1, a_2, and a_3, this array is written as

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Such an array is a special case of a *matrix*. Matrices are of fundamental importance to the study of Markov chains as well as being important in the study of other branches of mathematics. They will be studied in detail in the next chapter.

A second way to show the transition probabilities is by a *transition diagram*. Such a diagram is illustrated for a special case in Figure 22. The arrows from each state indicate the possible states to which a process can move from the given state.

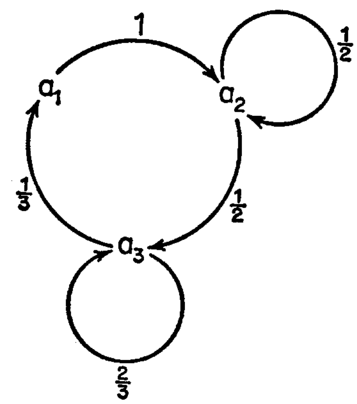


Figure 22

The matrix of transition probabilities which corresponds to this diagram is the matrix

$$P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix} \end{array}.$$

An entry of 0 indicates that the transition is impossible.

Notice that in the matrix P the sum of the elements of each row is 1. This must be true in any matrix of transition probabilities, since the elements of the ith row represent the probabilities for all possibilities when the process is in state a_i.

The kind of problem in which we are most interested in the study of Markov chains is the following. Suppose that the process starts in state i. What is the probability that after n steps it will be in state j? We denote this probability by $p_{ij}^{(n)}$. Notice that we do *not* mean by this the nth power of the number p_{ij}. We are actually interested in this probability for all possible starting positions i and all possible terminal positions j. We can represent these numbers conveniently again by a matrix. For example, for n steps in a three-state Markov chain we write these probabilities as the matrix

$$P^{(n)} = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & p_{13}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} \end{pmatrix}.$$

Example 1. Let us find for a Markov chain with transition probabilities indicated in Figure 22 the probability of being at the various possible states after three steps, assuming that the process starts at state a_1. We find these probabilities by constructing a tree and a tree measure as in Figure 23.

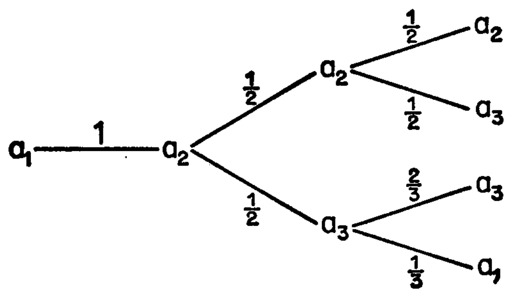


Figure 23

The probability $p_{13}^{(3)}$, for example, is the sum of the weights assigned by the tree measure to all paths through our tree which end at state a_3. That is,

$$1\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2}\cdot\tfrac{2}{3} = \tfrac{7}{12}.$$

Similarly

$$p_{12}^{(3)} = 1\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \quad\text{and}\quad p_{11}^{(3)} = 1\cdot\tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{1}{6}.$$

By constructing a similar tree measure, assuming that we start at state a_2, we could find $p_{21}^{(3)}, p_{22}^{(3)}$, and $p_{23}^{(3)}$. The same is true for $p_{31}^{(3)}, p_{32}^{(3)}$, and $p_{33}^{(3)}$. If this is carried out (see Exercise 7) we can write the results in matrix form as follows:

$$P^{(3)} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} \frac{1}{6} & \frac{1}{4} & \frac{7}{12} \\ \frac{7}{36} & \frac{7}{24} & \frac{37}{72} \\ \frac{4}{27} & \frac{7}{18} & \frac{25}{54} \end{pmatrix} \end{array}.$$

Again the rows add up to 1, corresponding to the fact that if we start at a given state we must reach some state after three steps. Notice now that all the elements of this matrix are positive, showing that it is possible to reach any state from any state in three steps. In the next chapter we will develop a simple method of computing $P^{(n)}$.

Example 2. Suppose that we are interested in studying the way in which a given state votes in a series of national elections. We wish to make long-term predictions and so will not consider conditions peculiar to a particular election year. We shall base our predictions only on past history of the outcomes of the elections, Republican or Democratic. It is clear that a knowledge of these past results would influence our predictions for the future. As a first approximation, we assume that the knowledge of the past beyond the last election would not cause us to change the probabilities for the outcomes on the next election. With this assumption we obtain a Markov chain with two states R and D and matrix of transition probabilities

$$\begin{array}{c c} & \begin{array}{cc} R & D \end{array} \\ \begin{array}{c} R \\ D \end{array} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}. \end{array}$$

The numbers a and b could be estimated from past results as follows. We could take for a the fraction of the previous years in which the outcome has changed from Republican in one year to Democratic in the next year, and for b the fraction of reverse changes.

We can obtain a better approximation by taking into account the previous two elections. In this case our states are RR, RD, DR, and DD, indicating the outcome of two successive elections. Being in state RR means that the last two elections were Republican victories. If the next election is a Democratic victory, we will be in state RD. If the election outcomes for a series of years is $DDDRDRR$, then our process has moved from state DD to DD to DR to RD to DR, and finally to RR. Notice that the first letter of the state to which we move must agree with the second letter of the state from which we came, since these refer to the same election year. Our matrix of transition probabilities will then have the form,

$$\begin{array}{c c} & \begin{array}{cccc} RR & DR & RD & DD \end{array} \\ \begin{array}{c} RR \\ DR \\ RD \\ DD \end{array} & \begin{pmatrix} 1-a & 0 & a & 0 \\ b & 0 & 1-b & 0 \\ 0 & 1-c & 0 & c \\ 0 & d & 0 & 1-d \end{pmatrix}. \end{array}$$

Again the numbers a, b, c, and d would have to be estimated. The study of this example is continued in Chapter V, Section 7.

Example 3. The following example of a Markov chain has been used in physics as a simple model for diffusion of gases. We shall see later that a similar model applies to an idealized problem in changing populations.

We imagine n black balls and n white balls which are put into two urns so that there are n balls in each urn. A single experiment consists in choosing a ball from each urn at random and putting the ball obtained from the first urn into the second urn, and the ball obtained from the second urn into the first. We take as state the number of black balls in the first urn. If at any time we know this number, then we know the exact composition of each urn. That is, if there are j black balls in urn 1, there must be $n - j$ black balls in urn 2, $n - j$ white balls in urn 1, and j white balls in urn 2. If the process is in state j, then after the next exchange it will be in state $j - 1$, if a black ball is chosen from urn 1 and a white ball from urn 2. It will be in state j if a ball of the same color is drawn from each urn. It will be in state $j + 1$ if a white ball is drawn from urn 1 and a black ball from urn 2. The transition probabilities are then given by (see Exercise 12)

$$
\begin{aligned}
p_{jj-1} &= \left(\frac{j}{n}\right)^2 && j > 0 \\
p_{jj} &= \frac{2j(n-j)}{n^2} && \\
p_{jj+1} &= \left(\frac{n-j}{n}\right)^2 && j < n \\
p_{jk} &= 0 && \text{otherwise.}
\end{aligned}
$$

A physicist would be interested, for example, in predicting the composition of the urns after a certain number of exchanges have taken place. Certainly any predictions about the early stages of the process would depend upon the initial composition of the urns. For example, if we started with all black balls in urn 1, we would expect that for some time there would be more black balls in urn 1 than in urn 2. On the other hand, it might be expected that the effect of this initial distribution would wear off after a large number of exchanges. We shall see later, in Chapter V, Section 7, that this is indeed the case.

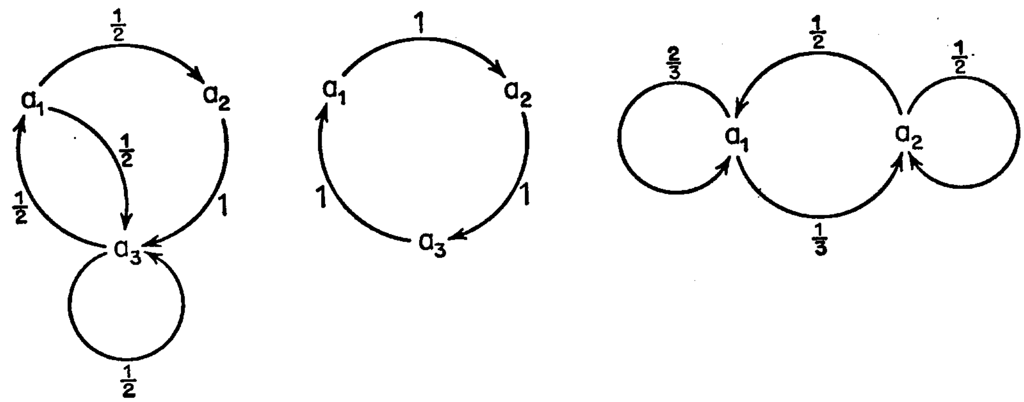
EXERCISES

1. Draw a state diagram for the Markov chain with transition probabilities given by the following matrices.

$$
\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad
\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix},
$$

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
$$

2. Give the matrix of transition probabilities corresponding to the following transition diagrams.



3. Find the matrix $P^{(2)}$ for the Markov chain determined by the matrix of transition probabilities

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \qquad \left[Ans.\ \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix}.\right]$$

4. What is the matrix of transition probabilities for the Markov chain in Example 3, for the case of two white balls and two black balls?

5. Find the matrices $P^{(2)}$, $P^{(3)}$, $P^{(4)}$ for the Markov chain determined by the transition probabilities

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find the same for the Markov chain determined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

6. Suppose that a Markov chain has two states, a_1 and a_2, and transition probabilities given by the matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

By means of a separate chance device we choose a state in which to start the process. This device chooses a_1 with probability $\frac{1}{2}$ and a_2 with probability $\frac{1}{2}$. Find the probability that the process is in state a_1 after the first step. Answer the same question in the case that the device chooses a_1 with probability $\frac{1}{3}$ and a_2 with probability $\frac{2}{3}$. [*Ans.* $\frac{5}{12}$; $\frac{4}{9}$.]

7. Referring to the Markov chain with transition probabilities indicated in Figure 22, construct the tree measures and determine the values of

$$p_{21}^{(3)}, p_{22}^{(3)}, p_{23}^{(3)}, \quad \text{and} \quad p_{31}^{(3)}, p_{32}^{(3)}, p_{33}^{(3)}.$$

8. A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage there is a probability p that the digit which enters this stage will be changed when it leaves. We form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?

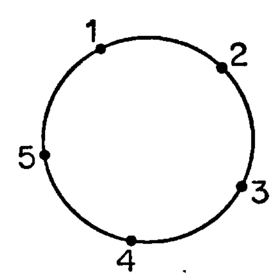
9. For the Markov chain in Exercise 8, draw a tree and assign a tree measure, assuming that the process begins in state 0 and moves through three stages of transmission. What is the probability that the machine after three stages produces the digit 0, i.e., the correct digit? What is the probability that the machine never changed the digit from 0?

10. Assume that a man's profession can be classified as professional, skilled laborer, or unskilled laborer. Assume that of the sons of professional men 80 per cent are professional, 10 per cent are skilled laborers, and 10 per cent are unskilled laborers. In the case of sons of skilled laborers, 60 per cent are skilled laborers, 20 per cent are professional, and 20 per cent are unskilled laborers. Finally, in the case of unskilled laborers, 50 per cent of the sons are unskilled laborers, and 25 per cent each are in the other two categories. Assume that every man has a son, and form a Markov chain by following a given family through several generations. Set up the matrix of transition probabilities. Find the probability that the grandson of an unskilled laborer is a professional man. [*Ans.* .375.]

11. In Exercise 10 we assumed that every man has a son. Assume instead that the probability a man has a son is .8. Form a Markov chain with four states. The first three states are as in Exercise 10, and the fourth state is such that the process enters it if a man has no son, and that the state cannot be left. This state represents families whose male line has died out. Find the matrix of transition probabilities and find the probability that an unskilled laborer has a grandson who is a professional man. [*Ans.* .24.]

12. Explain why the transition probabilities given in Example 3 are correct.

SUPPLEMENTARY EXERCISES



13. Five points are marked on a circle. A process moves clockwise from a given point to its neighbor with probability $\frac{2}{3}$, or counterclockwise to its neighbor with probability $\frac{1}{3}$.

(a) Considering the process to be a Markov chain process, find the matrix of transition probabilities.

(b) Given that the process starts in a state 3, what is the probability that it returns to the same state in two steps?

14. In northern New England, years for apples can be described as good, average, or poor. Suppose that following a good year the probabilities of good, average, or poor years are respectively .4, .4, and .2. Following a poor year the probabilities of good, average, or poor years are .2, .4, and .4 respectively. Following an average year the probabilities that the next year will be good or poor are each .2, and of an average year, .6.

(a) Set up the transition matrix of this Markov chain.

(b) 1965 was a good year. Compute the probabilities for 1966, 1967, and 1968. [*Partial Ans. For* 1967: .28, .48, .24.]

15. In Exercise 14 suppose that there is probability $\frac{1}{4}$ for a good year, $\frac{1}{2}$ for an average year, and $\frac{1}{4}$ for a poor year. What are the probabilities for the following year?

16. A teacher in an oversized mathematics class finds, after grading all homework papers for the first two assignments, that it is necessary to reduce the amount of time spent in such grading. He therefore designs the following system: Papers will be marked satisfactory or unsatisfactory. All papers of students receiving a mark of unsatisfactory on any assignment will be read on each of the two succeeding days. Of the remaining papers, the teacher will read one-fifth, chosen at random. Assuming that each paper has a probability of one-fifth of being classified "unsatisfactory,"

(a) Set up a three-state Markov chain to describe the process.

(b) Suppose that a student has just handed in a satisfactory paper. What are the probabilities for the next two assignments?

17. In another model for diffusion, it is assumed that there are two urns which together contain N balls numbered from 1 to N. Each second a number from 1 to N is chosen at random, and the ball with the corresponding number is moved to the other urn. Set up a Markov chain by taking as state the number of balls in urn 1. Find the transition matrix.

*14. THE CENTRAL LIMIT THEOREM

We continue our discussion of the independent trials process with two outcomes. As usual, let p be the probability of success on a trial, and $f(n, p; x)$ be the probability of exactly x successes in n trials.

In Figure 24 we have plotted bar graphs which represent $f(n, .3; x)$ for $n = 10$, 50, 100, and 200. We note first of all that the graphs are drifting off to the right. This is not surprising, since their peaks occur

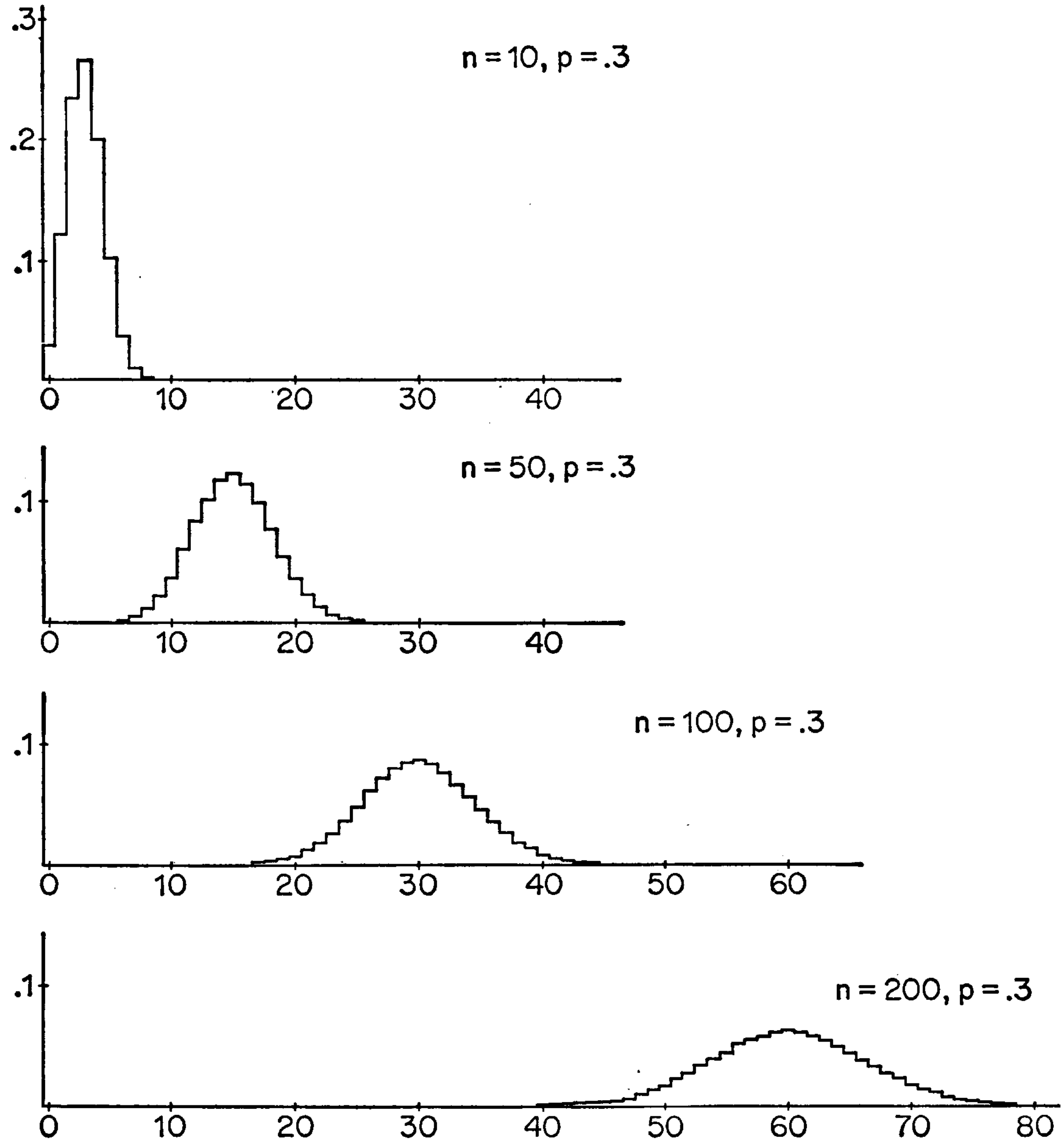


Figure 24

at np, which is steadily increasing. We also note that while the total area is always 1, this area becomes more and more spread out.

We want to redraw these graphs in a manner that prevents the drifting and the spreading out. First of all, we replace x by $x - np$, assuring that our peak always occurs at 0. Next we introduce a new unit for measuring the deviation, which depends on n, and which gives comparable scales. As we saw in Section 10, the standard deviation $\sqrt{npq}$ is such a unit.

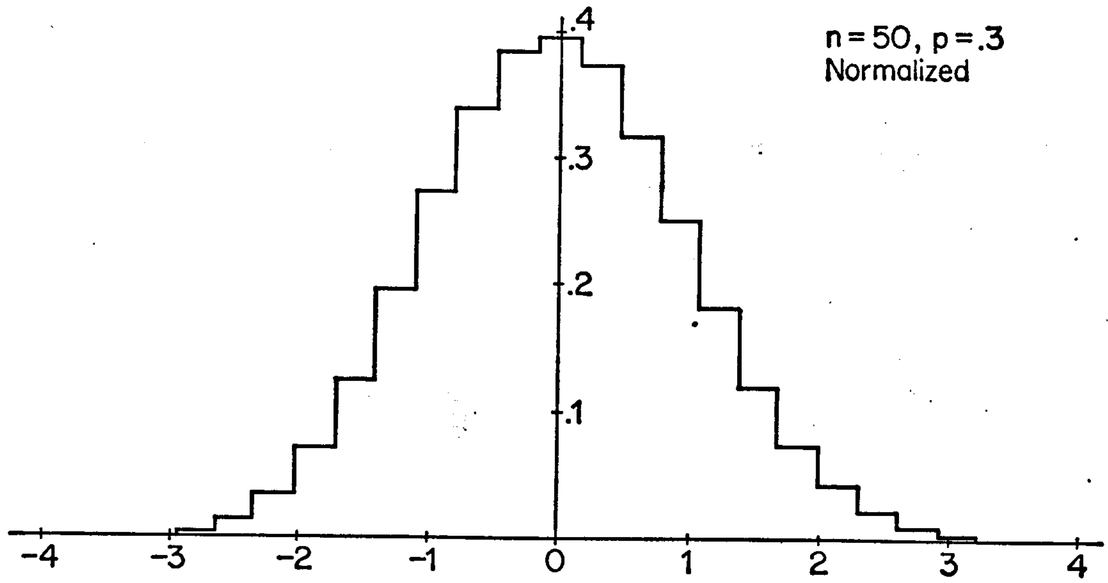


Figure 25

We must still insure that probabilities are represented by areas in the graph. In Figure 24 this is achieved by having a unit base for each rectangle, and having the probability $f(n, p; x)$ as height. Since we are now representing a standard deviation as a single unit on the horizontal axis, we must take $f(n, p; x)\sqrt{npq}$ as the heights of our rectangles. The resulting curves for $n = 50$ and 200 are shown in Figures 25 and 26, respectively.

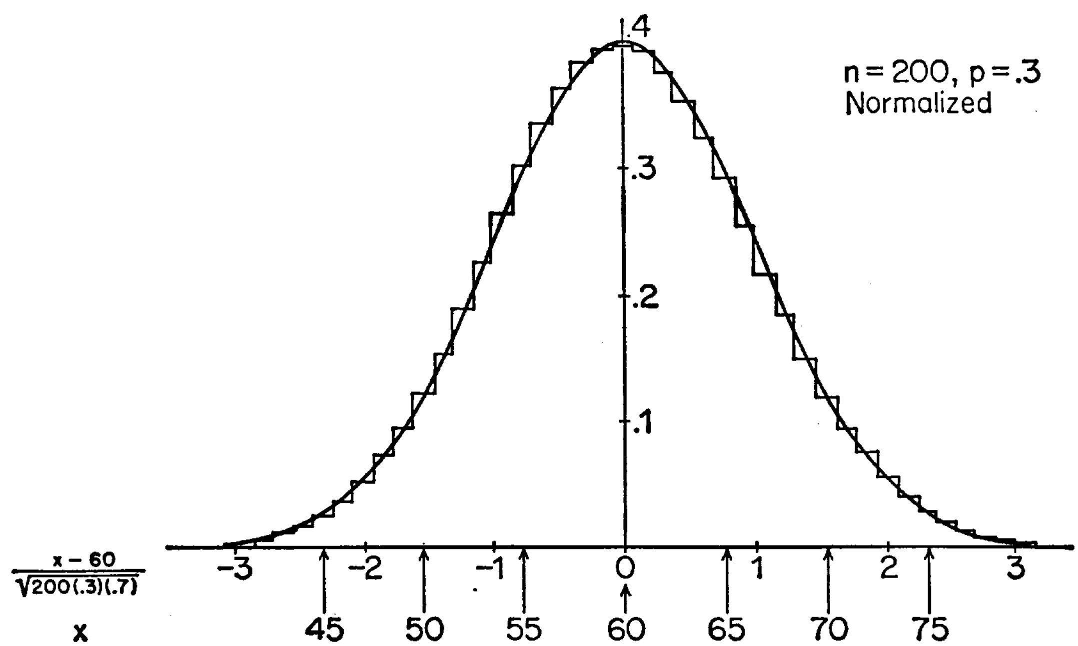


Figure 26

We note that the two figures look very much alike. We have also shown in Figure 26 that it can be approximated by a bell-shaped curve. This curve represents the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and is known as the *normal curve.* It is a fundamental theorem of probability theory that as n increases, the appropriately rescaled bar-graphs more and more closely approach the normal curve. The theorem is known as the *Central Limit Theorem,* and we have illustrated it graphically.

More precisely, the theorem states that for any two numbers a and b, with $a < b$,

$$\Pr\left[a < \frac{x - np}{\sqrt{npq}} < b\right]$$

approaches the area under the normal curve between a and b, as n increases. This theorem is particularly interesting in that the normal curve is symmetric about 0, while $f(n, p; x)$ is symmetric about the expected value np only for the case $p = \frac{1}{2}$. It should also be noted that we always arrive at the same normal curve, no matter what the value of p is.

In Figure 27 we give a table for the area under the normal curve between 0 and d. Since the total area is 1, and since it is symmetric about the origin, we can compute arbitrary areas from this table. For example, suppose that we wish the area between -1 and $+2$. The area between 0 and 2 is given in the table as .477. The area between -1 and 0 is the same as between 0 and 1, and hence is given as .341. Thus the total area is .818. The area outside the interval $(-1, 2)$ is then $1 - .818 = .182$.

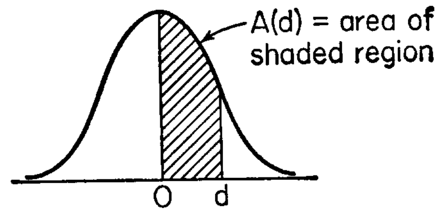
Example 1. Let us find the probability that x differs from the expected value np by as much as d standard deviations.

$$\Pr\,[|x - np| \geq d\sqrt{npq}] = \Pr\left[\left|\frac{x - np}{\sqrt{npq}}\right| \geq d\right],$$

and hence the approximate answer should be the area outside the interval $(-d, d)$ under the normal curve. For $d = 1, 2, 3$ we obtain

$$1 - (2 \times .341) = .318, 1 - (2 \times .477) = .046$$

and



d	$A(d)$	d	$A(d)$	d	$A(d)$	d	$A(d)$
.0	.000	1.1	.364	2.1	.482	3.1	.4990
.1	.040	1.2	.385	2.2	.486	3.2	.4993
.2	.079	1.3	.403	2.3	.489	3.3	.4995
.3	.118	1.4	.419	2.4	.492	3.4	.4997
.4	.155	1.5	.433	2.5	.494	3.5	.4998
.5	.191	1.6	.445	2.6	.495	3.6	.4998
.6	.226	1.7	.455	2.7	.497	3.7	.4999
.7	.258	1.8	.464	2.8	.497	3.8	.49993
.8	.288	1.9	.471	2.9	.498	3.9	.49995
.9	.316	2.0	.477	3.0	.4987	4.0	.49997
1.0	.341					5.0	.49999997

Figure 27

$$1 - (2 \times .4987) = .0026,$$

respectively. These agree with the values given in Section 10, to within rounding errors. In fact, the Central Limit Theorem is the basis of those estimates.

Example 2. In Section 10 we considered the example of throwing a coin 10,000 times. The expected number of heads that turn up is 5000, and the standard deviation is $\sqrt{10{,}000,\ \frac{1}{2}\cdot\frac{1}{2}} = 50$. We observed that the probability of a deviation of more than two standard deviations (or 100) was very unlikely. On the other hand, consider the probability of a deviation of less than .1 standard deviation. That is, of a deviation of less than five. The area from 0 to .1 under the normal curve is .040 and hence the probability of a deviation from 5000 of less than five is approximately .08. Thus, while a deviation of 100 is very unlikely, it is also very unlikely that a deviation of less than five will occur.

Example 3. The normal approximation can be used to estimate the individual probabilities $f(n, x; p)$ for large n. For example, let us estimate $f(200, 65; .3)$. The graph of the probabilities $f(200, x; .3)$ was given in Figure 26 together with the normal approximation. The desired probability is the area of the bar corresponding to $x = 65$. An inspection of the graph suggests that we should take the area under the normal curve between 64.5 and 65.5 as an estimate for this probability. In normalized units this is the area between

$$\frac{4.5}{\sqrt{200(.3)(.7)}} \quad \text{and} \quad \frac{5.5}{\sqrt{200(.3)(.7)}},$$

or between .6944 and .8487. Our table is not fine enough to find this area, but from more complete tables, or by machine computation, this area may be found to be .046 to three decimal places. The exact value to three decimal places is .045. This procedure gives us a good estimate.

If we check all of the values of $f(200, x; .3)$ we find in each case that we would make an error of at most .001 by using the normal approximation. There is unfortunately no simple way to estimate the error caused by the use of the Central Limit Theorem. The error will clearly depend upon how large n is, but it also depends upon how near p is to 0 or 1. The greatest accuracy occurs when p is near $\frac{1}{2}$.

Example 4. Suppose that a drug has been administered to a number of patients and found to be effective a fraction $\bar{p}$ of the time. Assuming an independent trials process, it is natural to take $\bar{p}$ as an estimate for the unknown probability p for success on any one trial. It is useful to have a method of estimating the reliability of this estimate. One method is the following. Let x be the number of successes for the drug given to n patients. Then by the Central Limit Theorem

$$\Pr\left[\left|\frac{x - np}{\sqrt{npq}}\right| \leq 2\right] \approx .95.$$

This is the same as saying

$$\Pr\left[\left|\frac{x/n - p}{\sqrt{pq/n}}\right| \leq 2\right] \approx .95.$$

Putting $\bar{p} = x/n$, we have

$$\Pr\left[|\bar{p} - p| \leq 2\sqrt{\frac{pq}{n}}\right] \approx .95.$$

Using the fact that $pq \leq \frac{1}{4}$ (see Exercise 12) we have

$$\Pr\left[|\bar{p} - p| \leq \frac{1}{\sqrt{n}}\right] \geq .95.$$

This says that no matter what p is, with probability $\geq .95$, the true value will not deviate from the estimate $\bar{p}$ by more than $1/\sqrt{n}$. It is customary then to say that

$$\bar{p} - \frac{1}{\sqrt{n}} \leq p \leq \bar{p} + \frac{1}{\sqrt{n}}$$

with confidence .95. The interval $\left[\bar{p} - \frac{1}{\sqrt{n}}, \bar{p} + \frac{1}{\sqrt{n}}\right]$ is called a 95 per cent *confidence interval.* Had we started with

$$\Pr\left[\left|\frac{x - np}{\sqrt{npq}}\right| \leq 3\right] \approx .99,$$

we would have obtained the 99 per cent confidence interval

$$\left[\bar{p} - \frac{3}{2\sqrt{n}}, \bar{p} + \frac{3}{2\sqrt{n}}\right].$$

For example, if in 400 trials the drug is found effective 124 times, or .31 of the times, the 95 per cent confidence interval for p is

$$[.31 - \tfrac{1}{20}, .31 + \tfrac{1}{20}] \quad \text{or} \quad [.26, .36]$$

and the 99 per cent confidence interval is

$$[.31 - \tfrac{3}{40}, .31 + \tfrac{3}{40}] \quad \text{or} \quad [.235, .385].$$

EXERCISES

1. Let x be the number of successes in n trials of an independent trials process with probability p for success. Let $x^* = \dfrac{x - np}{\sqrt{npq}}$. For large n estimate the following probabilities.

(a) $\Pr[x^* < -2.5]$. [*Ans.* .006.]

(b) $\Pr[x^* < 2.5]$.

(c) $\Pr[x^* \geq -.5]$.

(d) $\Pr[-1.5 < x^* < 1]$. [*Ans.* .774.]

2. A coin is biased in such a way that a head comes up with probability .8 on a single toss. Use the normal approximation to estimate the probability that in a million tosses there are more than 800,400 heads.

3. Plot a graph of the probabilities $f(10, x; .5)$. Plot a graph also of the normalized probabilities as in Figures 25 and 26.

4. An ordinary coin is tossed one million times. Let x be the number of heads which turn up. Estimate the following probabilities.

(a) Pr $[499{,}500 \leq x \leq 500{,}500]$.

(b) Pr $[499{,}000 \leq x \leq 501{,}000]$.

(c) Pr $[498{,}500 \leq x \leq 501{,}500]$.

[*Ans.* .682; .954; .997 (Approximate answers.).]

5. Assume that a baseball player has probability .37 of getting a hit each time he comes to bat. Find the probability of getting an average of .388 or better if he comes to bat 300 times during the season. (In 1957 Ted Williams had a batting average of .388 and Mickey Mantle had an average of .353. If we assume this difference is due to chance, we may estimate the probability of a hit as the combined average, which is about .37.) [*Ans.* .242.]

6. A true-false examination has 48 questions. Assume that the probability that a given student knows the answer to any one question is $\frac{3}{4}$. A passing score is 30 or better. Estimate the probability that the student will fail the exam.

7. In Example 3 of Section 10, assume that the school decides to admit 1296 students. Estimate the probability that they will have to have additional dormitory space. [*Ans.* Approximately .115.]

8. Peter and Paul each have 20 pennies. They agree to match pennies 400 times, keeping score but not paying until the 400 matches are over. What is the probability that one of the players will not be able to pay? Answer the same question for the case that Peter has 10 pennies and Paul has 30.

9. In tossing a coin 100 times, the probability of getting 50 heads is, to three decimal places, .080. Estimate this same probability using the Central Limit Theorem. [*Ans.* .080.]

10. A standard medicine has been found to be effective in 80 per cent of the cases where it is used. A new medicine for the same purpose is found to be effective in 90 of the first 100 patients on which the medicine is used. Could this be taken as good evidence that the new medication is better than the old?

11. In the Weldon dice experiment, 12 dice were thrown 26,306 times and the appearance of a 5 or a 6 was considered to be a success. The mean number of successes observed was, to four decimal places, 4.0524. Is this result significantly different from the expected average number of 4? [*Ans.* Yes.]

12. Prove that $pq \leq \frac{1}{4}$. [*Hint:* write $p = \frac{1}{2} + x$.]

13. Suppose that out of 1000 persons interviewed 650 said that they would vote for Mr. Big for mayor. Construct the 99 per cent confidence interval for p, the proportion in the city that would vote for Mr. Big.

14. Opinion pollsters in election years usually poll about 3000 voters. Suppose that in an election year 51 per cent favor candidate A and 49 per cent favor candidate B. Construct 95 per cent confidence limits for candidate A winning. [*Ans.* .492, .528.]

15. In an experiment with independent trials we are going to estimate p by the fraction $\bar{p}$ of successes. We wish our estimate to be within .02 of the correct value with probability .95. Show that 2500 observations will always suffice. Show that if it is known that p is approximately .1, then 900 observations would be sufficient.

16. An experimenter has an independent trials process and he has a hypothesis that the true value of p is p_0. He decides to carry out a number of trials, and from the observed $\bar{p}$ calculate the 95 per cent confidence interval for p. He will reject p_0 if it does not fall within these limits. What is the probability that he will reject p_0 when in fact it is correct? Should he accept p_0 if it does fall within the confidence interval?

17. A coin is tossed 100 times and turns up heads 61 times. Using the method of Exercise 16 test the hypothesis that the coin is a fair coin. [*Ans.* Reject.]

18. Two railroads are competing for the passenger traffic of 1000 passengers by operating similar trains at the same hour. If a given passenger is equally likely to choose one train as the other, how many seats should the railroad provide if it wants to be sure that its seating capacity is sufficient in 99 out of 100 cases? [*Ans.* 537.]

*15. GAMBLER'S RUIN

In this section we will study a particular Markov chain, which is interesting in itself and has far-reaching applications. Its name, "gambler's ruin," derives from one of its many applications. In the text we will describe the chain from the gambling point of view, but in the exercises we will present several other applications.

Let us suppose that you are gambling against a professional gambler, or gambling house. You have selected a specific game to play, on which you have probability p of winning. The gambler has made sure that the game is favorable to him, so that $p < \frac{1}{2}$. However, in most situations p will be close to $\frac{1}{2}$. (The cases $p = \frac{1}{2}$ and $p > \frac{1}{2}$ are considered in the exercises.)

At the start of the game you have A dollars, and the gambler has B dollars. You bet \$1 on each game, and play until one of you is ruined.

What is the probability that you will be ruined? Of course, the answer depends on the exact values of p, A, and B. We will develop a formula for the ruin-probability in terms of these three given numbers.

First we will set the problem up as a Markov chain. Let $N = A + B$, the total amount of money in the game. As states for the chain we choose the numbers $0, 1, 2, \ldots, N$. At any one moment the position of the chain is the amount of money *you* have. The initial position is shown in Figure 28.

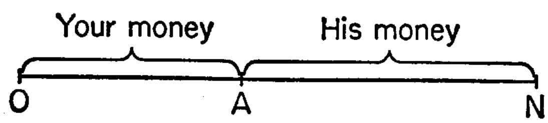


Figure 28

If you win a game, your money increases by \$1, and the gambler's fortune decreases by \$1. Thus the new position is one state to the right of the previous one. If you lose a game, the chain moves one step to the left. Thus at any step there is probability p of moving one step to the right, and probability $q = 1 - p$ of one step to the left. Since the probabilities for the next position are determined by the present position, it is a Markov chain.

If the chain reaches 0 or N, we stop. When 0 is reached, you are ruined. When N is reached, you have all the money, and you have ruined the gambler. We will be interested in the probability of *your* ruin, i.e., the probability of reaching 0.

Let us suppose that p and N are fixed. We actually want the probability of ruin when we start at A. However, it turns out to be easier to solve a problem that appears much harder: Find the ruin-probability for every possible starting position. For this reason we introduce the notation x_i, to stand for the probability of your ruin if you start in position i (that is, if you have i dollars).

Let us first solve the problem for the case $N = 5$. We have the unknowns x_0, x_1, x_2, x_3, x_4, and x_5. Suppose that we start at position 2. The chain moves to 3, with probability p, or to 1, with probability q. Thus

$$\Pr[\text{ruin}|\text{start at } 2] = \Pr[\text{ruin}|\text{start at } 3]\cdot p + \Pr[\text{ruin}|\text{start at } 1]\cdot q,$$

using the conditional probability formula, with a set of two alternatives. But once it has reached state 3, a Markov chain behaves just as if it had been started there. Thus

$$\Pr[\text{ruin}|\text{start at } 3] = x_3.$$

And, similarly,

$$\Pr\,[\text{ruin}|\text{start at } 1] = x_1.$$

We obtain the key relation

$$x_2 = px_3 + qx_1.$$

We can modify this as follows:

$$\begin{aligned}(p+q)x_2 &= px_3 + qx_1,\\ p(x_2 - x_3) &= q(x_1 - x_2),\\ x_1 - x_2 &= r(x_2 - x_3),\end{aligned}$$

where $r = p/q$, and hence $r < 1$. When we write such an equation for each of the four "ordinary" positions, we obtain

$$\begin{aligned}x_0 - x_1 &= r(x_1 - x_2)\\ x_1 - x_2 &= r(x_2 - x_3)\\ x_2 - x_3 &= r(x_3 - x_4)\\ x_3 - x_4 &= r(x_4 - x_5).\end{aligned} \tag{1}$$

We must still consider the two extreme positions. Suppose that the chain reaches 0. Then you are ruined, hence the probability of your ruin is 1. While if the chain reaches $N = 5$, the gambler drops out of the game, and you can't be ruined. Thus

$$x_0 = 1, \qquad x_5 = 0. \tag{2}$$

If we substitute the value of x_5 in the last equation of (1), we have $x_3 - x_4 = rx_4$. This in turn may be substituted in the previous equation, etc. We thus have the simpler equations

$$\begin{aligned}x_4 &= 1\cdot x_4\\ x_3 - x_4 &= rx_4\\ x_2 - x_3 &= r^2x_4\\ x_1 - x_2 &= r^3x_4\\ x_0 - x_1 &= r^4x_4.\end{aligned} \tag{3}$$

Let us add all the equations. We obtain

$$x_0 = (1 + r + r^2 + r^3 + r^4)x_4.$$

From (2) we have that $x_0 = 1$. We also use the simple identity

$$(1 - r)(1 + r + r^2 + r^3 + r^4) = 1 - r^5.$$

And then we solve for x_4:

$$x_4 = \frac{1 - r}{1 - r^5}.$$

If we add the first two equations in (3), we have that $x_3 = (1 + r)x_4$. Similarly, adding the first three equations, we solve for x_2, and adding the first four equations we obtain x_1. We now have our entire solution,

$$(4) \qquad x_1 = \frac{1 - r^4}{1 - r^5}, \quad x_2 = \frac{1 - r^3}{1 - r^5}, \quad x_3 = \frac{1 - r^2}{1 - r^5}, \quad x_4 = \frac{1 - r}{1 - r^5}.$$

The same method will work for any value of N. And it is easy to guess from (4) what the general solution looks like. If we want x_A, the answer is a fraction like those in (4). In the denominator the exponent of r is always N. In the numerator the exponent is $N - A$, or B. Thus the ruin-probability is

$$(5) \qquad x_A = \frac{1 - r^B}{1 - r^N}.$$

We recall that A is the amount of money you have, B is the gambler's stake, $N = A + B$, p is your probability of winning a game, and $r = p/(1 - p)$.

In Figure 29 we show some typical values of the ruin-probability. Some of these are quite startling. If the probability of p is as low as .45 (odds against you on each game 11:9) and the gambler has 20 dollars to put up, you are almost sure to be ruined. Even in a nearly fair game, say $p = .495$, with each of you having \$50 to start with, there is a .731 chance for your ruin.

It is worth examining the ruin-probability formula, (5), more closely. Since the denominator is always less than 1, your probability of ruin is at least $1 - r^B$. This estimate does not depend on how much money you have, only on p and B. Since r is less than 1, by making B large enough, we can make r^B practically 0, and hence make it almost certain that you will be ruined.

Suppose, for example, that a gambler wants to have probability .999 of ruining you. (You can hardly call him a gambler under those circumstances!) He must make sure that $r^B < .001$. For example, if $p = .495$, the gambler needs \$346 to have probability .999 of ruining you, even if you are a millionaire. If $p = .48$, he needs only \$87. And even for the almost fair game with $p = .499$, \$1727 will suffice.

There are two ways that gamblers achieve this goal. Small gambling houses will fix the odds quite a bit in their favor, making r much less than 1. Then even a relatively small bank of B dollars suffices to assure them of winning. Larger houses, with B quite sizable, can afford to let you play nearly fair games.

Ruin-probabilities for $p = .45, .48, .49, .495$.

$p = .45$

A \ B	1	5	10	20	50
1	.550	.905	.973	.997	1
5	.260	.732	.910	.988	1
10	.204	.666	.881	.984	1
20	.185	.638	.868	.982	1
50	.182	.633	.866	.982	1

$p = .48$

A \ B	1	5	10	20	50
1	.520	.865	.941	.981	.999
5	.202	.599	.788	.923	.994
10	.131	.472	.690	.878	.990
20	.095	.381	.606	.832	.985
50	.078	.334	.555	.801	.982

$p = .49$

A \ B	1	5	10	20	50
1	.510	.850	.926	.969	.994
5	.184	.550	.731	.871	.972
10	.110	.402	.599	.788	.951
20	.069	.287	.472	.690	.921
50	.045	.204	.363	.586	.881

$p = .495$

A \ B	1	5	10	20	50
1	.505	.842	.918	.961	.989
5	.175	.525	.699	.838	.948
10	.100	.367	.550	.731	.905
20	.058	.242	.402	.599	.839
50	.031	.143	.259	.438	.731

Figure 29

EXERCISES

1. An urn has nine white balls and 11 black balls. A ball is drawn, and replaced. If it is white, you win five cents, if black, you lose five cents. You have a dollar to gamble with, and your opponent has fifty cents. If you keep on playing till one of you loses all his money, what is the probability that you will lose your dollar? [*Ans.* .868.]

2. Suppose that you are shooting craps, and you always hold the dice. You have \$20, your opponent has \$10, and \$1 is bet on each game; estimate your probability of ruin.

3. Two government agencies, A and B, are competing for the same task. A has 50 positions, and B has 20. Each year one position is taken away from one of the agencies, and given to the other. If 52 per cent of the time the shift is from A to B, what do you predict for the future of the two agencies?
[*Ans.* One agency will be abolished. B survives with probability .8, A with probability .2.]

4. What is the approximate value of x_A if you are rich, and the gambler starts with \$1?

5. Consider a simple model for evolution. On a small island there is room for 1000 members of a certain species. One year a favorable mutant appears. We assume that in each subsequent generation either the mutants take one place from the regular members of the species, with probability .6, or the reverse happens. Thus, for example, the mutation disappears in the very first generation with probability .4. What is the probability that the mutants eventually take over? [*Hint:* See Exercise 4.] [*Ans.* $\frac{1}{3}$.]

6. Verify that the proof of the text is still correct when $p > \frac{1}{2}$. Interpret formula (5) for this case.

7. Show that if $p > \frac{1}{2}$, and both parties have a substantial amount of money, your probability of ruin is approximately $1/r^A$.

8. Modify the proof in the text to apply to the case $p = \frac{1}{2}$. What is the probability of your ruin? [*Ans.* B/N.]

9. You are matching pennies. You have 25 pennies to start with, and your opponent has 35. What is the probability that you will win all his pennies?

10. Mr. Jones lives on a short street, about 100 steps long. At one end of the street is his home, at the other a lake, and in the middle a bar. One evening he leaves the bar in a state of intoxication, and starts to walk at random. What is the probability that he will fall into the lake if

(a) He is just as likely to take a step to the right as to the left?

[*Ans.* $\frac{1}{2}$.]

(b) If he has probability .51 of taking a step towards his home?

[*Ans.* .119.]

11. You are in the following hopeless situation: You are playing a game in which you have only $\frac{1}{3}$ chance of winning. You have \$1, and your opponent has \$7. What is the probability of your winning all his money if

(a) You bet \$1 each time? [*Ans.* $\frac{1}{255}$.]

(b) You bet all your money each time? [*Ans.* $\frac{1}{27}$.]

12. Repeat Exercise 11 for the case of a fair game, where you have probability $\frac{1}{2}$ of winning.

13. Modify the proof in the text to compute y_i, the probability of reaching state $N = 5$.

14. Verify, in Exercise 13, that $x_i + y_i = 1$ for every state. Interpret.

Note: Exercises 15–18 deal with the following ruin problem: A and B play a game in which A has probability $\frac{2}{3}$ of winning. They keep playing until either A has won six times or B has won three times.

15. Set up the process as a Markov chain whose states are (a, b), where a is the number of times A won, and b the number of B wins.

16. For each state compute the probability of A winning from that position. [*Hint:* Work from higher a- and b-values to lower ones.]

17. What is the probability that A reaches his goal first? [*Ans.* $\frac{1024}{2187}$.]

18. Suppose that payments are made as follows: If A wins six games, he receives \$1, if B wins three games then A pays \$1. What is the expected value of the payment, to the nearest penny?

SUGGESTED READING

Cramer, Harold, *The Elements of Probability Theory*, Wiley, New York, 1955, Part I.

Feller, W., *An Introduction to Probability Theory and its Applications*, Wiley, New York, 1950.

Goldberg, S., *Probability: An Introduction*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1960.

Mosteller, F., *Fifty Challenging Problems in Probability with Solutions*, Addison-Wesley, Reading, Mass., 1965, 88 pp.

Neyman, J., *First Course in Probability and Statistics*, Holt, New York, 1950.

Parzen, E., *Modern Probability Theory and Its Applications*, Wiley, New York, 1960.

Whitworth, W. A., *Choice and Chance, with 1000 Exercises*, Stechert, New York, 1934.