Compound Statements

7

1

1 PURPOSE OF THE THEORY

A *statement* is a verbal or written assertion which can be determined to be either true or false. In the English language such assertions are made by means of declarative sentences. For example, "It is snowing" and "I made a mistake in signing up for this course" are statements.

The reader should note that questions such as "Who killed cock robin?" or exhortations such as "Tread softly but carry a big stick!" are *not* statements in our sense since they do not have a truth value.

The two statements quoted in the first paragraph above are *simple statements*. A combination of two or more simple statements is a *compound statement*. For example, "It is snowing, and I wish that I were out of doors, but I made the mistake of signing up for this course" is a compound statement.

It might seem natural that one should make a study of simple statements first, and then proceed to the study of compound ones. However, the reverse order has proved to be more useful. Because of the tremendous variety of simple statements, the theory of such statements is very complex. It has been found in mathematics that it is often fruitful to assume for the moment that a difficult problem has been solved and then to go on to the next problem. Therefore we shall proceed as if we knew all about simple statements and study only the way they are compounded. The latter is a relatively easy problem.

While the first systematic treatment of such problems is found in the writings of Aristotle, mathematical methods were first employed by George Boole more than a hundred years ago. The more polished techniques now available are the product of twentieth-century mathematical logicians.

The fundamental property of any statement is that it is either true or false (and that it cannot be both true and false). Naturally, we are interested in finding out which is the case. For a compound statement it is sufficient to know which of its components are true, since the truth values (i.e., the truth or falsity) of the components determine in a way to be described later the truth value of the compound.

Our problem then is twofold: (1) In how many different ways can statements be compounded? (2) How do we determine the truth value of a compound statement given the truth values of its components?

Let us consider ordinary mathematical statements. In any mathematical formula we find three kinds of symbols: *constants, variables,* and *auxiliary symbols.* For example, in the formula $(x + y)^2$ the plus sign and the exponent are constants, the letters x and y are variables, and the parentheses are auxiliary symbols. Constants are symbols whose meanings in a given context are fixed. Thus in the formula given above, the plus sign indicates that we are to form the sum of the two numbers x and y, while the exponent 2 indicates that we are to multiply $(x + y)$ by itself. Variables always stand for entities of a given kind, but they allow us to leave open just which particular entity we have in mind. In our example above the letters x and y stand for unspecified numbers. Auxiliary symbols function somewhat like punctuation marks. Thus if we omit the parentheses in the expression above we obtain the formula $x + y^2$, which has quite a different meaning than the formula $(x + y)^2$.

In this chapter we shall use variables of only one kind. We indicate these variables by the letters p, q, r, etc., which will stand for unspecified statements. These statements frequently will be simple statements but may also be compound. In any case we know that, since each variable stands for a statement, it has an (unknown) truth value.

The constants that we shall use will stand for certain connectives used in the compounding of statements. We shall have one symbol for forming the negation of a statement and several symbols for combining two statements. It will not be necessary to introduce symbols for the compounding of three or more statements, since we can show that the same combination can also be formed by compounding them two at a time. In practice only a small number of basic constants are used and the others are defined in terms of these. It is even possible to use only a single connective! (See Section 2, Exercises 6–8.)

The auxiliary symbols that we shall use are, for the most part, the same ones used in elementary algebra. Any usage of a different symbol will be explained when it first occurs.

EXAMPLES As examples of simple statements, let us take "The weather is nice" and "It is very hot." We will let p stand for the former and q for the latter.

Suppose we wish to make the compound statement that both are true. "The weather is nice *and* it is very hot." We shall symbolize this statement by $p \wedge q$. The symbol $\wedge$, which can be read "and," is our first connective.

In place of the strong assertion above we might want to make the weak (cautious) assertion that one or the other of the statements is true. "The weather is nice *or* it is very hot." We symbolize this assertion by $p \vee q$. The symbol $\vee$, which can be read "or," is the second connective which we shall use.

Suppose we believed that one of the statements above was false, for example, "It is *not* very hot." Symbolically we would write $\sim q$. Our third connective is then $\sim$, which can be read "not."

More complex compound statements can now be made. For example, $p \wedge \sim q$ stands for "The weather is nice *and* it is *not* very hot."

EXERCISES

1. The following are compound sentences or may be so interpreted. Find their simple components.
   - **(a)** It is quite hot and I would like to go swimming.
   - **(b)** It is raining or it is very humid.
   - **(c)** Jones did not have time to go, but Smith went instead.
     [*Ans.* "Jones did have time to go"; "Smith went instead."]
   - **(d)** The murderer is Jones or Smith.
   - **(e)** Jack and Jill went up the hill.
   - **(f)** Either Bill has not arrived or he left before we got here.
   - **(g)** Neither the post office nor the bank is open today.

2. In Exercise 1 assign letters to the various components, and write the statements in symbolic form. [*Ans.* (c) $\sim p \wedge q$.]

3. Write the following statements in symbolic form.
   - **(a)** Fred likes George. (Statement p.)
   - **(b)** George likes Fred. (Statement q.)
   - **(c)** Fred and George like each other.
   - **(d)** Fred and George dislike each other.
   - **(e)** Fred likes George, but George does not reciprocate.
   - **(f)** George is liked by Fred, but Fred is disliked by George.
   - **(g)** Neither Fred nor George dislikes the other.
   - **(h)** It is not true that Fred and George dislike each other.

4. Assume that Fred dislikes George and George likes Fred. Which of the eight statements in Exercise 3 are true?

5. Write the following statements in symbolic form, letting p be "Fred is smart" and q be "George is smart."
   - **(a)** Fred is smart and George is not smart.
   - **(b)** George is smart or George is not smart.
   - **(c)** Neither Fred nor George is smart.
   - **(d)** Either Fred is smart or George is not smart.
   - **(e)** Fred is not smart, but George is smart.
   - **(f)** It is not true that both Fred and George are not smart.

6. If Fred and George are both smart, which of the six compound statements in Exercise 5 are true?

7. For each statement in Exercise 5 give a condition under which it is false, if it is possible to do so. [*Ans.* (a) George is smart.]
8. Let p be "Stock prices are high" and q be "Stocks are rising." Give a verbal translation for each of the following.
   (a) $p \vee q$.
   (b) $p \wedge q$.
   (c) $\sim p \vee \sim q$.
   (d) $\sim(p \wedge q)$.
   (e) $\sim(\sim p \vee q)$.
   (f) $\sim(\sim p \wedge \sim q)$.
9. Using your answers to Exercise 8, parts (d), (e), and (f), find simpler symbolic statements expressing the same idea. [*Ans.* (d) $\sim p \vee \sim q$.]
10. Let p be "I will win" and q be "You will lose." Using the methods of Exercises 8 and 9, find a simpler statement for

$$[\sim\sim q] \wedge \sim[\sim p \vee \sim q].$$

2 THE MOST COMMON CONNECTIVES

The truth value of a compound statement is determined by the truth values of its components. When discussing a connective we shall want to know just how the truth of a compound statement made from this connective depends upon the truth of its components. A very convenient way of tabulating this dependency is by means of a *truth table.*

Let us consider the compound $p \wedge q$. Statement p could be either true or false and so could statement q. Thus there are four possible pairs of truth values for these statements and we want to know in each case whether or not the statement $p \wedge q$ is true. The answer is straightforward: If p and q are both true, then $p \wedge q$ is true, and otherwise $p \wedge q$ is false. This seems reasonable since the assertion $p \wedge q$ says no more and no less than that p and q are both true.

Figure 1 gives the truth table which defines $p \wedge q$, the *conjunction* of p and q. The truth table contains all the information that we need to know about the connective $\wedge$, namely it tells us the truth value of the conjunction of two statements given the truth values of each of the statements.

We next look at the compound statement $p \vee q$, the *disjunction* of p and q. Here the assertion is that one or the other of these statements is true. Clearly, if one statement is true and the other false, then the disjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1

p	q	$p \vee q$
T	T	?
T	F	T
F	T	T
F	F	F

Figure 2

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 3

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

Figure 4

is true, while if both statements are false, then the disjunction is certainly false. Thus we can fill in the last three rows of the truth table for disjunction (see Figure 2).

Observe that one possibility is left unsettled, namely, what happens if both components are true? Here we observe that the everyday usage of "or" is ambiguous. Does "or" mean "one or the other or both" or does it mean "one or the other but not both"?

Let us seek the answer in examples. The sentence "This summer I will visit France or Italy" allows for the possibility that the speaker may visit both countries. However, the sentence "I will go to Dartmouth or to Princeton" indicates that only one of these schools will be chosen. "I will buy a TV set or a phonograph next year" could be used in either sense; the speaker may mean that he is trying to make up his mind which one of the two to buy, but it could also mean that he will buy *at least one* of these—possibly both. We see that sometimes the context makes the meaning clear, but not always.

A mathematician would never waste his time on a dispute as to which usage "should" be called the disjunction of two statements. Rather he recognizes two perfectly good usages, and calls one the *inclusive disjunction* (p or q or both) and the other the *exclusive disjunction* (p or q but not both). The symbol $\vee$ will be used for inclusive disjunction, and the symbol $\veebar$ will be used for exclusive disjunction. The truth tables for each of these are found in Figures 3 and 4. Unless we state otherwise, our disjunctions will be inclusive disjunctions.

p	$\sim p$
T	F
F	T

Figure 5

The last connective which we shall discuss in this section is *negation*. If p is a statement, the symbol $\sim p$, called the negation of p, asserts that p is false. Hence $\sim p$ is true when p is false, and false when p is true. The truth table for negation is shown in Figure 5.

Besides using these basic connectives singly to form compound statements, several can be used to form a more complicated compound statement, in much the same way that complicated algebraic expressions can be formed by means of the basic arithmetic operations. For example, $\sim(p \wedge q)$, $p \wedge \sim p$, and $(p \vee q) \vee \sim p$ are all compound statements. They are to be read "from the inside out" in the same way that algebraic expressions are, namely, quantities inside the innermost parentheses are first grouped together, then these parentheses are grouped together, etc. Each compound statement has a truth table which can be constructed in a routine way. The following examples show how to construct truth tables.

EXAMPLE 1 Consider the compound statement $p \vee \sim q$. We begin the construction of its truth table by writing in the first two columns the four possible pairs of truth values for the statements p and q. Then we write the proposition in question, leaving plenty of space between symbols so that we can fill in columns below. Next we copy the truth values of p and q in the columns below their occurrences in the proposition. This completes step 1 (see Figure 6).

p	q	p	$\vee$	$\sim q$
T	T	T		T
T	F	T		F
F	T	F		T
F	F	F		F
Step No.		1		1

Figure 6

Next we treat the innermost compound, the negation of the variable q, completing step 2 (see Figure 7).

p	q	p	$\vee$	$\sim$	q
T	T	T		F	T
T	F	T		T	F
F	T	F		F	T
F	F	F		T	F
Step No.		1		2	1

Figure 7

Finally we fill in the column under the disjunction symbol, which gives us the truth value of the compound statement for various truth values of its variables. To indicate this we place two parallel lines on each side of the final column, completing step 3 as in Figure 8.

p	q	p	$\vee$	$\sim$	q
T	T	T	T	F	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	F	T	T	F
Step No.		1	3	2	1

Figure 8

The next two examples show truth tables of more complicated compounds worked out in the same manner. There are only two basic rules which the student must remember when working these: first, work from the "inside out"; second, the truth values of the compound statement are found in the last column filled in during this procedure.

EXAMPLE 2 The truth table for the statement $(p \vee \sim q) \wedge \sim p$ together with the numbers indicating the order in which the columns are filled in appears in Figure 9.

p	q	$(p$	$\vee$	$\sim$	$q)$	$\wedge$	$\sim$	p
T	T	T	T	F	T	F	F	T
T	F	T	T	T	F	F	F	T
F	T	F	F	F	T	F	T	F
F	F	F	T	T	F	T	T	F
Step No.		1	3	2	1	4	2	1

Figure 9

Two compound statements having the same variables are said to be *equivalent* if and only if they have exactly the same truth table. It is always permissible, and sometimes desirable, to replace a given statement by an equivalent one.

EXAMPLE 3 Augustus DeMorgan was a well-known English mathematician and logician of the nineteenth century and was the first person to state two important equivalences, or "laws." The first of DeMorgan's laws asserts that the statements $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are equivalent. The truth tables in Figure 10 show that this is indeed true. The reader will notice that we wrote

p	q	$\sim$	$(p \wedge q)$	$\sim p$	$\vee$	$\sim q$
T	T	F	T	F	F	F
T	F	T	F	F	T	T
F	T	T	F	T	T	F
F	F	T	F	T	T	T
Step No.		2	1	1	2	1

Figure 10

the truth tables for $p \wedge q$, $\sim p$, and $\sim q$ directly on the first step to shorten the work. Notice that the two columns marked on step 2 are identical, so that $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are equivalent statements.

Let us give an interpretation of the equivalence just mentioned. Consider "It is false that business is good and stocks are high." The equivalent statement derived from DeMorgan's laws is: "Either business is bad or stocks are low." Intuitively the equivalence of these two compound statements is clear.

The other of DeMorgan's laws is that the statements $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are equivalent. This law is discussed in Exercise 12.

EXAMPLE 4 The truth table for the statement $\sim[(p \wedge q) \vee (\sim p \wedge \sim q)]$ together with the numbers indicating the order in which the columns are filled appears in Figure 10a. We note that the compound statement has the same truth table as $p \veebar q$. These two statements are therefore equivalent.

p	q	$\sim$	$[(p$	$\wedge$	$q)$	$\vee$	$(\sim$	p	$\wedge$	$\sim$	$q)]$
T	T	F	T	T	T	T	F	T	F	F	T
T	F	T	T	F	F	F	F	T	F	T	F
F	T	T	F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	T	T	F	T	T	F
Step No.		5	1	2	1	4	2	1	3	2	1

Figure 10a

To illustrate this equivalence, consider the statement "I will attend either Dartmouth or Princeton, but not both." This is equivalent to the *denial* of the statement "I will either attend both Dartmouth and Princeton [symbolized by $(p \wedge q)$] or I will attend neither Dartmouth nor Princeton [symbolized by $(\sim p \wedge \sim q)$]."

EXERCISES

1. Construct a truth table for each of the following:
   (a) $q \vee \sim q$. [*Ans.* TT.]
   (b) $(p \wedge q) \vee \sim q$.
   (c) $\sim p \vee q$.
   (d) $[\sim(p \vee q) \wedge (\sim p \vee \sim q)]$. [*Ans*, FFFT.]
2. Using only $\sim$, $\vee$, and $\wedge$, give a compound statement which symbolically states "p or q but not both."
3. Construct a truth table for your answer to Exercise 2, and compare it with Figure 4.
4. Let p stand for "Smith went skiing," and let q stand for "Smith broke his leg." Translate into symbolic form the statement "It is not the case that either Smith did not go skiing or Smith did not break his leg." Construct a truth table for this symbolic statement.
5. Find a simpler verbal statement about Smith whose symbolic form has the same truth table as the one in Exercise 4.
6. Let $p \downarrow q$ express that "both p and q are false." Write a symbolic expression for $p \downarrow q$ using $\sim$ and $\wedge$. Write a truth table for $p \downarrow q$.
7. Write a truth table for $p \downarrow p$.
8. Write a truth table for $(p \downarrow p) \downarrow (q \downarrow q)$.
9. Construct a truth table for each of the following:
   (a) $(\sim p \vee q) \wedge (p \vee \sim q)$. [*Ans.* TFFT.]
   (b) $\sim(p \downarrow q)$.
   (c) $\sim(p \veebar \sim q)$.
   (d) $(p \wedge q) \vee (q \wedge p)$.

10. Construct symbolic statements, using only $\sim$, $\vee$, and $\wedge$, which have the following truth tables (a) and (b), respectively:

p	q	(a)	(b)
T	T	T	F
T	F	T	T
F	T	F	T
F	F	T	F

11. Using only $\sim$ and $\veebar$, construct a compound statement having the same truth table as:
    (a) $(p \wedge q) \vee (\sim(p \vee q))$. [*Ans.* $\sim(p \veebar q)$.]
    (b) $p \vee q$. [*Ans.* Impossible.]
    (c) $\sim p \vee q$.

12. Use truth tables to show that $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are equivalent.

3 OTHER CONNECTIVES

Suppose we did not wish to make an outright assertion but rather an assertion containing a condition. As examples, consider the following sentences. "If the weather is nice, I will take a walk." "If the following statement is true, then I can prove the theorem." "If the cost of living continues to rise, then the government will impose rigid curbs." Each of these statements is of the form "*if p then q.*" The *conditional* is then a new connective which is symbolized by the arrow $\rightarrow$.

Of course the precise definition of this new connective must be made by means of a truth table. If both p and q are true, then to make logic coincide with ordinary usage $p \rightarrow q$ is certainly true, and if p is true and q false, then $p \rightarrow q$ is certainly false for the same reason. Thus the first two lines of the truth table can easily be filled in—see Figure 11a. Suppose now that p is false; how shall we fill in the last two lines of the truth table in Figure 11a? At first thought one might suppose that it would be best to leave it completely undefined. However, to do so would violate our basic principle that a statement is either true or false.

Therefore we make the completely arbitrary decision that the conditional, $p \rightarrow q$, is *true* whenever p is false, regardless of the truth value of q. This

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	?
F	F	?

Figure 11a

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 11b

decision enables us to complete the truth table for the conditional and it is given in Figure 11b. A glance at this truth table shows that the conditional $p \to q$ is considered false only if p is true and q is false. If we wished, we might rationalize the arbitrary decision made above by saying that if statement p happens to be false, then we give the conditional $p \to q$ the "benefit of the doubt" and consider it true. (For another reason, see Exercise 1.)

In everyday conversation it is customary to combine simple statements only if they are somehow related. Thus we might say "It is raining today and I will take an umbrella," but we would not say "I read a good book and I will take an umbrella." However, the rather ill-defined concept of relatedness is difficult to enforce. Concepts related to each other in one person's mind need not be related in another's. In our study of compound statements no requirement of relatedness is imposed on two statements in order that they be compounded by any of the connectives. This freedom sometimes produces strange results in the use of the conditional. For example, according to the truth table in Figure 11b, the statement "If $2 \times 2 = 5$, then black is white" is true, while the statement "If $2 \times 2 = 4$, then cows are monkeys" is false. Since we use the "if . . . then . . ." form usually only when there is a causal connection between the two statements, we might be tempted to label both of the above statements as nonsense. At this point it is important to remember that no such causal connection is intended in the usage of $\to$; the meaning of the conditional is contained in Figure 11b and nothing more is intended. This point will be discussed again in Section 6 in connection with implication.

Closely connected to the conditional connective is the *biconditional* statement, $p \leftrightarrow q$, which may be read "*p if and only if q.*" The biconditional statement asserts that if p is true, then q is true, and if p is false, then q is false. Hence the biconditional is true in these cases and false in the others, so that its truth table can be filled in as in Figure 12.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 12

The biconditional is the last of the five connectives which we shall use in this chapter. The table below gives a summary of them together with the numbers of the figures giving their truth tables. Remember that the complete definition of each of these connectives is given by its truth table. The examples at the top of the next page show the use of the two new connectives.

Name	Symbol	Translated as	Truth Table
Conjunction	$\wedge$	"and"	Figure 1
Disjunction (inclusive)	$\vee$	"or"	Figure 3
Negation	$\sim$	"not"	Figure 5
Conditional	$\rightarrow$	"if . . . then . . ."	Figure 11b
Biconditional	$\leftrightarrow$	". . . if and only if . . ."	Figure 12

EXAMPLE 1 In Figures 13 and 14 the truth tables of two statements are worked out following the procedure of Section 2.

p	q	p	$\rightarrow$	$(p$	$\vee$	$q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F
Step No.		1	3	1	2	1

Figure 13

p	q	$\sim$	p	$\leftrightarrow$	$(p$	$\rightarrow$	$\sim$	$q)$
T	T	F	T	T	T	F	F	T
T	F	F	T	F	T	T	T	F
F	T	T	F	T	F	T	F	T
F	F	T	F	T	F	T	T	F
Step No.		2	1	4	1	3	2	1

Figure 14

EXAMPLE 2 It is also possible to form compound statements from three or more simple statements. The next example is a compound formed from three simple

p	q	r	$[p$	$\rightarrow$	$(q$	$\vee$	$r)]$	$\wedge$	$\sim$	$[p$	$\leftrightarrow$	$\sim$	$r]$
T	T	T	T	T	T	T	T	T	T	T	F	F	T
T	T	F	T	T	T	T	F	F	F	T	T	T	F
T	F	T	T	T	F	T	T	T	T	T	F	F	T
T	F	F	T	F	F	F	F	F	F	T	T	T	F
F	T	T	F	T	T	T	T	F	F	F	T	F	T
F	T	F	F	T	T	T	F	T	T	F	F	T	F
F	F	T	F	T	F	T	T	F	F	F	T	F	T
F	F	F	F	T	F	F	F	T	T	F	F	T	F
Step No.			1	3	1	2	1	5	4	1	3	2	1

Figure 15

statements p, q, and r. Notice that there will be a total of eight possible triples of truth values for these three statements so that the truth table for our compound will have eight rows as shown in Figure 15.

EXAMPLE 3 It is interesting to consider statements that are equivalent to the conditional $p \to q$. In Exercise 14 you will be asked to show that the following statements have the same truth table as $p \to q$:

$$\sim p \vee q, \qquad \sim(p \wedge \sim q), \qquad \sim q \to \sim p.$$

It follows that the following English statements are equivalent:

If I win a prize then I must have bought a lottery ticket.
Either I didn't win or I bought a lottery ticket.
It is impossible to win a prize without buying a lottery ticket.
If I did not buy a lottery ticket then I won't win a prize.

Exercise 15 considers statements that are equivalent to the biconditional $p \leftrightarrow q$.

EXERCISES

1. One way of filling in the question-marked positions in Figure 11a is given in Figure 11b. There are three other possible ways.
   - **(a)** Write the other three truth tables.
   - **(b)** Show that each of these truth tables has an interpretation in terms of the connectives now available to us.
   - **(c)** Show that the choice of Figure 11b is the only one possible so that $(p \wedge q) \to q$ is always true.
2. Construct truth tables for each of the following:
   - **(a)** $(\sim p \vee q) \to r$. [*Ans.* TFTTTFTF.]
   - **(b)** $(p \wedge q) \to (p \vee q)$.
   - **(c)** $[(p \vee q) \wedge (p \vee r)] \to p$. [*Ans.* TTTTFTTT.]
   - **(d)** $\sim(p \wedge q) \wedge \sim r$.
   - **(e)** $(p \wedge (p \to q)) \to q$. [*Ans.* TTTT.]
   - **(f)** $\sim[(p \wedge q) \to r] \leftrightarrow [\sim(p \to r) \vee \sim(q \to r)]$.
3. The truth table for a statement compounded from two simple statements has four rows, and the truth table for a statement compounded from three simple statements has eight rows. How many rows would the truth table have for a statement compounded from four simple statements? From five? From n?
4. Let p stand for "He ate spinach," q stand for "He ate dessert," and r stand for "He read a logic book." Find a symbolic form for each of the following statements, and construct its truth table.
   - **(a)** If he did not eat spinach, then he did not eat dessert.
   - **(b)** He ate spinach but did not read a logic book.

(c) If he read a logic book, then either he ate dessert or did not eat spinach.
(d) He ate spinach if and only if he ate dessert and read a logic book.

5. Construct a truth table for each of the following:
   (a) $(p \to q) \lor \sim p$.
   (b) $((p \to q) \to q) \to q$. [*Ans.* TFTT.]
   (c) $(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow (p \leftrightarrow q))$.
   (d) $(\sim p \lor q) \leftrightarrow (\sim q \lor p)$. [*Ans.* TFFT.]
6. Write truth tables for $q \lor p, q \land p, q \to p, q \leftrightarrow p$. Compare these with the truth tables in Figures 3, 1, 11b, and 12, respectively. When is it possible to interchange variables in a statement and get an equivalent statement?
7. Construct a truth table for $[((p \lor r) \to q) \land q] \to (p \lor r)$.
8. Find a simpler statement having the same truth table as the one found in Exercise 7.
9. Let p be "She will graduate," and let q be "She will find a job." Put each of the following into symbolic form, and construct the truth table for each symbolic statement.
   (a) If she graduates, then she will find a job.
   (b) If she graduates, then she will find a job, and if she finds a job, then she will graduate.
   (c) If she does not graduate, then she will not find a job.
   (d) Either she will graduate and find a job, or, if she does not graduate, then she will not find a job.
   (e) It is not the case that if she does not find a job then she will not graduate.
10. Construct the truth tables for:
   (a) $\sim(p \land q) \leftrightarrow (\sim r \lor \sim s)$. [*Ans.* TFFFFTTTFTTTFTTT.]
   (b) $[\sim(p \to q) \lor (s \leftrightarrow (r \land p))] \land [p \to (q \to \sim r)]$.
   [*Ans.* FFFTTTTTFTFTFTFT.]
11. Using only $\land$, $\lor$, and $\sim$, write a statement which has the same truth table as:
   (a) $p \to q$
   (b) $\sim(p \to q)$
   (c) $p \leftrightarrow q$.
   Using only $\land$ and $\sim$, write a statement having the same truth table as $p \lor q$. What have we proved?
12. Look back at Exercises 6, 7, and 8 of Section 2. What compound statement has the same truth table as $p \downarrow p$? As $(p \downarrow p) \downarrow (q \downarrow q)$?
13. Using the results of Exercises 11 and 12, show that any truth table can be represented using only the single connective $\downarrow$. Using only that connective, write statements having the same truth table as:
   (a) $\sim p \land q$. [*Ans.* $[(p \downarrow p) \downarrow (p \downarrow p)] \downarrow (q \downarrow q)$.]
   (b) $p \lor q$.
   (c) $p \to q$.
   (d) $q \to p$.

14. Show that $p \to q$, $\sim p \lor q$, $\sim(p \land \sim q)$, and $\sim q \to \sim p$ are all equivalent.
15. Show that the statements $p \leftrightarrow q$, $\sim p \leftrightarrow \sim q$, $(p \land q) \lor (\sim p \land \sim q)$, and $(\sim p \lor q) \land (\sim q \lor p)$ are all equivalent.
16. Let p be the statement "I win a prize" and q be the statement "I bought a lottery ticket." Give verbal equivalents of the statements in Exercise 15.

4 LOGICAL POSSIBILITIES

One of the most important contributions that mathematics can make to the solution of a scientific problem is to provide an exhaustive analysis of the logical possibilities for the problem. The role of science is then to discover facts which will eliminate all but one possibility. Or, if this cannot be achieved, at least science tries to estimate the probabilities of the various possibilities.

So far we have considered only a very special case of the analysis of logical possibilities, namely truth tables. We started with a small number of given statements, say p, q, and r, and we assumed that all the truth table cases were possible. This amounts to assuming that the three statements are logically unrelated. Then we could determine the truth or falsity of every compound statement formed from p, q, and r for every truth table case (every logical possibility).

But there are many more statements whose truth cannot be analyzed in terms of the eight truth table cases discussed above. For example, $\sim p \lor (q \land r \land \sim s)$ requires a finer analysis, a truth table with 16 cases.

Many of these ideas are applicable in a more general setting. Let us suppose that we have an analysis of logical possibilities. That is, we have a list of eventualities, such that one and only one of them can possibly be true. We know this partly from the framework in which the problem is considered, and partly as a matter of pure logic. We then consider *statements relative to this set of possibilities*. These are statements whose truth or falsity can be determined for each logical possibility. For example, the set of possibilities may be the eight truth table cases, and the statements relative to these possibilities are the compound statements formed from p, q, and r. But we should consider a more typical example.

EXAMPLE 1 Let us consider the following problem, which is of a type often studied in probability theory. "There are two urns; the first contains two black balls and one white ball, while the second contains one black ball and two white balls. Select an urn at random and draw two balls in succession from it. What is the probability that . . .?" Without raising questions of probability, let us ask what the possibilities are. Figures 16 and 17 give us two ways of analyzing the logical possibilities.

In Figure 16 we have analyzed the possibilities as far as colors of balls drawn was concerned. Such an analysis may be sufficient for many pur-

Case	Urn	First Ball	Second Ball
1	1	black	black
2	1	black	white
3	1	white	black
4	2	black	white
5	2	white	black
6	2	white	white

Figure 16

Case	Urn	First Ball	Second Ball
1	1	black no. 1	black no. 2
2	1	black no. 2	black no. 1
3	1	black no. 1	white
4	1	black no. 2	white
5	1	white	black no. 1
6	1	white	black no. 2
7	2	black	white no. 1
8	2	black	white no. 2
9	2	white no. 1	black
10	2	white no. 2	black
11	2	white no. 1	white no. 2
12	2	white no. 2	white no. 1

Figure 17

poses. In Figure 17 we have carried out a finer analysis, in which we distinguished between balls of the same color in an urn. For some purposes the finer analysis may be necessary.

It is important to realize that the possibilities in a given problem may be analyzed in many different ways, from a very rough grouping to a highly refined one. The only requirements on an analysis of logical possibilities are:

(1) That under any conceivable circumstances one and only one of these possibilities must be the case, and
(2) that the analysis is fine enough so that the truth value of each statement under consideration in the problem is determined in each case.

It is easy to verify that both analyses (Figures 16 and 17) satisfy the first condition. Whether the rougher analysis will satisfy the second condition depends on the nature of the problem. If we can limit ourselves to statements like "Two black balls are drawn from the first urn," then it suffices. But if we wish to consider "The first black ball is drawn after the second black ball from the first urn," then the finer analysis is needed.

Given the analysis of logical possibilities, we can ask for each assertion about the problem, and for each logical possibility, whether the assertion is true in this case. Normally, for a given statement there will be many cases in which it is true and many in which it is false. Logic will be able to do no more than to point out the cases in which the statement is true. In Example 1, the statement "One white ball and one black ball is drawn" is true (in Figure 16) in cases 2, 3, 4, and 5, and false in cases 1 and 6. However, there are two notable exceptions, namely, a statement that is true in every logically possible case, and one that is false in every case. Here logic alone suffices to determine the truth value.

A statement that is true in every logically possible case is said to be *logically true.* The truth of such a statement follows from the meaning of the words and the form of the statement, together with the context of the problem about which the statement is made. We shall see several examples of logically true statements below. A statement that is false in every logically possible case is said to be *logically false,* or to be a *self-contradiction.* For example, the conjunction of any statement with its own negation will always be a self-contradiction, since it cannot be true under any circumstances.

In Example 1, the statement "At most two black balls are drawn" is true in every case, in either analysis. Hence this statement is logically true. It follows from the very definition of the problem that we cannot draw more than two balls. Hence, also, the statement "Draw three white balls" is logically false.

What the logical possibilities are for a given set of statements will depend on the context, i.e., on the problem that is being considered. Unless we know what the possibilities are, we have not understood the task before us. This does not preclude that there may be several ways of analyzing the logical possibilities. In Example 1 above, for example, we gave two different analyses, and others could be found. In general, the question "How many cases are there in which p is true?" will depend on the analysis given. (This will be of importance in our study of probability theory.) However, note that a statement that is logically true (false) according to one analysis will be logically true (false) according to every other analysis of the given problem.

The truth table analysis is often the roughest possible analysis. There may be hundreds of logical possibilities, but if all we are interested in are compounds formed from p and q, we need only know when p and q are true or false. For example, a statement of the form $p \to (p \vee q)$ will have to be true in every conceivable case. We may have a hundred cases, giving varying truth values for p and q, but every such case must correspond to one of the four truth table cases, as far as the compound is concerned. In each of these four cases the compound is true, and therefore such a statement is logically true. An example of it is "If Jones is smart, then he is smart or lucky."

However, if the components are logically related, then a truth table analysis may not be adequate. Let p be the statement "Jim is taller than Bill," while q is "Bill is taller than Jim." And consider the statement "Either Jim is not taller than Bill or Bill is not taller than Jim," i.e., $\sim p \vee \sim q$. If we work the truth table of this compound, we find that it is false in the first case. But this case is not logically possible, since under no circumstances can p and q both be true! Our compound is logically true, but a truth table will not show this. Had we made a careful analysis of the possibilities as to the heights of the two men, we would have found that the compound statement is true in every case.

EXAMPLE 2 The Miracle Filter Company conducts an annual survey of the smoking habits of adult Americans. The results of the survey are organized into 25 files, corresponding to the 25 cases in Figure 18.

First, figures are kept separately for men and women. Secondly, the educational level is noted according to the following code:

0 did not finish high school
1 finished high school, no college
2 some college, but no degree
3 college graduate, but no graduate work
4 did some graduate work

Finally, there is a rough occupational classification: housewife, salaried professional, or salaried nonprofessional.

They have found that this classification is adequate for their purposes. For instance, to get figures on all adults in their survey who did not go beyond high school, they pull out the files numbered 1, 2, 3, 4, 11, 12, 13, 14, 15, and 16. Or they can locate data on male professional workers by looking at files 1, 3, 5, 7, and 9.

According to their analysis, the statement "The person is a housewife, professional, or nonprofessional" is logically true, while the statement "The person has educational level greater than 3, is neither professional nor nonprofessional, but not a female with graduate education" is a self-contradiction. The former statement is true about all 25 files, the latter about none.

Of course, they may at some time be forced to consider a finer analysis of logical possibilities. For instance, "The person is a male with annual income over $10,000" is *not* a statement relative to the given possibilities. We could choose a case—say case 6—and the given statement may be either true or false in this case. Thus the analysis is not fine enough.

Of all the logical possibilities, one and only one represents the facts as

Case	Sex	Educational Level	Occupation
1	male	0	prof.
2	male	0	nonprof.
3	male	1	prof.
4	male	1	nonprof.
5	male	2	prof.
6	male	2	nonprof.
7	male	3	prof.
8	male	3	nonprof.
9	male	4	prof.
10	male	4	nonprof.
11	female	0	housewife
12	female	0	prof.
13	female	0	nonprof.
14	female	1	housewife
15	female	1	prof.
16	female	1	nonprof.
17	female	2	housewife
18	female	2	prof.
19	female	2	nonprof.
20	female	3	housewife
21	female	3	prof.
22	female	3	nonprof.
23	female	4	housewife
24	female	4	prof.
25	female	4	nonprof.

Figure 18

they are. That is, for a given person, one and only one of the 25 cases is a correct description. To know which one, we need factual information. When we say that a certain statement is "true," without qualifying it, we mean that it is true in this one case. But, as we have said before, what the case actually is lies outside the domain of logic. Logic can tell us only what the circumstances (logical possibilities) are under which a statement is true.

EXERCISES

1. Prove that the negation of a logically true statement is logically false, and the negation of a logically false statement is logically true.
2. Prove that if p and $p \to q$ are logically true, then so is q.
3. Classify each of the following as logically true, logically false, or neither:
   (a) $(p \wedge (p \to q)) \to q$. [*Ans.* Logically true.]

(b) $[(p \wedge q) \rightarrow r] \leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$.
(c) $p \rightarrow (q \vee \sim q)$.
(d) $(p \rightarrow q) \wedge (q \rightarrow r) \wedge \sim(p \rightarrow r)$.
(e) $((p \wedge q) \vee (p \wedge r)) \rightarrow p$.
(f) $(p \vee q) \wedge (p \vee r)$. [*Ans.* Neither.]
(g) $(p \rightarrow q) \wedge \sim(\sim q \rightarrow \sim p)$. [*Ans.* Logically false.]

4. Find all cases in Figure 18 about which the following statement is true: "The person is a nonprofessional and, if male, has had at least some college training."

5. In the example in Figure 18, give two logically true and two logically false statements (other than those in the text).

6. A hat is filled with slips numbered 1 through 20, and two slips are drawn. Which of the following analyses satisfy the first condition for logical possibilities? What is wrong with the others?
The sum of the numbers on the slips is:
(a) (1) even, (2) odd.
(b) (1) prime, (2) greater than 37.
(c) (1) less than 3, (2) even, (3) prime.
(d) (1) divisible by 3, (2) not divisible by 3.
(e) (1) less than 17, (2) 17, (3) greater than 17.
(f) (1) greater than 2, (2) less than 40.
(g) (1) 4, 8, 12, 16, or 20, (2) larger than 20, (3) smaller than 20 and odd.

7. In a college using grades A, B, C, D, and F how many logically possible report cards are there for a student taking four courses? What if the only grades are Pass and Fail?

8. A drive-in restaurant sells hamburgers for 35 cents, cheeseburgers for 45 cents, french fries for 20 cents, and milkshakes for 25 cents. How many logical possibilities are there for orders totaling 85 cents? What are they? [*Partial Ans.* There are three possibilities.]

9. Concerning the answer to Exercise 8, which of the following are logically true? Which are logically false? Which are neither?
(a) The order contains no french fries? [*Ans.* Neither.]
(b) The order contains more than one of some item.
(c) The order contains two hamburgers. [*Ans.* Logically false.]
(d) The order contains french fries if and only if it contains a milkshake.
(e) The order contains exactly two different types of food.

10. Suppose in Exercise 8 we are further told that the order is for a man who is on a diet and therefore cannot eat milkshakes. What can we conclude?

11. In Example 1, with the logical possibilities given by Figure 17, state the cases in which the following are true:
(a) Exactly one white ball is drawn.
(b) Either the first urn is selected and a white ball is chosen on the first draw, or two white balls are chosen.

(c) A white ball is drawn, and then a black ball.
(d) If the first ball is black, then the urn selected is not number 1 and the second ball is black.
(e) The balls are of the same color if and only if the first is black.
[*Ans.* 1, 2, 5, 6, 9, 10.]

12. A survey of families having three children is taken. The sex of each child is noted, beginning with the oldest. Construct a list of the logical possibilities. [*Hint:* There are eight cases.]

13. In Exercise 12, in which cases is each statement below true?
(a) There are more girls than boys, but at least one boy.
(b) There is a boy if and only if there is a girl.
[*Ans.* Every case *except* BBB and GGG.]
(c) The oldest child is a boy if the youngest is a girl.

14. How does the list of possibilities in Exercise 12 change if we neglect the order in which the children were born?

5 TREE DIAGRAMS

A very useful tool for the analysis of logical possibilities is the drawing of a "tree." This device will be illustrated by several examples.

EXAMPLE 1 Consider again the survey of the Miracle Filter Company. They keep two large filing cabinets, one for men and one for women. Each cabinet has five drawers, corresponding to the five educational levels. Each drawer is subdivided according to occupations; drawers in the filing cabinet for men have two large folders, while in the other cabinet each drawer has three folders.

When a clerk files a new piece of information, he first has to find the right cabinet, then the correct drawer, and then the appropriate folder. This three-step process of filing is shown in Figure 19. For obvious reasons we shall call a figure like this, which starts at a point and branches out, a *tree*.

Observe that the tree contains all the information relevant to classifying a person interviewed. There are 25 ways of starting at the bottom and following a path to the top. The 25 paths represent the 25 cases in Figure

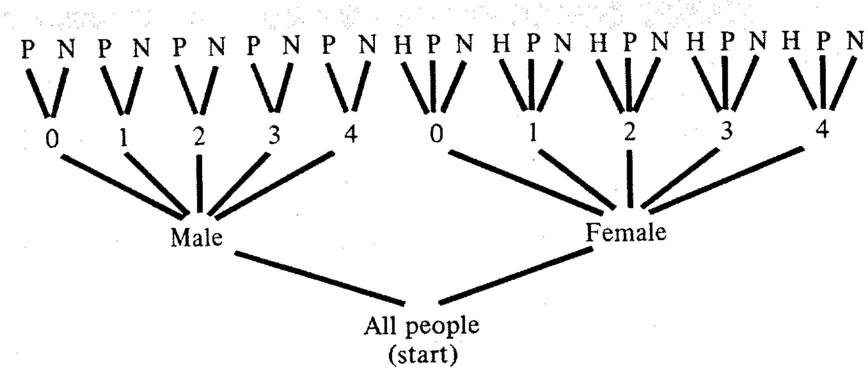

Figure 19

18. The order in which we performed the classification is arbitrary. We might as well have classified first according to educational level, then according to occupation, and then according to sex. We would still obtain a tree representing the 25 logical possibilities, but the tree would look quite different. (See Exercise 1.)

EXAMPLE 2 Next let us consider the example of Figure 16. This is a three-stage process; first we select an urn, then draw a ball and then draw a second ball. The tree of logical possibilities is shown in Figure 20. We note that six is the correct number of logical possibilities. The reason for this is: If we choose

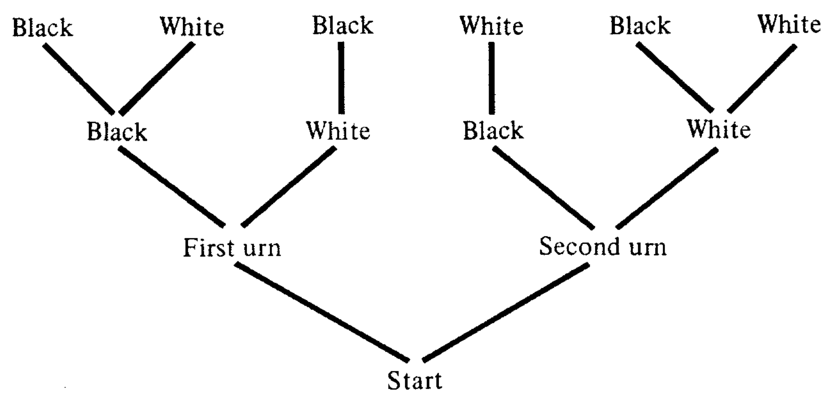

Figure 20

the first urn (which contains two black balls and one white ball) and draw from it a black ball, then the second draw may be of either color; however, if we draw a white ball first, then the second ball drawn is necessarily black. Similar remarks apply if the second urn is chosen.

EXAMPLE 3 As a final example, let us construct the tree of logical possibilities for the outcomes of a World Series played between the Pirates and the Orioles. In Figure 21 is shown half of the tree, corresponding to the case when the Pirates win the first game (the dotted line at the bottom leads to the other half of the tree). In the figure a "P" stands for a Pirate win and "O" for

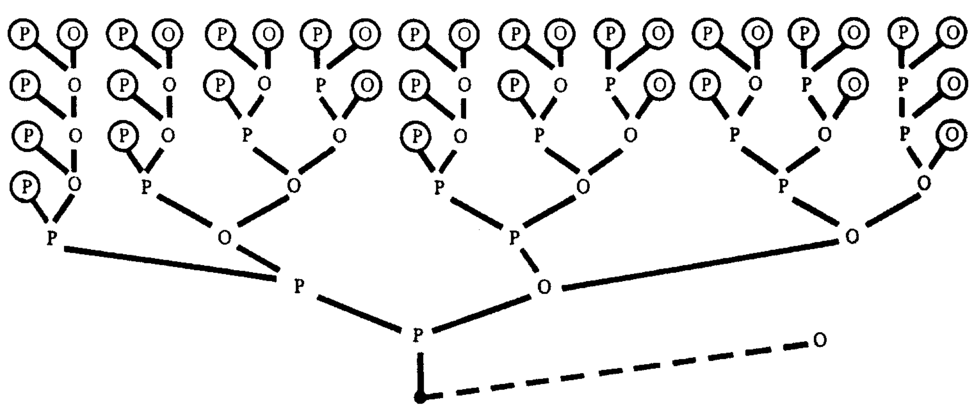

Figure 21

an Oriole win. There are 35 possible outcomes (corresponding to the circled letters) in the half-tree shown, so that the World Series can end in 70 ways.

This example is different from the previous two in that the paths of the tree end at different levels, corresponding to the fact that the World Series ends whenever one of the teams has won four games.

Not always do we wish as detailed an analysis as that provided in the examples above. If, in Example 2, we wanted to know only the color and order in which the balls were drawn and not which urn they came from, then there would be only four logical possibilities instead of six. Then in Figure 20 the second and fourth paths (counting from the left) represent the same outcome, namely, a black ball followed by a white ball. Similarly, the third and fifth paths represent the same outcome. Finally, if we cared only about the color of the balls drawn, not the order, then there are only three logical possibilities: two black balls, two white balls, or one black and one white ball.

A less detailed analysis of the possibilities for the World Series is also possible. For example, we can analyze the possibilities as follows: Pirates in four, five, six, or seven games, and Orioles in four, five, six, or seven games. The new classification reduced the number of possibilities from 70 to eight. The other possibilities have not been eliminated but merely grouped together. Thus the statement "Pirates in four games" can happen in only one way, while "Pirates in seven games" can happen in 20 ways (see Figure 21). A still less detailed analysis would be a classification according to the number of games in the series. Here there are only four logical possibilities.

You will find that it often requires several trials before the "best" way of listing logical possibilities is found for a given problem.

EXERCISES

1. Construct a tree for Example 1, if people are first classified according to educational level, then according to profession, and finally according to sex. Is the shape of the tree the same as in Figure 19? Does it represent the same possibilities?

2. We set up an experiment similar to that of Figure 20, but urn 1 contains two black balls and four white balls, while urn 2 has one white ball and five black balls. An urn is selected and three balls are drawn. Construct the tree of logical possibilities. How many cases are there? [*Ans.* 11.]

3. From the tree constructed in Exercise 2 answer the following questions.
   (a) In how many cases do we draw three white balls?
   (b) In how many cases do we draw three black balls?
   (c) In how many cases do we draw two white balls and a black ball?
   (d) How many cases does this leave? What cases are these?

4. In 1965 the Dodgers lost the first two games of the World Series, but won the series in the end. In how many ways can the Series go so that the winning team loses the first two games? [*Ans.* 10.]
5. In how many ways can the World Series be played (see Figure 21) if the Pirates win the first game and
   (a) The Pirates win the series? [*Ans.* 20.]
   (b) No team wins two games in a row?
   (c) The losing team wins three games in a row?
   (d) The losing team wins four games in a row?
6. The following is a typical process in genetics: Each parent has two genes for a given trait, AA or Aa or aa. The child will inherit one gene from each parent. What are the possibilities for a child if both parents are AA? What if one is AA and the other aa? What if one is AA and the other Aa? What if both are Aa? Construct a tree for each process. (Let stage 1 be the choice of a gene from the first parent, stage 2 from the second parent. Then see how many different types the resulting branches represent.)
7. It is often the case that types AA and Aa (see Exercise 6) are indistinguishable from the outside but easily distinguishable from type aa. What are the logical possibilities if the two parents are of noticeably different types?
8. A certain businessman has three favorite bars. After work he goes to one of these bars at which he orders either whiskey or scotch. If he likes the drink he goes directly home. If he does not like it, he goes to one of the other two bars and again orders either whiskey or scotch; he then goes home after the second drink. Draw the tree of logical possibilities, labeling the bars, A, B, and C. How many possibilities are there?
9. In Exercise 8, how many possibilities are there in which
   (a) He drinks only whiskey? [*Ans.* 9.]
   (b) He visits bar B?
   (c) He visits bars A and B?
   (d) He visits Bar C and has both scotch and whiskey?
10. In Exercise 2 we wish to make a rougher classification of logical possibilities. What branches (in the tree there constructed) become identical if
    (a) We do not care about the order in which the balls are drawn?
    (b) We care neither about the order of balls, nor about the number of the urn selected?
    (c) We care only about what urn is selected, and whether the balls drawn are all the same color?
11. A menu lists a choice of soup, fruit, or orange juice for an appetizer; a choice of steak, chicken, or fish for the entree; and a choice of pie or cake for dessert. A complete dinner consists of one choice for each course. Draw a tree for the possible complete dinners.
    (a) How many different complete dinners are possible? [*Ans.* 18.]

(b) If a man refuses to eat chicken or cake, how many different complete dinners can be choose?

(c) A certain customer eats pie for dessert if and only if he did not have fruit or orange juice for an appetizer. How many different complete meals are available to him?

12. A man is considering the purchase of one of four types of stocks. Each stock may go up, go down, or stay the same after his purchase. Draw the tree of logical possibilities.

13. For the tree constructed in Exercise 12 give a statement which

(a) Is true in half the cases.

(b) Is false in all but one case.

(c) Is true in all but one case.

(d) Is logically true.

(e) Is logically false.

14. In how many different ways can 70 cents change be given, using quarters, dimes, and nickels? Draw a tree. [*Hint:* To eliminate duplication, require that larger coins be handed out before smaller ones. Let the branches of the tree be labeled with the number of coins of each type handed out.] [*Ans.* 16.]

15. Redraw the tree of Exercise 14, requiring that smaller coins be handed out before larger ones.

16. What is the answer to Exercise 14 if only two dimes are available?

17. A college valedictorian plans to speak on brotherhood, integrity, or "the System" at commencement. The college president will speak on brotherhood, integrity, or the challenge of the future, but will not pick the same topic as the valedictorian. The college chaplain always speaks on brotherhood, unless the president does, in which case he chooses one of the other three topics.

(a) Using a tree, determine the number of logical possibilities. [*Ans.* 11.]

(b) In how many of the different programs will there be a speech on "the System"? [*Ans.* 6.]

(c) How many different programs are there in which the audience will have to listen to more than one speech on the same topic?

18. In Exercise 17, how many logical possibilities are there if we take into account only speech topics and number of times a given topic is used and disregard the speakers and order of the speeches?

6 LOGICAL RELATIONS

Until now we have considered statements in isolation. Sometimes, however, we want to consider a relationship between pairs of statements. The most interesting such relation is that one statement (logically) *implies* another one. We *define* implication as follows: r implies s if s is true whenever r is true, i.e., if s is true in all the logically possible cases in which r is true. We shall use the notation $r \Rightarrow s$ for the relation r *implies* s.

If p implies q we also say that q follows from p, or that q is (logically) deducible from p. For example, in any mathematical theorem the hypothesis implies the conclusion.

Note that $r \Rightarrow s$ is a relation and *not a statement*. However, it follows from the definition that $r \Rightarrow s$ holds if and only if the conditional $r \to s$ is logically true.

For compound statements having the same components, truth tables provide a convenient method for testing this relation. In Figure 22 we

p	q	$p \leftrightarrow q$	$p \to q$	$p \vee q$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	T	T	F

Figure 22

illustrate this method. Let us take $p \leftrightarrow q$ as our hypothesis r. Since it is true only in the first and fourth cases, and $p \to q$ is true in both these cases, we see that the statement $p \leftrightarrow q$ implies $p \to q$. On the other hand, the statement $p \vee q$ is false in the fourth case and hence it is not implied by $p \leftrightarrow q$. Again, a comparison of the last two columns of Figure 22 shows that the statement $p \to q$ does not imply and is not implied by $p \vee q$.

Let us now take up the "paradoxes" of the conditional. Conditional statements sound paradoxical when the components are not related. For example, it sounds strange to say that "If it is a nice day then chalk is made of wood" is true on a rainy day. It must be remembered that the conditional statement just quoted means no more and no less than that one of the following holds: (1) It is a nice day and chalk is made of wood, or (2) It is not a nice day and chalk is made of wood, or (3) It is not a nice day and chalk is not made of wood. (See Figure 11b.) And on a rainy day number (3) happens to be correct.

But it is by no means true that "It is a nice day" implies that "Chalk is made of wood." It is logically possible for the former to be true and for the latter to be false (indeed, this is the case on a nice day, with the usual method of chalk manufacture), hence the implication does not hold. Thus, while the conditional quoted in the previous paragraph is true on a given day, it is not logically true.

In common parlance "if . . . then . . ." is usually asserted on logical grounds. Hence any usage in which such an assertion happens to be true, but is not logically true, sounds paradoxical. Similar remarks apply to the common usage of "if and only if."

The second relation we shall consider is *equivalence*. We shall say that r is equivalent to s, denoted by $r \Leftrightarrow s$, if r is true whenever s is true and vice versa. In other words, $r \Leftrightarrow s$, if and only if $r \leftrightarrow s$ is logically true. We have already noted that equivalent statements have the same truth table.

p	q	$\sim p \wedge \sim q$	$\sim(p \vee q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Figure 23

Figure 23 establishes that $\sim p \wedge \sim q$ is equivalent to $\sim(p \vee q)$, which is one of *DeMorgan's laws*. (See Figure 10 for the other DeMorgan law.)

An implication $r \Rightarrow s$ or an equivalence $p \Leftrightarrow q$ can be established on purely logical grounds. From these we can construct valid arguments, as we shall see in Section 8. In Section 9 other ways of stating implications and equivalences will be discussed which are (sometimes) more convenient forms in which to carry out such arguments.

A third important relationship is that of inconsistency. Statements r and s are *inconsistent* if it is impossible for both of them to be true, in other words, if $r \wedge s$ is a self-contradiction. For example, the statements $p \wedge q$ and $\sim q$ are inconsistent (see Figure 24). An important use of logic is to check for inconsistencies in a set of assumptions or beliefs.

p	q	$p \wedge q$	$\sim q$
T	T	T	F
T	F	F	T
F	T	F	F
F	F	F	T

Figure 24

We conclude this section by listing several important implications and equivalences:

(1) $$p \wedge q \Rightarrow p.$$
(2) $$p \wedge q \Rightarrow p \vee q.$$
(3) $$(p \leftrightarrow q) \Rightarrow (p \rightarrow q).$$
(4) $$p \wedge (p \rightarrow q) \Rightarrow q.$$
(5) $$(p \rightarrow q) \wedge \sim q \Rightarrow \sim p.$$
(6) $$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r).$$
(7) $$(p \leftrightarrow q) \wedge (q \leftrightarrow r) \Rightarrow (p \leftrightarrow r).$$
(8) $$(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow (p \leftrightarrow q).$$

In the Exercises you will be asked to establish some of these relations.

EXERCISES

1. Show that $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$ is logically true, but that $(p \leftrightarrow q) \rightarrow (p \vee q)$ is not logically true. Interpret this in terms of implications.
2. Is it true that $p \Rightarrow \sim p$? Explain why this does or does not tell us that $p \rightarrow \sim p$ is logically false.
3. If p is logically true, prove that
   (a) $p \vee q$ is logically true.
   (b) $\sim p \wedge q$ is logically false.
   (c) $p \wedge q$ is equivalent to q.
   (d) $\sim p \vee q$ is equivalent to q.
4. Construct truth tables for the following compounds and test for implications and equivalences.
   (a) $p \vee \sim q$.
   (b) $\sim p \leftrightarrow \sim q$.
   (c) $q \rightarrow p$.
   (d) $p \wedge \sim q$.
   (e) $\sim(p \rightarrow q)$. [*Partial Ans.* (a) $\Leftrightarrow$ (c); (e) $\Rightarrow$ (a), (c), (d).]
5. Construct truth tables for the following compounds, and arrange them in order so that each compound implies all the following ones.
   (a) $\sim p \leftrightarrow q$.
   (b) $p \rightarrow (\sim p \rightarrow q)$.
   (c) $\sim[p \rightarrow (q \rightarrow p)]$.
   (d) $p \vee q$.
   (e) $\sim p \wedge q$. [*Ans.* (c), (e), (a), (d), (b).]
6. Which of the following are equivalent: $p, \sim p, p \vee p, p \wedge p, p \rightarrow p, p \leftrightarrow p$? Prove (using truth tables) your answers. Which are inconsistent?
7. Construct a compound equivalent to $p \leftrightarrow q$ using only the connectives $\rightarrow$ and $\wedge$. Interpret your result in terms of equivalences and implications.
   [*Partial Ans.* Saying that two statements are equivalent is the same as saying that each implies the other.]
8. Show that $\sim p \wedge q \Leftrightarrow \sim(q \rightarrow p)$.
9. If p is logically true, q is logically false, and p and r are inconsistent, what is the status of $\sim p \Leftrightarrow \sim(q \vee r)$?
10. The statements r and s are compounds of p and q and have the following truth tables:

p	q	r	s	t
T	T	T	T	
T	F	F	F	
F	T	T	T	
F	F	T	F	

Find a statement t which is a compound of p and q satisfying each of the following properties:

(a) $t \Leftrightarrow r$. [*Ans.* $p \to q$.]

(b) t is inconsistent with s. [*Ans.* $\sim q$.]

(c) $t \Leftrightarrow s$.

(d) $t \leftrightarrow r$ is logically true.

(e) $t \to s$ is neither logically true nor logically false.

(f) $t \leftrightarrow s$ is logically false.

(g) $r \Rightarrow t$.

(h) $t \to r$ is logically false.

11. In Exercise 10,

(a) What is the relation between r and s? [*Ans.* $s \Rightarrow r$.]

(b) How many nonequivalent statements t which are compounds of p and q can be found which satisfy the condition that s implies t and t implies s? What are they?

12. If r and s are compounds of p and q such that r is logically false and s is logically true, what compounds t will satisfy both the conditions $r \Rightarrow t$ and $t \Rightarrow s$.

13. Pick out an inconsistent pair from among the following four compound statements.

r: $p \to q$.

s: q.

t: $\sim(q \to p)$. u: $\sim p \leftrightarrow \sim q$.

14. In Exercise 13 is there an inconsistent pair among r, s, and t? Is it possible that all three statements are true?

15. What relation exists between two logically true statements? Between two self-contradictions?

16. Verify the implications (1)–(4) stated at the bottom of page 28.

17. Verify the implications (5)–(7) and equivalence (8) stated at bottom of page 28.

18. Let $p|q$ be defined as "p and q are not both true."

(a) Construct a truth table for $p|q$. [*Ans.* FTTT.]

(b) Show that $(p|q)|(p|q)$ is equivalent to $p \wedge q$.

(c) Find a compound using only $|$ which is equivalent to $\sim p$.

19. We shall call a connective "adequate" if $\sim p$, $p \wedge q$, $p \vee q$, $p \leftrightarrow q$, and $p \to q$ can be expressed in terms of p, q, and that connective.

(a) Using the results of Exercise 11 in Section 3 and Exercise 18 above, show that the connective $|$ is adequate.

(b) In Exercise 13, Section 3, it was shown that the connective $\downarrow$ is adequate. Could we define a third different connective such that it would also be adequate? (*Hint:* Consider the truth table that it must have. If we call the new connective $\updownarrow$, then $p \updownarrow p$ must be false if p is true, since otherwise any expression involving p and $\updownarrow$ would be true if p is true and we would not be able to express $\sim p$. Thus $p \updownarrow q$ is false if p and q are both true.]

*7 VALID ARGUMENTS

One of the most important tasks of a logician is the checking of *arguments*. By an argument we shall mean the assertion that a certain statement (the *conclusion*) follows from other statements (the *premises*). An argument will be said to be (logically) *valid*, if and only if the conjunction of the premises implies the conclusion; i.e., if the premises are all true, the conclusion *must* also be true.

It is important to realize that the truth of the conclusion is irrelevant as far as the test of the validity of the argument goes. A true conclusion is neither necessary nor sufficient for the validity of the argument. The two examples below show this, and they also show the form in which we shall state arguments, i.e., first we state the premises, then draw a line, and then state the conclusion.

EXAMPLE 1

If the United States is a democracy, then its citizens have the right to vote.
Its citizens do have the right to vote.

Therefore the United States is a democracy.

The conclusion is, of course, true. However, the argument is not valid since the conclusion does not follow from the two premises, as we shall show later.

EXAMPLE 2

To pass this math course you must be a genius.
Every player on the football team has passed this course.
The captain of the football team is not a genius.

Therefore the captain of the football team does not play on the team.

Here the conclusion is false, but the argument is valid since the conclusion follows from the premises. If we observe that the first premise is false, the paradox disappears. There is nothing surprising in the correct derivation of a false conclusion from false premises.

If an argument is valid, then the conjunction of the premises implies the conclusion. Hence if all the premises are ture, then the conclusion is also true. However, if one or more of the premises is false, so that the conjunction of all the premises is false, then the conclusion may be either true or false. In fact, all the premises could be false, the conclusion true, and the argument valid, as the following example shows.

EXAMPLE 3

All dogs have two legs.
All two-legged animals are carnivorous.

Therefore, all dogs are carnivorous.

Here the argument is valid and the conclusion is true, but both premises are false!

Each of these examples underlines the fact that neither the truth value nor the content of the statements appearing in an argument affect the validity of the argument. In Figures 25a and 25b are two valid forms of arguments.

$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

Figure 25a

$$\begin{array}{l} p \to q \\ \underline{\sim q \qquad} \\ \therefore \sim p \end{array}$$

Figure 25b

The symbol $\therefore$ means "therefore." The truth tables for these argument forms appear in Figure 26.

Figure 26

p	q	$p \to q$	p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	T	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	F	T	F	F	T	T	T

For the argument of Figure 25a, we see in Figure 26 that there is only one case in which both premises are true, namely, the first case, and that in this case the conclusion is true, hence the argument is valid. Similarly, in the argument of Figure 25b, both premises are true in the fourth case only, and in this case the conclusion is also true; hence the argument is valid.

Another way of stating that the argument in Figure 25a is valid is that the implication $[(p \to q) \wedge q] \Rightarrow q$ is true. Similarly for Figure 25b we note that the implication $[(p \to q) \wedge \sim q] \Rightarrow \sim p$ is true. Actually any true implication gives rise to a valid argument and vice versa.

An argument that is not valid is called a *fallacy*. Two examples of fallacies are the following argument forms.

$$\begin{array}{l} p \to q \\ \underline{q \qquad} \\ \therefore p \end{array} \qquad \textit{Fallacies} \qquad \begin{array}{l} p \to q \\ \underline{\sim p \qquad} \\ \therefore \sim q \end{array}$$

In the first fallacy, both premises are true in the first and third cases of Figure 26, but the conclusion is false in the third case, so that the argument is invalid. (This is the form of Example 1.) Similarly, in the second fallacy we see that both premises are true in the last two cases, but the conclusion is false in the third case.

We say that an argument depends only upon its form in that it does not matter what the componenets of the argument are. The truth tables in Figure 26 show that if both premises are true, then the conclusions of the arguments in Figures 25a and 25b are also true. For the fallacies above, the truth tables show that it is possible to choose both premises true without making the conclusion true, namely, choose a false p and a true q.

EXAMPLE 4 Consider the following argument.

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

The truth table of the argument appears in Figure 27.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Figure 27

Both premises are true in the first, fifth, seventh, and eighth rows of the truth table. Since in each of these cases the conclusion is also true, the argument is valid—that is, the implication $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$ is true. (Example 3 can be written in this form.)

Once we have discovered that a certain form of argument is valid, we can use it in drawing conclusions. It is then no longer necessary to compute truth tables. Presumably, this is what we do when we reason in everyday life; we apply a variety of valid forms known to us from previous experience. However, the truth table method has one great advantage: it is always applicable and purely automatic. We can even get a computer to test the validity of arguments involving compound statements.

EXERCISES

1. Test the validity of the following arguments:

(a) $$\begin{array}{r} p \leftrightarrow q \\ p \\ \hline \therefore q \end{array}$$

(b) $$\begin{array}{r} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

(c) $$\begin{array}{r} p \wedge q \\ \sim p \rightarrow q \\ \hline \therefore \sim q \end{array}$$

[*Ans.* (a), (b) are valid.]

2. Test the validity of the following arguments:

(a)
$$\begin{array}{rl} & p \to q \\ & \sim q \to \sim r \\ \hline \therefore & r \to p \end{array}$$

(b)
$$\begin{array}{rl} & p \to q \\ & \sim r \to \sim q \\ \hline \therefore & \sim r \to \sim p \end{array}$$

[*Ans.* (b) is valid.]

3. Test the validity of the argument

$$\begin{array}{rl} & p \leftrightarrow q \\ & q \vee r \\ & \sim r \\ \hline \therefore & \sim p \end{array}$$

[*Ans.* Not valid.]

4. Test the validity of the argument

$$\begin{array}{rl} & p \veebar q \\ & \sim q \to r \\ & \sim p \vee \sim r \\ \hline \therefore & \sim p \end{array}$$

5. Test the validity of the argument

$$\begin{array}{rl} & p \to q \\ & \sim p \to \sim q \\ & p \wedge \sim r \\ \hline \therefore & s \end{array}$$

6. Given are the premises $\sim p \to q$ and $\sim r \to \sim q$. We wish to find a valid conclusion involving p and r (if there is any).
   (a) Construct truth tables for the two premises.
   (b) Note the cases in which the conclusion must be true.
   (c) Construct a truth table for a combination of p and r only, filling in T wherever necessary.
   (d) Fill in the remainder of the truth table, making sure that you do not end up with a logically true statement.
   (e) What combination of p and r has this truth table? This is a valid conclusion. [*Ans.* $p \vee r$.]
7. Translate the following argument into symbolic form, and test its validity.

   If this is a good course, then it is worth taking.
   Either the grading is lenient, or the course is not worth taking.
   But the grading is not lenient.
   ____
   Therefore, this is not a good course.

   [*Ans.* Valid.]
8. Show that the following method may be used for testing the validity of an argument: Find the cases in which the conclusion is false, and show that in each case at least one premise is false.

9. Use the method of Exercise 8 to test Example 4.
10. Redo Exercise 1 using the method of Exercise 8.
11. Redo Exercise 4 using the method of Exercise 8.
12. Draw a valid conclusion from the following premises:

    He is either a man or a mouse.
    He has no skill in athletics.
    To be a man it is necessary to command respect.
    A man can command respect only if he has some athletic skill.
13. Draw a valid conclusion from the following premises:

    Either he will go to graduate school or he will be drafted.
    If he does not go to graduate school, he will get married.
    If he gets married, he will need a good income.
    He will not have a good income in the army.
14. Write the following argument in symbolic form, and test its validity.

    "For the candidate to win, it is sufficient that he carry New York. He will carry New York only if he takes a strong stand on civil rights. He will not take a strong stand on civil rights. Therefore, he will not win."
15. Write the following argument in symbolic form and test its validity.

    "Father praises me only if I can be proud of myself. Either I do well in sports or I cannot be proud of myself. If I study hard, then I cannot do well in sports. Therefore, if father praises me, then I do not study hard."

*8 VARIANTS OF THE CONDITIONAL

The conditional of two statements differs from the biconditional and from disjunctions and conjunctions of these two in that it lacks symmetry. Thus $p \vee q$ is equivalent to $q \vee p$, $p \wedge q$ is equivalent to $q \wedge p$, and $p \leftrightarrow q$ is equivalent to $q \leftrightarrow p$; *but* $p \rightarrow q$ *is not equivalent to* $q \rightarrow p$. The latter statement, $q \rightarrow p$, is called the *converse* of $p \rightarrow q$. Many of the most common fallacies in thinking arise from a confusion of a statement with its converse.

On the other hand, the conditional $p \rightarrow q$ is equivalent to the conditional $\sim q \rightarrow \sim p$, which is known as the *contrapositive.* The relationships among these three statements is demonstrated in Figure 28.

EXAMPLE 1 Let a be a positive real number, p the statement "$a < 7$," and q the statement "$a^2 < 100$." Then the conditional $p \rightarrow q$ is "If $a < 7$, then $a^2 < 100$." This is logically true, i.e., true for every positive real number. But the converse, "If $a^2 < 100$ then $a < 7$," is *not* logically true, and hence cannot be equivalent to the original statement. To show that it fails to be logically true, we must exhibit at least one logical possibility for which it is false. For example, if $a = 9$, then q is true ($9^2 < 100$) but p ($9 < 7$) is false.

The contrapositive is "If $a^2 \geq 100$, then $a \geq 7$," which *is* logically true.

		Conditional	Converse	Contra-positive
p	q	$p \to q$	$q \to p$	$\sim q \to \sim p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Figure 28

Since a positive number a can have $a^2 \geq 100$ only if $a \geq 10$, it is necessarily true that $a \geq 7$.

A mathematical statement that one suspects to be true, but whose truth or falsity has not yet been established by a proof, is known as a *conjecture*. One then attempts one of two procedures. One may attempt to construct a proof, which establishes the logical truth of the proposition. Or one may attempt to construct a *counterexample*, that is, a single logically possible case for which the proposition is false, which shows that the statement is not logically true. In either case the conjecture is settled, either positively or negatively.

The use of conditionals seems to cause more trouble than the use of the other connectives, perhaps because of the lack of symmetry, but also perhaps because there are so many different ways of expressing conditionals. In many cases only a careful analysis of a conditional statement shows whether the person making the assertion means the given conditional or its converse. Indeed, sometimes he means both of these, i.e., he means the biconditional. (See Exercise 5.)

The statement "I will go for a walk only if the sun shines" is a variant of a conditional statement. A statement of the form "p only if q" is closely related to the statement "If p then q," but just how? Actually the two express the same idea. The statement "p only if q" states that "If $\sim q$ then $\sim p$" and hence is equivalent to "If p then q." Thus the statement at the beginning of the paragraph is equivalent to the statement "If I go for a walk, then the sun will be shining."

Other phrases, in common use by mathematicians, which indicate a conditional statement are: "a necessary condition" and "a sufficient condition." To say that p is a sufficient condition for q means that if p takes place, then q will also take place. Hence the sentence "p is a sufficient condition for q" is equivalent to the sentence "If p then q."

Similarly, the sentence "p is a necessary condition for q" is equivalent to "q only if p." Since we know that the latter is equivalent to "If q then p," it follows that the assertion of a necessary condition is the converse of the assertion of a sufficient condition.

Finally, if both a conditional statement and its converse are asserted, then effectively the biconditional statement is being asserted. Hence the assertion

Basic Statement	Equivalent Forms
If p then q	p only if q p is a sufficient condition for q
If q then p	q only if p p is a necessary condition for q
p if and only if q	p is a necessary and sufficient condition for q

Figure 29

"p is a necessary and sufficient condition for q" is equivalent to the assertion "p if and only if q."

These various equivalences are summarized in Figure 29.

EXAMPLE 1 (continued) We can restate "If $a < 7$, then $a^2 < 100$" as follows:

$a < 7$ is a sufficient condition for $a^2 < 100$.
$a < 7$ only if $a^2 < 100$.
$a^2 < 100$ is a necessary condition that $a < 7$.

EXAMPLE 2 Let a be an integer, let p be the statement "a is odd," and let q be the statement "a^2 is odd." Then the biconditional $p \leftrightarrow q$ is the statement "Integer a is odd if and only if a^2 is odd," which can easily be proved to be true (see Section 9.) We can restate this as follows:

a is odd is necessary and sufficient that a^2 is odd.

Since $p \leftrightarrow q$ and $\sim p \leftrightarrow \sim q$ are equivalent, we can also state the theorem as:

a is even if and only if a^2 is even; or
a is even is necessary and sufficient that a^2 is even.

EXERCISES

1. Let p stand for "I will pass this course" and q for "I will do homework regularly." Put the following statements into symbolic form.
   - (a) I will pass the course only if I do homework regularly.
   - (b) Doing homework regularly is a necessary condition for me to pass this course.
   - (c) Passing this course is a sufficient condition for me to do homework regularly.
   - (d) I will pass this course if and only if I do homework regularly.
   - (e) Doing homework regularly is a necessary and sufficient condition for me to pass this course.

2. Take the statement in part (a) of the previous exercise. Form its converse, its contrapositive, and the converse of the contrapositive. For each of these give both a verbal and a symbolic form.
3. Let p stand for "It snows" and q for "The train is late." Put the following statements into symbolic form.
   (a) Snowing is a sufficient condition for the train to be late.
   (b) Snowing is a necessary and sufficient condition for the train to be late.
   (c) The train is late only if it snows.
4. Take the statement in part (a) of the previous exercise. Form its converse, its contrapositive, and the converse of its contrapositive. Give a verbal form of each of them.
5. Prove that the conjunction of a conditional and its converse is equivalent to the biconditional.
6. To what is the conjunction of the contrapositive and its converse equivalent? Prove it.
7. Prove that
   (a) $\sim\sim p$ is equivalent to p.
   (b) The contrapositive of the contrapositive is equivalent to the original conditional.
8. "For a matrix to have an inverse it is necessary that its determinant be different from zero." Which of the following statements follow from this? (No knowledge of matrices is required.)
   (a) For a matrix to have an inverse it is sufficient that its determinant be zero.
   (b) For its determinant to be different from zero it is sufficient for the matrix to have an inverse.
   (c) For its determinant to be zero it is necessary that the matrix have no inverse.
   (d) A matrix has an inverse if and only if its determinant is not zero.
   (e) A matrix has a zero determinant only if it has no inverse.
   [*Ans.* (b); (c); (e).]
9. "A function that is differentiable is continuous." This statement is true for all functions, but its converse is not always true. Which of the following statements are true for all functions? (No knowledge of functions is required.)
   (a) A function is differentiable only if it is continuous.
   (b) A function is continuous only if it is differentiable.
   (c) Being differentiable is a necessary condition for a function to be continuous.
   (d) Being differentiable is a sufficient condition for a function to be continuous.
   (e) Being differentiable is a necessary and sufficient condition for a function to be differentiable. [*Ans.* (a); (d); (e).]
10. Prove that the negation of "p is a necessary and sufficient condition

for q" is equivalent to "p is a necessary and sufficient condition for $\sim q$."

11. Supply a conclusion to the following argument, making it a valid argument. [Adapted from Lewis Carroll.]

"If he goes to a party, he does not fail to brush his hair.
To look fascinating it is necessary to be tidy.
If he is an opium eater, then he has no self-command.
If he brushes his hair, he looks fascinating.
He wears white kid gloves only if he goes to a party.
Having no self-command is sufficient to make one look untidy.
Therefore. . . ."

*9 THE INDIRECT METHOD OF PROOF

A mathematical *theorem* is an implication of the form $p \Rightarrow q$, where p is the conjunction of hypotheses and q is the conclusion. A *proof* is an argument that shows the conditional statement $p \rightarrow q$ is logically true. Such an argument usually depends on axioms, known theorems, etc. The construction of mathematical proofs frequently requires great ingenuity.

Instead of showing that $p \rightarrow q$ is logically true it is sometimes more convenient to show that an equivalent statement is logically true. We call such arguments *indirect proofs*. For instance, if we show that the contrapositive

(1) $$\sim q \rightarrow \sim p$$

is logically true, then, since it is equivalent to $p \rightarrow q$, we have also proved the latter to be logically true.

EXAMPLE 1 Let x and y be positive integers.

Theorem If xy is an odd number, then x and y are both odd.

Proof Suppose, on the contrary, that they are not both odd. Then one of them is even, say $x = 2z$. Then $xy = 2zy$ is an even number, contrary to hypothesis. Hence we have proved our theorem.

EXAMPLE 2 "He did not know the first name of the president of the Jones Corporation, hence he cannot be an employee of that firm. Why? Because every employee of that firm calls the boss by his first name (behind his back). Therefore, if he were really an employee of Jones, then he would know Jones's first name."

These are simple examples of a very common form of argument, frequently used both in mathematics and in everyday discussions. Let us try to unravel the form of the argument.

Given:	xy is an odd number.	He doesn't know Jones's first name.	p
To prove:	x and y are both odd numbers.	He doesn't work for Jones.	q
Suppose:	x and y are not both odd numbers.	He does work for Jones.	$\sim q$
Then:	xy is an even number.	He must know what Jones's first name is.	$\sim p$

In each case we assume the denial of the conclusion and derive, by a valid argument, the denial of the hypothesis. This is one form of the *indirect* method of proof.

There are several other important variants of this method of proof. It is easy to check that the following statements have the same truth table as—i.e., are equivalent to—the conditional $p \to q$.

$$(p \wedge \sim q) \to \sim p. \tag{2}$$

$$(p \wedge \sim q) \to q. \tag{3}$$

$$(p \wedge \sim q) \to (r \wedge \sim r). \tag{4}$$

Statement (2) shows that in the indirect method of proof we may make use of the original hypothesis in addition to the contradictory assumption $\sim q$. Statement (3) shows that we may also use this double hypothesis in the direct proof of the conclusion q. Statement (4) shows that if, from the double hypothesis p and $\sim q$ we can arrive at a contradiction of the form $r \wedge \sim r$, then the proof of the original statement is complete. This last form of the method is often referred to as *reductio ad absurdum*.

These last forms of the method are very useful for the following reasons: First of all we see that we can always take $\sim q$ as a hypothesis in addition to p. Second we see that besides q there are two other conclusions. ($\sim p$ or a contradiction) which are just as good.

EXAMPLE 3 Let a and b be integers, p the statement "$a + b$ is odd and a is even," and q the conclusion "b is odd." We prove $p \to q$ by means of (2) as follows:

To prove:	$p \to q$	If $a + b$ is odd and a is even, then b is odd.
Suppose:	$p \wedge \sim q$	$a + b$ is odd, a is even, and b is even.
Then:	$\sim p$	$a + b$ is even (since the sum of two even numbers is even).

We can also illustrate (4) by starting the same way but ending with

Then:	$\sim p \wedge p$	$a + b$ is even (as above) and (by hypothesis) $a + b$ is odd.

EXAMPLE 4 Let p be the statement "$3a^3 - 2a + 4 = 0$" and q the statement "a is not 0." We use (3):

To prove:	$p \rightarrow q$	If $3a^3 - 2a + 4 = 0$, then $a \neq 0$.
Assume:	$p \wedge \sim q$	$3a^3 - 2a + 4 = 0$ and $a = 0$.
Then:	q	(Since $a = 0$, put $3a^3 = 0$ into the equation to get $-2a + 4 = 0$ or $a = 2$.) Hence $a \neq 0$.

We can also again illustrate (4) starting the same way but ending

Then:	$r \wedge \sim r$	(Put $3a^3 = 2a = 0$ into the equation, giving $4 = 0$.) Since we know $4 \neq 0$, we have the absurdity $(4 = 0) \wedge (4 \neq 0.)$

EXERCISES

1. Construct indirect proofs for the following assertions:
   (a) If x^2 is odd, then x is odd (x an integer).
   (b) If I am to pass this course, I must do homework regularly.
2. Give a symbolic analysis of the following argument:

   "If he is to succeed, he must be both competent and lucky. Because, if he is not competent, then it is impossible for him to succeed. If he is not lucky, something is sure to go wrong."
3. Construct indirect proofs for the following assertions.
   (a) If $p \vee q$ and $\sim q$, then p.
   (b) If $p \leftrightarrow q$ and $q \rightarrow \sim r$ and r, then $\sim p$.
4. Give a symbolic analysis of the following argument:

   "If Jones is the murderer, then he knows the exact time of death and the murder weapon. Therefore, if he does not know the exact time or does not know the weapon, then he is not the murderer."
5. Verify that forms (2), (3), and (4) given above are equivalent to $p \rightarrow q$.
6. Let x and y be integers. Construct indirect proofs for the following assertions.
   (a) If $x + y$ is even, then x and y are both odd or both even.
   (b) If $x + y$ is odd, then either x is odd and y even or x is even and y odd.
7. Consider the conditional $p \rightarrow (q \vee r)$ corresponding to a theorem in which the conclusion is a disjunction. Discuss the four forms of indirect proof for this statement. [*Hint:* Use Exercise 6 as an example.]
8. Give an example of an indirect proof of some statement in which a contradiction is derived from p and $\sim q$.
9. Give a statement equivalent to $(p \wedge q) \rightarrow r$ which is in terms of $\sim p$, $\sim q$, and $\sim r$. Show how this can be used in a proof where there are two hypotheses given.

10. Use the indirect method to establish the validity of the following argument.

$$
\begin{array}{rcl}
p & \veebar & q \\
\sim p & \rightarrow & r \\
r & \rightarrow & s \\
q & \rightarrow & \sim s \\
\hline
\therefore p & &
\end{array}
$$

11. Use the indirect method on Exercise 7 of Section 7.

SUGGESTED READING

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