Frobenius's last proof

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Abstract

Around about 1917, Issai Schur rediscovered the Rogers-Ramanujan identities, and proved a system of polynomial identities that imply them. Schur wrote that Georg Frobenius (his former advisor) had shown him a simple, direct proof of these polynomial identities. Schur did not see fit to reveal Frobenius's proof, preferring his own rather complicated proof. But it is easy enough to guess what this 'simple, direct' proof must have been. As Frobenius died in 1917, we may call this 'Frobenius's last proof'.

1 Introduction

Around about 1917, Issai Schur [4] rediscovered the Rogers-Ramanujan identities, and proved a system of polynomial identities that imply them. Schur [4, p. 131] wrote that Georg Frobenius (his former advisor) had shown him a simple, direct proof ('einen enfachen direkten Beweis') of these polynomial identities: Schur did not see fit to reveal Frobenius's proof, preferring his own rather complicated proof. But it is easy enough to guess what this 'simple, direct' proof must have been. As Frobenius died in 1917, we may call this 'Frobenius's last proof'.

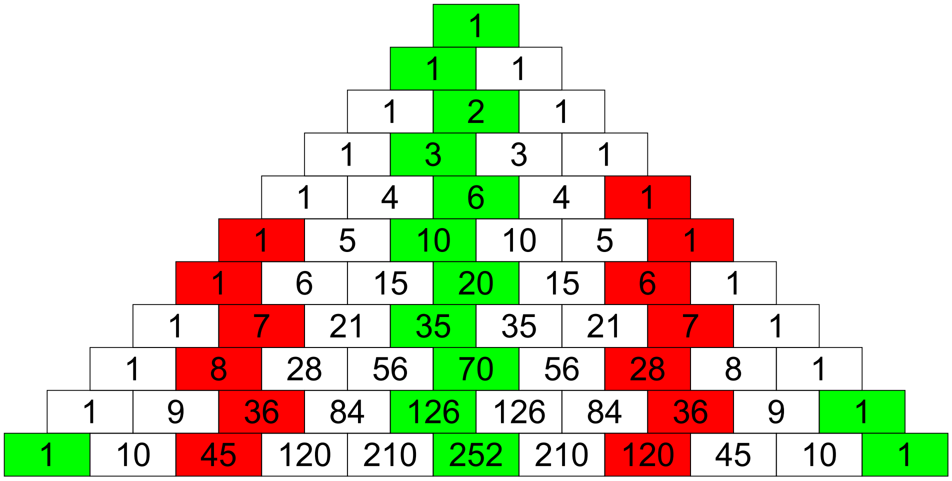

Figure 1: First GS identity.

2 The alpine identities

Told that Figure 1 illustrates an identity involving binomial coefficients, along with the hint that the specific colors red and green are significant, many a sophomore will be able to guess the identity.

What gives sophomores a leg up here is that the Fibonacci sequence is apt to loom large in their imagination. This steers them to the answer: Totaling the numbers in each row, taking the green numbers as positive and the red as negative, yields the Fibonacci sequence.

Schur [4, p. 131] thanks Frobenius for showing him the following way to formulate the identity. (We're shifting the sequence one notch from what is in Schur, to make $p(0) = p(1) = 1$.)

Proposition 1 (First GS identity — amateur version) *The sequence*

$$p(n) = \sum_{-\lfloor \frac{n}{5} \rfloor \leq \lambda \leq \lfloor \frac{n+1}{5} \rfloor} (-1)^\lambda \binom{n}{\lfloor \frac{n+5\lambda}{2} \rfloor}$$

satisfies the recurrence

$$p(n) = p(n-1) + p(n-2)$$

with

$$p(-1) = 0; \; p(0) = 1.$$

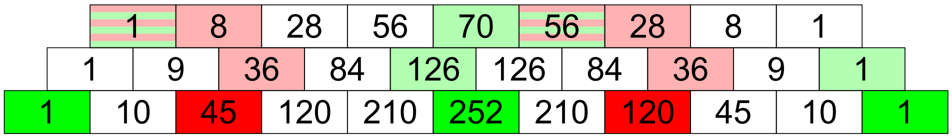

Figure 2: Proof of the first GS identity.

In the summation here, λ may range over all integers; the limits simply indicate the non-vanishing terms.

We're calling this the 'GS identity', short for 'Giant Slalom identity', because of the way the highlighted entries slalom their way down Pascal's triangle. This is the 'amateur version' of the identity, formulated for standard binomial coefficients $\binom{n}{k}$. Schur's original 'pro version' involves Gauss's polynomial analogs $\begin{bmatrix} n \\ k \end{bmatrix}$ of the binomial coefficients. We start with the amateur identity because the pro identity is just a gussied-up version of the amateur identity. The same goes for the proof.

And as for the proof, many a sophomore (though this time, not as many) can find it. It is illustrated in Figure 2. This proof is as simple and direct as one could wish. We unhesitatingly identify this as 'Frobenius's last proof'.

In symbols, the proof comes down to this:

$$\begin{aligned} \binom{n}{k-1} &= \binom{n-1}{k-2} + \binom{n-1}{k-1} \\ &= \binom{n-1}{k-2} + \binom{n-2}{k-2} + \binom{n-2}{k-1}, \end{aligned}$$

while

$$\begin{aligned} \binom{n}{k+1} &= \binom{n-1}{k} + \binom{n-1}{k+1} \\ &= \binom{n-2}{k-1} + \binom{n-2}{k} + \binom{n-1}{k+1}. \end{aligned}$$

Subtracting, we have

$$\begin{aligned} \binom{n}{k-1} - \binom{n}{k+1} &= \binom{n-1}{k-2} - \binom{n-1}{k+1} + \\ &\quad \binom{n-2}{k-2} - \binom{n-2}{k}, \end{aligned}$$

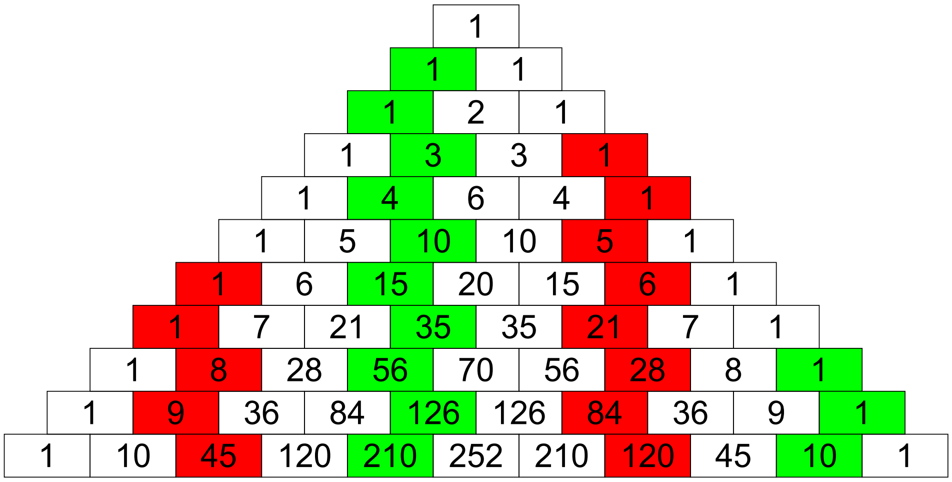

Figure 3: Second GS identity.

because the terms $\binom{n-2}{k-1}$ cancel.

As the name ‘First GS identity’ indicates, there is a second GS identity, illustrated in Figure 3. Here again the row totals give the Fibonacci sequence, this time starting with $0, 1$ instead of $1, 1$.

Proposition 2 (Second GS identity — amateur version) *The sequence*

$$q(n) = \sum_{-\left\lfloor \frac{n-1}{5} \right\rfloor \le \lambda \le \left\lfloor \frac{n+2}{5} \right\rfloor} (-1)^\lambda \binom{n}{\left\lfloor \frac{n-1+5\lambda}{2} \right\rfloor}$$

satisfies the recurrence

$$q(n) = q(n-1) + q(n-2),$$

with

$$q(0) = 0;\ q(1) = 1.$$

The proof is the same.

Along with the GS identities comes the simpler ‘slalom identity’: Totalling the entries in any row of Figure 4 yields 1. Here is the identity as Frobenius formulated it:

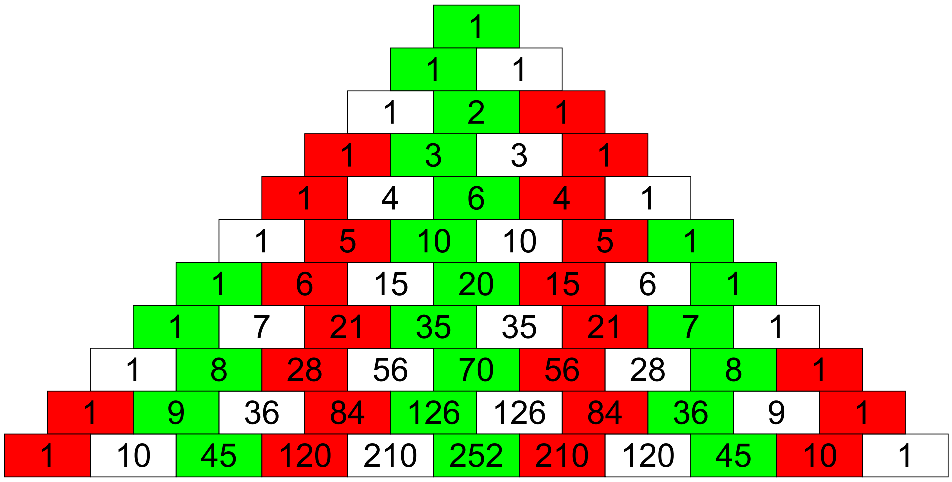

Figure 4: Slalom identity.

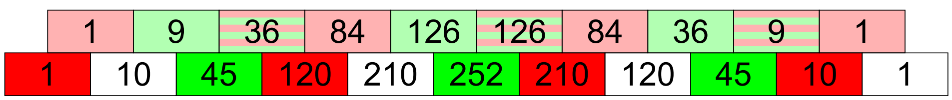

Figure 5: Proof of the slalom identity.

Proposition 3 (Slalom identity — amateur version) *The sequence*

$$r(n) = \sum_{-\lfloor \frac{n}{3} \rfloor \leq \lambda \leq \lfloor \frac{n+1}{3} \rfloor} (-1)^\lambda \binom{n}{\lfloor \frac{n+3\lambda}{2} \rfloor}$$

is the constant sequence $1, 1, 1, \ldots$. *Or, as we prefer to say, it satisfies the recurrence*

$$r(n) = r(n-1),$$

with

$$r(0) = 1.$$

The proof of the slalom identity is like that of the GS identities, only simpler: See Figure 5. In symbols,

$$\begin{aligned}
&\binom{n}{k} - \binom{n}{k+1} \\
= &\binom{n-1}{k-1} + \binom{n-1}{k} - \binom{n-1}{k} - \binom{n-1}{k+1} \\
= &\binom{n-1}{k-1} - \binom{n-1}{k+1}.
\end{aligned}$$

3 Conclusion

Shown the amateur alpine identities, any number of combinatorialists will know to look for the pro analogs involving Gaussian binomial coefficients, and will easily find them; the proofs precisely follow those of amateur versions to which they reduce. And then, just as Schur did, they will know how derive the Rogers-Ramanujan identities from the GS identities, either before or after deriving Euler's pentagonal number theorem from the slalom identity. We'll give details of this below, as Addenda. We conclude here because, from the point of view of any number of combinatorialists, we're already done.

4 Addendum: The pro alpine identities

The polynomial analogs of binomial coefficients are the Gaussian binomial coefficients, introduced by Gauss [3, p. 16]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(x^n-1)(x^{n-1}-1)\dots(x^{n-k+1}-1)}{(x-1)(x^2-1)\dots(x^k-1)}$$

Here we are to understand that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if any of $n, k, n-k$ are negative. (The notation $\begin{bmatrix} n \\ k \end{bmatrix}$ was introduced by Schur [4, p. 128] in the paper we're discussing.)

These are indeed polynomials because they satisfy the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + x^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

with

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1.$$

Because
$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$
we have the alternative recurrence
$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n \\ n-k \end{bmatrix} \\
&= \begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix} + x^{n-k} \begin{bmatrix} n-1 \\ n-k \end{bmatrix} \\
&= x^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}.
\end{aligned}$$

Proposition 4 (First GS identity) *Setting*
$$a(\lambda) = \frac{5\lambda^2 - \lambda}{2},$$
the sequence
$$P(n) = \sum_{-\lfloor \frac{n}{5} \rfloor \le \lambda \le \lfloor \frac{n+1}{5} \rfloor} (-1)^\lambda x^{a(\lambda)} \begin{bmatrix} n \\ \lfloor \frac{n+5\lambda}{2} \rfloor \end{bmatrix}$$
satisfies the recurrence
$$P(n) = P(n-1) + x^{n-1} P(n-2),$$
with
$$P(-1) = 0;\ P(0) = 1.$$

Proof. We repeat the proof of the amateur identity, sprinkling in appropriate powers of x here and there.
$$\begin{aligned}
\begin{bmatrix} n \\ k-1 \end{bmatrix} &= \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + x^{k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
&= \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + x^{k-1} \left(x^{n-k} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} \right) \\
&= \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + x^{n-1} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} + x^{k-1} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix},
\end{aligned}$$

while

$$\begin{aligned}
\begin{bmatrix} n \\ k+1 \end{bmatrix} &= x^{n-k-1}\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k+1 \end{bmatrix} \\
&= x^{n-k-1}\left(\begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + x^k \begin{bmatrix} n-2 \\ k \end{bmatrix}\right) + \begin{bmatrix} n-1 \\ k+1 \end{bmatrix} \\
&= x^{n-k-1}\begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + x^{n-1}\begin{bmatrix} n-2 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}.
\end{aligned}$$

Taking a linear combination with coefficients $\alpha, -\beta$, we get

$$\begin{aligned}
\alpha\begin{bmatrix} n \\ k-1 \end{bmatrix} - \beta\begin{bmatrix} n \\ k+1 \end{bmatrix} &= \alpha\begin{bmatrix} n-1 \\ k-2 \end{bmatrix} - \beta\begin{bmatrix} n-1 \\ k+1 \end{bmatrix} + \\
&\quad x^{n-1}\left(\alpha\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \beta\begin{bmatrix} n-2 \\ k \end{bmatrix}\right) + \\
&\quad (\alpha x^{k-1} - \beta x^{n-k-1})\begin{bmatrix} n-2 \\ k-1 \end{bmatrix} .
\end{aligned}$$

Now consider any pair of entries $\left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right]$ occupying cells of opposite color, meaning that

$$k-1 = \left\lfloor \frac{n+5\lambda}{2} \right\rfloor ; \; k+1 = \left\lfloor \frac{n+5(\lambda+1)}{2} \right\rfloor .$$

This happens just when n and λ have the same parity, making

$$k-1 = \frac{n+5\lambda}{2}$$

and

$$\lambda = \frac{2k-n-2}{5},$$

so that

$$a(\lambda+1) - a(\lambda) = 5\lambda + 2 = 2k - n.$$

This means that if we take $\alpha = x^{a(\lambda)}$, $\beta = x^{a(\lambda+1)}$ in the expressions above, the final term vanishes because the coefficient is

$$x^{a(\lambda)}x^{k-1} - x^{a(\lambda+1)}x^{n-k-1} = x^{a(\lambda)}x^{k-1}(1 - x^{a(\lambda+1)-a(\lambda)}x^{n-2k}) = 0.$$

This establishes the recurrence. ♠

Proposition 5 (Second GS identity) *Setting*

$$b(\lambda) = \frac{5\lambda^2 - 3\lambda}{2},$$

the sequence

$$Q(n) = \sum_{-\lfloor \frac{n-1}{5} \rfloor \le \lambda \le \lfloor \frac{n+2}{5} \rfloor} (-1)^\lambda x^{b(\lambda)} \begin{bmatrix} n \\ \lfloor \frac{n-1+5\lambda}{2} \rfloor \end{bmatrix}$$

satisfies the recurrence

$$Q(n) = Q(n-1) + Q(n-2),$$

with

$$Q(0) = 0;\ Q(1) = 1.$$

Proof. Same as above, only now the entries $\left[{n \atop k-1}\right]$ and $\left[{n \atop k+1}\right]$ have opposite color just when n and λ have opposite parity, making

$$k - 1 = \frac{n - 1 + 5\lambda}{2}$$

and

$$\lambda = \frac{2k - 1 - n}{5},$$

so that once again

$$b(\lambda + 1) - b(\lambda) = 5\lambda + 1 = 2k - n. \quad \spadesuit$$

Proposition 6 (Slalom identity) *Setting*

$$c(\lambda) = \frac{3\lambda^2 - \lambda}{2},$$

the sequence

$$R(n) = \sum_{-\lfloor \frac{n}{5} \rfloor \le \lambda \le \lfloor \frac{n+1}{5} \rfloor} (-1)^\lambda x^{c(\lambda)} \begin{bmatrix} n \\ \lfloor \frac{n+3\lambda}{2} \rfloor \end{bmatrix}$$

is the constant sequence $1, 1, 1, \ldots$. *Or, as we prefer to say, it satisfies the recurrence*

$$R(n) = R(n-1),$$

with

$$R(0) = 1.$$

Proof.

$$\begin{aligned}
& \alpha\begin{bmatrix} n \\ k \end{bmatrix} - \beta\begin{bmatrix} n \\ k+1 \end{bmatrix} \\
= \quad & \alpha\left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + x^k\begin{bmatrix} n-1 \\ k \end{bmatrix}\right) - \beta\left(x^{n-k-1}\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}\right) \\
= \quad & \alpha\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \beta\begin{bmatrix} n-1 \\ k+1 \end{bmatrix} + \\
& (\alpha x^k - \beta x^{n-k-1})\begin{bmatrix} n-1 \\ k \end{bmatrix}
\end{aligned}$$

If the entries $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right]$ are in cells of opposite color, then n and λ have the same parity, making

$$k = \frac{n+3\lambda}{2}$$

and

$$\lambda = \frac{2k-n}{3},$$

so that

$$c(\lambda+1) - c(\lambda) = 3\lambda + 1 = 2k - n + 1,$$

so that with $\alpha = c(\lambda)$, $\beta = c(\lambda+1)$, the coefficient of $\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ above is

$$x^{c(\lambda)}x^k - x^{c(\lambda+1)}x^{n-k-1} = x^{c(\lambda)}x^k(1 - x^{c(\lambda+1)-c(\lambda)}x^{n-2k-1}) = 0. \quad \spadesuit$$

5 Addendum: Euler and Rogers-Ramanujan

Schur deduces the Euler pentagonal number theorem and Rogers-Ramanujan identities by combining the alpine identities with the Jacobi triple product formula (cf. Gauss [2]) in a way that by now is very well known (cf. Andrews and Eriksson [1]). We briefly review this here.

Attribute to an integer partition $a = (a_1, \ldots, a_n)$,

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq 1$$

the *weight* $x^{a_1+\ldots+a_n}$. Totaling the weights of all partitions gives

$$E = \prod_i \frac{1}{(1-x^i)}.$$

The Gaussian binomial $\begin{bmatrix} n \\ k \end{bmatrix}$ totals the weights of partitions with at most $n-k$ parts, each of size at most k. If $k, n-k$ both tend to infinity, we get E:

$$\lim_{k,n-k\to\infty} \begin{bmatrix} n \\ k \end{bmatrix} = E.$$

Taking n to infinity in the slalom identity, we get

$$1 = \sum_{\lambda} (-1)^{\lambda} x^{\frac{3\lambda^2-\lambda}{2}} E,$$

or

$$\prod_i (1-x^i) = \sum_{\lambda} (-1)^{\lambda} x^{\frac{3\lambda^2-\lambda}{2}}.$$

This is Euler's pentagonal number theorem.

Call a a *kangaroo partition* if

$$\min(a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n) \geq 2.$$

$P(n)$ is the weight sum for kangaroo partitions with maximum part at most $n-1$. Taking n to infinity in GS1, on the left we get the weight sum $P(\infty)$ over all kangaroo partitions, while on the right all the Gaussian binomials turn into E:

$$P(\infty) = JE,$$

where

$$J = \sum_{\lambda} (-1)^{\lambda} x^{\frac{5\lambda^2-\lambda}{2}}.$$

But from the Jacobi triple product identity we have

$$J = \prod_{k\geq 0} (1-x^{5k+2})(1-x^{5k+3})(1-x^{5k+5}),$$

so

$$P(\infty) = \prod_{k\geq 0} \frac{1}{(1-x^{5k+1})(1-x^{5k+4})}.$$

The right hand side here enumerates partitions with all parts congruent to $\pm 1 \bmod 5$. This is the first Rogers-Ramanujan identity.

We get the second Rogers-Ramanujan identity from GS2 in like manner:

$$Q(\infty) = \prod_{k\geq 0} (1-x^{5k+2})(1-x^{5k+3}).$$

On the left we have kangaroo partitions with minimum part size at least 2, while on the right we have partitions into parts congruent to $\pm 2 \bmod 5$.

6 Addendum: Bijective proofs of the alpine identities

From our algebraic proofs of the alpine identities we can extract bijections that pair up terms of opposite sign, leaving a single positive term in each row for the slalom, or a Fibonacci number of positive terms for GS1 and GS2.

For GS1, if you trace through what cancels with what, you will find that the terms that remain correspond to partitions built from L-shaped pieces with equal or all-but-equal prongs, as indicated in Figure 6. All other partitions are matched up into pairs of opposite sign as shown in Figure 7. Here's pseudo-code for bijections that do this pairing:

```
leftmatch[{A, steps___}] := Join[rightmatch[{steps}], {B}]
leftmatch[{B, steps___, A}] := Join[{B}, leftmatch[{steps}], {A}]
leftmatch[{B, steps___, B}] := {A, steps, A}
leftmatch[{steps___}] := {steps}

rightmatch[{steps___, B}] := Join[{A}, leftmatch[{steps}]]
rightmatch[{B, steps___, A}] := Join[{B}, rightmatch[{steps}], {A}]
rightmatch[{A, steps___, A}] := {B, steps, B}
rightmatch[{steps___}] := {steps}
```

Apply `rightmatch` or `leftmatch` according as the parity of $n + \lambda$ is even or odd.

Verifying directly that these bijections have the stated properties gives us bijective proofs of the alpine identities. It would be a mistake to regard these bijective proofs as distinct from the algebraic proofs from which they are extracted. They are, collectively, just another manifestation of Frobenius's last proof.

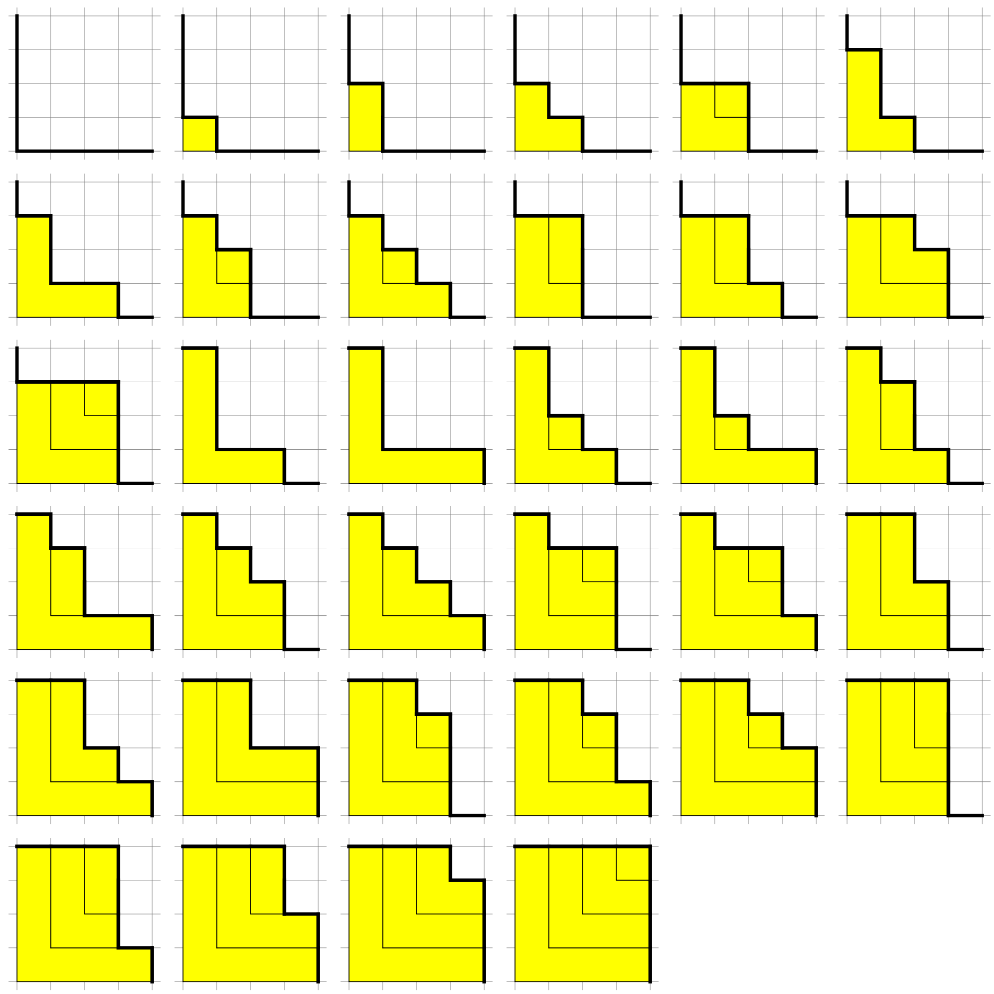

Figure 6: The 34 integer partitions belonging to $P(8)$.

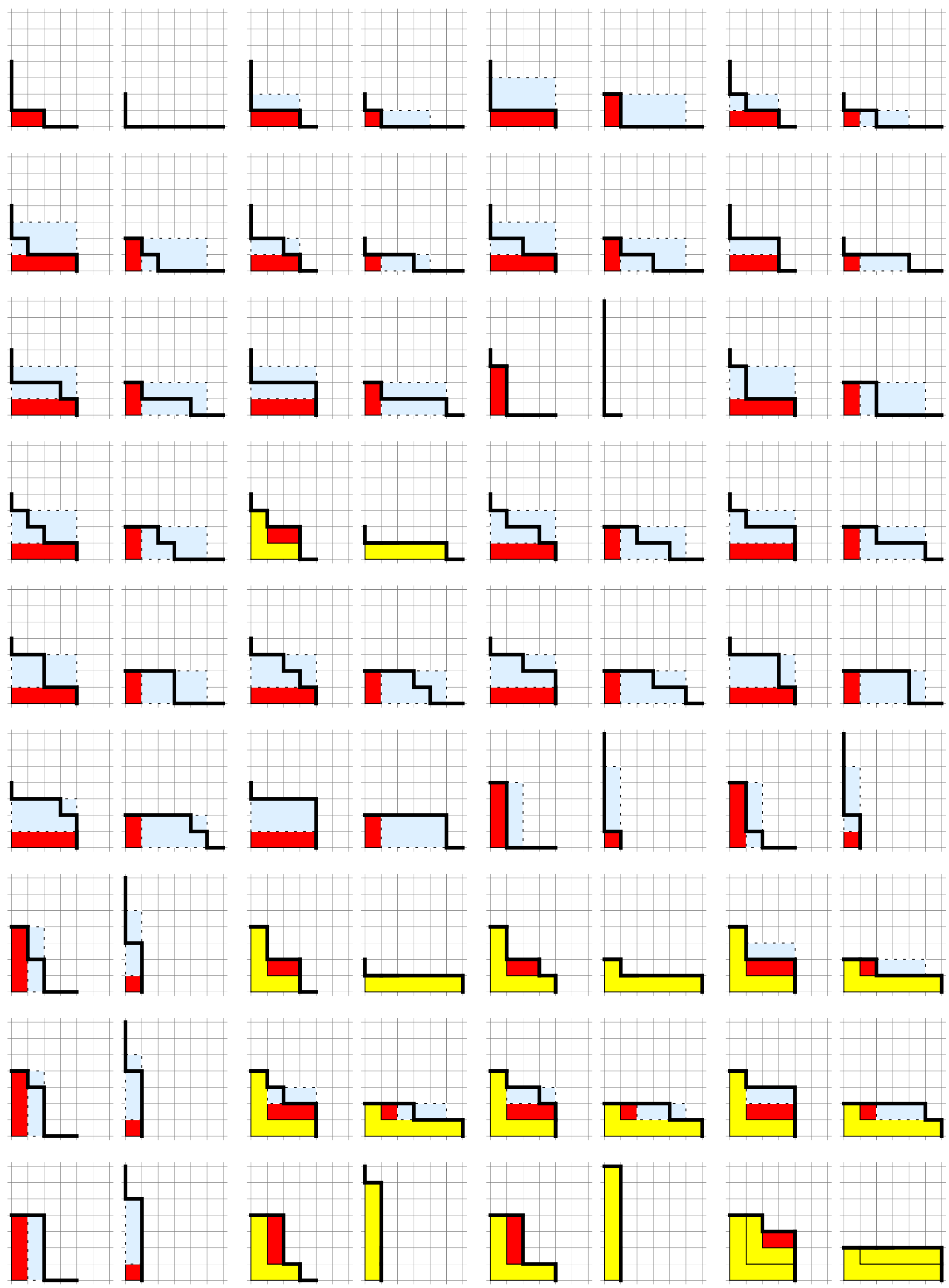

Figure 7: The 36 canceling pairs on the right of the GS1 identity for P(8).

References

[1] George E. Andrews and Kimmo Eriksson. *Integer Partitions.* 2004.

[2] C. F. Gauss. Hundert Theoreme über die neuen Transscendenten. In *Werke*, volume 3, pages 461–469. 1876.

[3] C. F. Gauss. Summatio quarumdam serierum singularium. In *Werke*, volume 2, pages 2–45. 1876.

[4] Issai Schur. Ein Beitrag zur additiven Zahlentheorie and zur Theorie der Kettenbrüche. *S.-B. der Preuss. Akad. Wiss., Phys.-Math. Kl.*, pages 302–321, 1917. In *Gesammelte Abhandlungen*, vol. 2, pages 117–136.