Hedging Huygens

TSILB*

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Sources and acknowledgements

- This work includes a reworking of part of the 1714 English translation of Huygens's 1657 work 'Libellus de ratiociniis in ludo aleae'; this translation has been attributed to John Arbuthnot. We found a copy online on the University of York history of statistics website:

  `http://www1.york.ac.uk/depts/maths/histstat/huygens.htm`

- An account of Huygens's approach to expected value is given in Grinstead and Snell's 'Introduction to Probability'.
- Special thanks are due to Stephen Stigler for his guest lecture on Huygens in Greg Leibon's Dartmouth CHANCE class, in the fall of 2002.

The fair market value of a gamble

In 1657, in the first probability book ever published, Christiaan Huygens gave an argument for taking the expected value $\bar{X}$ of a gamble X to be the 'fair value' of X. We show here that by slightly modifying and extending Huygens's argument, we can convert it into a 'hedging' argument, on the basis of which we may take $\bar{X}$ to be the 'fair market value' of X.

The terms 'fair value' and 'fair market value' sound similar, but the sounds are more similar than the concepts being named. That's because

*This Space Intentionally Left Blank. Contributors include: Peter Doyle.

the ear tends to forget that, in the phrase 'fair market value', it is the market rather than the value that is supposed to be 'fair'. In saying that $\bar{X}$ is the fair market value of X, we are predicting that this is the price for which the gamble X would sell in a fair market.

We will say more about all this anon. But first, let us see how this argument goes.

The postulates

AS a Foundation to the following Propositions, I take Leave to lay down these truths:

POSTULATE I. *Any Chance or Expectation is worth no more than such a Sum, as would procure in the same Chance or Expectation at a fair Lay.*

POSTULATE II. *Any Chance or Expectation is worth no less than such a Sum, as I would be able to procure with Certainty by taking that Chance or Expectation together with a fair Lay.*

Here the terms 'Chance' or 'Expectation' just mean what we would nowadays call a 'gamble', or 'random variable'. A 'fair Lay' means some kind of bet which we regard as manifestly fair, or a possibly dependent combination of such manifestly fair bets. We're using these quaint terms in honor of Huygens. This is probably a bad idea, since nowadays we're likely to take 'Chance' to mean 'probability'; and 'Expectation' to mean 'expected value'; and 'fair Lay' to mean nothing much at all. So let's rephrase our postulates in more modern terminology:

Postulate 1. *Any gamble X is worth no more than such a sum, as it would take to synthesize a gamble equivalent to X (one having the same payoffs, with the same probabilities), by placing some number of bets which are individually manifestly fair, but which may be dependent upon one another.*

Postulate 2. *Any gamble X is worth no less than such a sum, as could be procured with certainty by taking the payoff X in combination with the payoff of some number of bets which are individually manifestly fair, but which may be dependent upon one another, and upon X.*

The rationale for these postulates should be clear. Regarding Postulate 1, I am certainly not going to pay more for the gamble X than it would take to synthesize an equivalent gamble. This holds whether I plan to take the payoff X myself, or plan to resell the gamble to someone else. If I plan to take the payoff X, and X is selling for more than it would take to put together an equivalent gamble, I'll go with the equivalent gamble, thank you very much. And as I presume the same goes for my potential customers, I have no hope of reselling X for more than it would take a potential customer to synthesize an equivalent bet.

As for Postulate 2, if by taking the payoff X in combination with some number of bets will guarantee a certain payoff, I can't hope to be able to buy the gamble X for less, because I will be outbid by others who will offer more than I am offering but less than they will be able to recoup with certainty.

In addition to the usual assumptions about the 'efficiency' of the market, the key assumption behind these two postulates is that there is no problem lining up takers for certain 'manifestly fair bets'. As we'll see very shortly, such 'manifestly fair' bets will be taken to include fair coin flips; symmetrical lotteries; and close relatives. While we will only consider bets that would be readily acknowledged to be 'fair', on grounds of symmetry, in fact it matters not a whit to us whether, or in what sense, these bets really are fair. All that matters is that there is no problem finding takers for these bets.

The assumption that there will always be takers for 'manifestly fair' bets is a good point of departure for the analysis, but this assumption should not be regarded as being self-evident, or even true. We will have much to say about the limitations to the analysis that stem from the falsity of the postulates upon which it is based. But for the moment, we will treat the postulates as true.

Putting the postulates to work

WITH the Aid of these Postulates, I may find the Value of all Chances in Games of Fortune; Cards, Dice, Wagers, Lotteries, etc.

AS for Example, if Alice puts 3 Shillings in one Hand, without letting me know which, and 7 in the other, and gives me Choice of either of them; I say, it is the same thing as if Alice were to give me 5 Shillings. Because with 5 Shillings I may, at a fair Lay, procure the same even Chance or Expectation to win 3 or 7 Shillings; So this Chance can be worth no more than 5 Shillings. But if Alice gives me this Chance, then I may play with Laurie at a fair Lay, so as to obtain with Certainty 5 Shillings; So this Chance can be worth no less than 5 Shillings.

PROP. I. *If I expect a or b from Alice, and have an equal Chance of gaining either of them, my Expectation is worth $\frac{a+b}{2}$.*

THE Demonstration of which is very easy: For supposing without Loss of Generality that $a \leq b$; Then, starting with $\frac{a+b}{2}$ in Hand, I may stake $\frac{b-a}{2}$

with Laurie, who should likewise stake $\frac{b-a}{2}$, upon the Toss of a Coin. If I lose the Toss I will have a, but if I win the Toss I will have b. So by Postulate I, my Expectation from Alice can be worth no more than $\frac{a+b}{2}$.

But having this Expectation from Alice, I may play Laurie, that if Alice gives me a Laurie will give me $\frac{b-a}{2}$; while if Alice gives me b I will give Laurie this same sum $\frac{b-a}{2}$; and then with Certainty I will come off with $\frac{a+b}{2}$. So my Expectation from Alice can be worth no less than $\frac{a+b}{2}$. *Q.E.D.*

IN Numbers. If I have an equal Chance to 3 or 7, then my Expectation is, by this *Proposition*, worth 5, and it is certain I can procure 5, For if I and Laurie stake 2 a piece upon this Condition, That if Alice gives me 3 Laurie will give me 2, and if Alice gives me 7 I will give Laurie 2, 'tis plain the Lay is just and that I am certain to come off with 5.

PROP. II. *If I expect a, b, or c, and each of them is equally likely to fall to my Share, my Expectation is worth* $\frac{a+b+c}{3}$.

FOR supposing I play with two others upon this Condition, That every one of us stake x, the Winner to take the entire Stake $3x$; and I agree with one of them, that which soever of the Two wins, will give the Loser b; and with the other, that which soever of the Two wins, will give the Loser c. It appears evidently, that the Lay is very fair, and that I have by this means an equal Chance to gain b, if the first wins; or c if the second wins; or $3x-b-c$, if I win my self. But if $3x-b-c=a$, we will find $x=\frac{a+b+c}{3}$. So with $\frac{a+b+c}{3}$ I am able, by fair Gaming, to obtain an equal Expectation of a, b, or c; so this Expectation can be worth no more than $\frac{a+b+c}{3}$.

BUT having an equal Expectation of a, b, or c, and supposing for Simplicity that $a \leq \frac{a+b+c}{3} \leq b \leq c$ I may agree with Laurie, that if I get b from Alice, I will pay Laurie $b-\frac{a+b+c}{3}$; while if I get a from Alice, Laurie will pay me this same Stake; and at the same Time I may agree with Martha, that if I get c from Alice, I will pay Martha $c-\frac{a+b+c}{3}$; while if I get a from Alice, Martha will pay me this same Stake. And thus I will come off with $\frac{a+b+c}{3}$ no Matter what I get from Alice. *Q.E.D.*

AFTER the same Manner, an even Chance to a, b, c or d, will be found worth $\frac{a+b+c+d}{4}$. And so on.

Looking closely at the postulates

The big assumption behind our two postulates is the notion that there is no shortage of opportunities to place certain manifestly fair bets, called 'fair

Lays'. Among these fair Lays we have included coin flips; symmetrical lotteries; and 'can flips', by which we mean a bet on the flip of a 'fair can', which in addition to landing 'heads' or 'tails' may land 'sides', with a probability depending on the aspect ratio of the can. We do not assume that these fair Lays are independent.

Problem: Determine the dependence of the probability of 'sides' on the aspect ratio of the can.

Now, as far as the mathematical theory goes, it would suffice to consider as manifestly fair only (dependent) combinations of coin flips, as from these we can synthesize any symmetrical lottery or can flip.

But while considering only coin flips might make the theory mathematically leaner and meaner, it behooves us to consider lotteries and can flips as well. That's because the postulates we're using are *wrong*: In practice, not all 'manifestly fair' bets will be available, and I won't be able to line up any set of dependent coin flips that my heart might desire. So from a practical point of view, the better argument is not that which uses only those bets which are mathematically simplest, but that which contemplates those bets for which it seems most likely that it will be able to find takers.

As an example, let us discuss the possible short-comings of our hedging arguments when applied to a variant of the St. Petersburg paradox.

Peter's gamble

Peter asks me for a bid on the following gamble. I get to flip a coin up to 10 times. If I get heads on the kth flip, $1 \leq k \leq 10$, I collect 2^{k-1} and stop. If I manage to flip tails 10 times in a row, I collect 1024.

How much should I offer Peter for this gamble? In theory, the value of this gamble is

$$\begin{aligned} & (1/2 \cdot 1 + 1/4 \cdot 2 + 1/8 \cdot 4 + \ldots + 1/2^{10} \cdot 2^{9}) + 1/2^{10} \cdot 2^{10} \\ = \; & 10 \cdot 1/2 + 1 \\ = \; & 6. \end{aligned}$$

This means that with the aid of side bets, I can in theory arrange to net 6 from this gamble no matter what.

Here's how it might go: On the first flip, I'll make a side bet on heads with Laurie, for 5. If I flip heads, I'll collect 5 from Laurie and 1 from Peter, so I'll wind up with 6, as promised. If I flip tails, I'll pay 5 to Laurie, making

Flip	Side Bet	Heads fortune	Tails fortune
1	5	5+1=6	-5
2	9	-5+9+2=6	-5-9=-14
3	16	-14+16+4=6	-14-16=-30
4	28	-30+28+8=6	-30-28=-58
5	48	-58+48+16=6	-58-48=-106
6	80	-106+80+32=6	-106-80=-186
7	128	-186+128+64=6	-186-128=-314
8	192	-314-192+128=6	-314-192=-506
9	256	-506+256+256=6	-506-256=-762
10	256	-762+256+512=6	-762-256+1024=6

Table 1: Side bets.

my fortune -5, and I'll move on to my next flip. On subsequent flips, I will keep adjusting my side bet with Laurie so that if I flip heads I'll wind up with 6. As Table 1 shows, no matter what happens I'll wind up ahead by 6.

How plausible is this scenario? To play things this way, I need to have 762 at least in available credit, because that is how far I may have to go in the hole before emerging with my 6. I also have to rely on Laurie's willingness to keep betting. No matter what happens, Laurie risks losing at most 5, which means that I'm not exposing him to a lot of risk. But what if he should decide to quit once he us up by a few hundred?

For example, after I've flipped tails for the seventh time, Laurie will be up (and I'll be down) 314. I offer him a bet of 192 on the next flip. What if he says, 'No thanks'? If I now flip heads, I'll get 128, so I'll wind up behind by $314-128 = 186$; if I flip tails, then heads, I'll get 256, and wind up behind by $314 - 256 = 58$. Of course I may flip tails, tails, heads, and get 512, to wind up ahead 198; or even better, I may flip tails, tails, tails, to wind up ahead 710. My 'expected value' is still 6. But I have no interest in being in this exposed position!

The moral is, a gamble is not necessarily worth its theoretical 'expected value', because the hedging argument showing that it is worth at least this much fails to take into account possible limits to my credit, or to my ability to locate gamesters with the will and the wherewithal to make large bets on the flip of a coin. The fault lies not in the argument itself, but in the postulates upon which the argument is based.

Back to Huygens

Where we have used two postulates, Huygens used only one, which he took leave to lay down as a self-evident truth:

HUYGENS'S POSTULATE. *That any one Chance or Expectation to win any thing is worth just such a Sum, as would procure in the same Chance and Expectation at a fair Lay.*

In fact, we can derive Huygens's Postulate from our Postulates 1 and 2. The way to do so will probably be clear by now. In applying our two postulates to derive the fair market value of a gamble X, our first step has been to write

$$X = \bar{X} + Y,$$

with Y a fair Lay; from this we conclude that the value of X was at most $\bar{X}$. Our second step has been to write

$$\bar{X} = X + Z,$$

with Z a fair lay; from this we conclude that the value of X was at least $\bar{X}$.

Was this second step really necessary? If

$$X = \bar{X} + Y$$

then

$$\bar{X} = X + (-Y).$$

The question is whether $-Y$ is a fair lay whenever Y is. Negating a coin flip gives another coin flip, with the parties having traded sides; the same goes for a can flip. Negating a lottery doesn't give a lottery—but as observed earlier, a lottery is equivalent to a collection of dependent coin flips, and since negating coin flips preserves fairness, so does negating a lottery. Hence the negative of a fair Lay is a fair Lay (at least if it only involves coin flips, can flips, and lotteries), and hence Huygens's Postulate follows from our Postulates 1 and 2.

An alternative approach to negating a lottery would be to observe that the result is an 'anti-lottery', whereby a randomly selected player makes a fixed payment to all the other players. Such an arrangement would qualify as 'manifestly fair', and well might arise in practice, for example if a group of drinkers were to draw straws to determine who will buy the next round.

But we must emphasize once more that what is important to us in not whether an anti-lottery is 'fair', but whether we can line up takers when we need to.

More generally, since we are expecting difficulties lining up fair bets under various circumstances, this whole theoretical argument deducing Huygens's Postulate from our Postulates 1 and 2 is somewhat beside the point, which is that Postulates 1 and 2 are false, practically speaking.

Disclaimer. Nothing that has been said here should be taken as a criticism of Huygens's approach. As we have seen, Huygens's Postulate follows from our Postulates 1 and 2. If our aim were more theoretical, as Huygens's was, we might well have followed Huygens's example, and taken leave to put forth Huygens's Postulate as a 'self-evident truth'.

The law of large numbers

The law of large numbers provides an alternative approach to determining the fair market value of a gamble. It implies that sufficiently rich persons can make good money (with near certainty) by taking on a large number of small, independent, slightly favorable bets. Consequently, if a gamble X is for sale, we may expect that there will be people who are willing to pay something close to the expected value $\bar{X}$ for X, even if they don't plan to hedge the gamble.

The willingness of sufficiently rich persons to take on small, independent, nearly fair bets goes a long way toward justifying our use of Postulates I and II. It also helps clarify the circumstances under which we might expect our postulates to break down, namely, if we need takers for large bets, or dependent bets, or too-nearly-fair bets.

Problem. It appears that from the law of large numbers, it follows immediately that $\bar{X}$ is the fair market value of X. If so, why bother to derive this result from Postulates I and II, as we have done above?

Willingness to pay

It is frequently asserted that since $\bar{X}$ is the 'fair value' of the gamble X, it necessarily follows that I, or we, or you, or anyone at all 'should be willing' to pay $\bar{X}$ for X. An uncritical acceptance of this assertion leads to preposterous conclusions.

Let us suppose that $\bar{X}$ really is the fair market value of X, which as we have discussed may or may not be the case. Then certainly I should not expect to buy X for less than $\bar{X}$. If for some reason I do get a chance to buy X for substantially less than $\bar{X}$, I will do so, with the idea of turning right around and reselling it for $\bar{X}$. Or, if I am sure I'll be able to line up the necessary bets, I may decide to buy X at its discounted price; hedge the gamble myself; and collect the constant payoff of $\bar{X}$. Or—and this is the most likely alternative—I may figure there must be something fishy going on, and steer clear of the whole business. What I will NOT do is to buy X at the discounted price, and then simply accept the unhedged random payoff X. To suggest that I 'should be willing' to do so is simply wrong.

If the going rate for the privilege of sleeping over at the White House is \$100,000, and I have an opportunity to buy a ticket for \$80,000, it makes sense for me to do so, provided that I turn right around and sell the ticket to someone else for \$100,000. But the fact that someone is willing to pay \$100,000 for a night in the Lincoln bedroom does not imply that I 'should be willing' to pay \$100,000, or even \$80,000, for a chance to sleep there myself.