

# Max-min approach to Perron-Frobenius

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## Abstract

We give a max-min formula for the Perron-Frobenius eigenvalue of a positive matrix.

Let  $A$  be a square matrix with positive entries, or more generally, with non-negative entries, enough of which are positive so that the Perron-Frobenius theorem will apply to guarantee that  $A$  has positive row and column eigenvectors. These eigenvectors  $\bar{\nu}$  and  $\bar{\phi}$  are unique up to positive scalar multiples, and they share a common positive eigenvalue  $\lambda$ :

$$\bar{\nu}A = \lambda\bar{\nu},$$

$$A\bar{\phi} = \lambda\bar{\phi}.$$

Given vectors  $\nu$  and  $\phi$ , we denote by  $\nu * \phi$  their element-by-element product:

$$(\nu * \phi)_i = \nu_i \phi_i.$$

We write  $\nu\phi$  for the usual scalar product:

$$\nu\phi = \sum_i \nu_i \phi_i.$$

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This is justified because if we think of  $\nu$  and  $\phi$  as row and column vectors, respectively,  $\nu\phi$  is just the usual matrix product.

**Proposition.**

$$\lambda = \max_{\mu} \min_{\nu * \phi = \mu} \frac{\nu A \phi}{\nu \phi}.$$

Here and throughout  $\mu$ ,  $\nu$ , and  $\phi$  denote vectors with strictly positive entries.

**Proof.** For any  $\nu$  we have

$$\frac{\nu A \bar{\phi}}{\nu \bar{\phi}} = \lambda,$$

so taking  $\phi = \bar{\phi}$  shows that

$$\max_{\mu} \min_{\nu * \phi = \mu} \frac{\nu A \phi}{\nu \phi} \leq \lambda$$

To prove the inequality in the other direction, take  $\mu = \bar{\nu} * \bar{\phi}$ , and suppose that  $\nu * \phi = \mu = \bar{\nu} * \bar{\phi}$ . Write

$$\nu_i = \bar{\nu}_i f_i,$$

so that

$$\phi_i = \bar{\phi}_i / f_i.$$

Then

$$\begin{aligned} & \nu A \phi - \bar{\nu} A \bar{\phi} \\ = & \sum_{ij} \bar{\nu}_i A_{ij} \bar{\phi}_j \left( \frac{f_i}{f_j} - 1 \right) \\ \geq & \sum_{ij} \bar{\nu}_i A_{ij} \bar{\phi}_j \log \frac{f_i}{f_j} \\ = & \sum_{ij} \bar{\nu}_i A_{ij} \bar{\phi}_j \log f_i - \sum_{ij} \bar{\nu}_i A_{ij} \bar{\phi}_j \log f_j \\ = & \sum_i \bar{\nu}_i \lambda \bar{\phi}_i \log f_i - \sum_j \lambda \bar{\nu}_j \bar{\phi}_j \log f_j \\ = & \lambda \left( \sum_i \bar{\nu}_i \bar{\phi}_i \log f_i - \sum_j \bar{\nu}_j \bar{\phi}_j \log f_j \right) \\ = & 0. \end{aligned}$$

Thus

$$\nu A\phi \geq \bar{\nu} A\bar{\phi} = \lambda \bar{\nu}\bar{\phi} = \lambda \nu\phi,$$

so

$$\max_{\mu} \min_{\nu * \phi = \mu} \frac{\nu A\phi}{\nu\phi} \geq \lambda. \quad \blacksquare$$

**Corollary.**

$$\lambda = \max_{\mu} \min_{\phi} \frac{\sum_i \mu_i \frac{(A\phi)_i}{\phi_i}}{\sum_i \mu_i}. \quad \blacksquare$$

**Note.** I found this result and proof back in 1984 by painstakingly discretizing Charles Holland’s variational characterization of the principal eigenvalue of a second-order linear elliptic equation [3, 4]. This theorem applies in particular to diffusion-with-drift processes. To discretize Holland’s result, I created a continuous process on a 1-dimensional simplicial complex with ‘fat nodes’, and took a limit under which diffusion-with-drift on this 1-complex approached a continuous-time but discrete-space Markov chain. Taking the limit of Holland’s proof, so to speak, yielded the proof here. This discretization process took far longer than you would imagine, and it was made more painful by the ever-present suspicion that it should be possible just to write down the discrete analog of Holland’s result. I could likely have done this if I had paid more attention to the closely related work of Donsker and Varadhan [1, 2].

## References

- [1] M. D. Donsker and S. R. S. Varadhan. On a variational formula for the principal eigenvalue for operators with maximum principle. *Proc. Nat. Acad. Sci. U.S.A.*, 72:780–783, 1975.
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